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A CHRONOLOGICAL ANNOTATED BIBLIOGRAPHY ON ORDER STATISTICS. VOL--ETC(U)
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(6) A CHRONOLOGICAL ANNOTATED BIBLIOGRAPHY ON ORDER STATISTICS,

VOLUME 1, PRE-1950,

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(10) by
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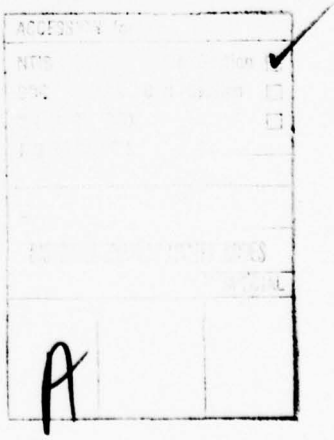
## PREFACE

In the preface to the first of the author's two 1970 volumes on *Order Statistics and their Use in Testing and Estimation*, the following statement is made: "Volume 3 will be devoted to a chronological annotated bibliography, with a brief summary of the contents of each article, together with lists of references to earlier work and citations by later authors." At the time those words were written (February 1969), the author had collected a list of slightly fewer than a thousand relevant publications, so it appeared feasible to summarize all of them in a single volume. Many things have happened in the intervening eight years to change the details of the plan, though the basic design has remained the same. When work actually began on the compilation of the bibliography, many more relevant publications were found. At times it seemed that for every publication read, two new ones were discovered, so that the question arose as to whether the process was a converging one. Moreover, the pressure of other work made it impossible to spend as much time on this task as had been anticipated and, as time passed, many new publications on order statistics appeared. As a result, several volumes will be required to complete the bibliography. It was decided that the first volume should be Volume 1 of a new series instead of Volume 3 of *Order Statistics and their Use in Testing and Estimation*, as was planned originally. Volume 1 covers relevant publications up through 1949 (a total of 937 publications). A diligent effort has been made to provide as nearly complete coverage as possible, but inevitably some relevant publications will have been overlooked.

The author wishes to acknowledge the assistance of a number of persons without whose help completion of this volume would have been difficult, if not impossible. Patricia Wittgruber, Mary Cross and Josephine Mickins of the Aerospace Research Laboratories Library provided invaluable assistance in obtaining copies of relevant publications. In later stages (since the disestablishment of ARL and the author's transfer to the Air Force Flight Dynamics Laboratory in 1975), Elizabeth Holloway and Christine Eynon of the AFFDL Library performed the same service. Copies of many publications not available at Wright-Patterson Air Force Base were obtained from the University of Illinois Library, with the assistance of Howard A. Messman, John K. Aikin and Mardelle Austin of the U. of I. Library staff and Ralph M. Riegelsperger of the ARL Support Division. Mark Breiter and David A. Lee of ARL translated several Russian papers, and copies and/or translations of a number of foreign publications were obtained from the Foreign Technology Division, with the assistance of Richard A. DiLorenzo, FTD Interfacer at AFFDL. Eva Brandenburg of ARL and Rhonda Baab of AFFDL typed the manuscript. Sharon Foley of the Aeronautical Systems Division Computer Center used a computer to generate lists of citations and author and subject indexes. Robert G. Merkle and Robert L. Neulieb of AFFDL and Albert H. Moore of the Air Force Institute of Technology read the manuscript and offered helpful suggestions. The author wishes to express his thanks to all these persons and his apologies to any others whose contributions may have been inadvertently overlooked.

Wright-Patterson Air Force Base, Ohio
August 1977





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CONTENTS


| | | PAGE |
|---|---|---|
| | INTRODUCTION | 1 |
| I. | CHRONOLOGICAL BIBLIOGRAPHY OF PRE-1950 PUBLICATIONS, WITH SUMMARIES, REFERENCES AND CITATIONS | 3 |
| | References to Post-1949 Literature (to be summarized in later volumes) | 436 |
| | Abbreviations of Names of Abstracting and Reviewing Journals | 436 |
| II. | AUTHOR INDEX (ALPHABETICAL, WITH YEARS OF PUBLICATION) | 437 |
| III. | SUBJECT INDEX (PERTINENT PUBLICATIONS LISTED CHRONOLOGICALLY) | 451 |

# INTRODUCTION

Let $x_1, x_2, \ldots, x_n$ be a sample of size n from a specified or unspecified population, arranged in nondecreasing order so that $x_1 \leq x_2 \leq \ldots \leq x_n$. Then $x_1, x_2, \ldots, x_n$ are collectively called the order statistics (plural) of the sample, and $x_i$ ($i = 1,2, \ldots, n$) is called the $i^{th}$ order statistic of the sample. In another sense of the expression, order statistics (singular) is that branch of the subject of statistics which deals with the mathematical properties of order statistics (plural) and with statistical methods based upon them. Order statistics has often been considered (incorrectly) to be a branch of nonparametric statistics (or of distribution-free statistics, which differs from nonparametric statistics in ways too subtle to discuss here). In fact, the majority of publications on order statistics deal with the order statistics of specific populations, and many of them involve the use of the sample order statistics to estimate one or more population parameters. The compiler of this bibliography (a designation which he will use in the sequel to distinguish between himself and the author of the particular publication under discussion) makes a sharp distinction between order statistics and rank statistics, which is a branch of nonparametric statistics. Publications dealing with the ranks of the observations in the sample but not with the values (or expected values) of any of them will not in general be included in this bibliography; however, a few that are needed for comparison of methods based on ranks with alternatives based on order statistics will be included. Similarly, although the method of least squares is not based on order statistics, a substantial number of publications on least squares will be included because most alternative methods are based on order statistics and comparisons with the classical method of least squares are inevitable. Likewise, since the "weakest-link" theory (based on the theory of extreme values, a branch of order statistics) plays an important role in the size effect on material strength, a few publications on the size effect which do not specifically mention the "weakest-link" theory will be included for comparative purposes.

This volume is divided into three sections. The first and largest of these is a chronological bibliography (alphabetical within years) of 937 pre-1950 publications, with summaries, references and citations. At the end of each bibliographic listing, the source(s) of any known abstract(s) and/or review(s) is (are) given. A summary of each publication (or the portion of it relevant to our subject) is given, sometimes including extensive quotations of the author's own words (or translations of them, if the publication is in a language other than English). Comments and/or notes by the compiler are included when appropriate, separate from and following the summary. The 937 publications summarized in this volume form a closed set with respect to the references and citations; references to and citations by other publications in the set are listed chronologically (alphabetically within years) for each of the 937 publications in the set. If a publication has no list of references (citations), there were none to (by) other publications included in the set. When the reference is to an edition other than the first, the date of the edition actually cited is given in brackets. References which are included only in later editions are enclosed in braces. A date in brackets in a citation list is the date of the first edition in which the citation occurs. A list of references to post-1949 literature (to be summarized in later volumes) and a list of abbreviations used for the names of abstracting and reviewing journals are given at the end of Section I. Section II is an alphabetical list of authors, with year(s) of publication for each author in parentheses following his name. Each author's name is alphabetized according to the first capital letter in his surname. Thus, de Finetti is alphabetized under F, but De Morgan is alphabetized under D;d'Addario is alphabetized under A, l'Hermite under H, von Mises under M, van den Berg under B, etc. In the case of joint authorship, the complete listing is alphabetized according to the name of the first author, and cross references are alphabetized according to the name(s) of the other author(s). Cross references are also given to different forms or spellings of authors' names. Section III is a subject index in which the subjects are listed alphabetically, with cross references when appropriate, and the publications pertinent to each subject are listed chronologically (alphabetically within years).

# SECTION I

# CHRONOLOGICAL BIBLIOGRAPHY OF PRE-1950 PUBLICATIONS, WITH SUMMARIES, REFERENCES AND CITATIONS

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* Ptolemaeus, Claudius (circa 150). *Mathematiches Syntaxeos Biblia* (Greek). German translation and explanatory notes by K. Manitius, *Ptolemäus Handbuch der Astronomie*, 2 vols., B. G. Teubner, Leipzig, 1913; revised edition, with foreword and corrections by O. Neugebauer, 1963.

**Summary:** The author gives a comprehensive account of what was known to the Greek astronomers, including some information concerning the methods they used in analyzing their observational data. He reports (Vol. I, p. 133) that Hipparchus (second century B. C.) noticed inequalities in the intervals of time between successive passages of the sun through the same solstitial point, which led him to question whether the length of the year is constant. Subsequently (see Vol. I, pp. 136-137; Plackett (1958b), p. 130), Hipparchus estimated the maximum variation in length as 3/4 day, apparently by taking half the range of his observations of the distance of the star called Spica west of the autumnal point of lunar eclipses (maximum 6-1/2°, minimum 5-1/4°, with an additional allowance of 1/4° for error in his observations and the calculations based on them).

**Note:** All references to volume and page numbers are to those of the German translation by Manitius (1913), the same pagination being retained in the revised (1963) edition.

* Galilei, Galileo (1632). *Dialogo sopra i due massimi sistemi del mondo: Tolemaico, e Copernicano.* Landini, Florence. [English translation, *Dialogue concerning the two chief world systems, Ptolemaic and Copernican*, by Stillman Drake (with foreword by A. Einstein). Univ. of Calif. Press, Berkeley, 1953].

**Summary:** Galileo, having been forbidden by the church to teach certain theories, seeks to evade this injunction by writing this dialogue in which he puts his own words into the mouths of the characters. An example is the following, chosen because of its relevance to our subject: "Then these observers being capable, and having erred for all that, and their errors needing to be corrected in order for us to get the best possible information from their observations, it will be appropriate for us to apply the minimum amendments and smallest corrections that we can—just enough to remove the observations from impossibility and restore them to possibility . . . ." [p. 290 of the English translation]. A part of the discussion of the third day of the dialogue concerns the question of determining the distance from the earth of a new star, given observations on its maximum and minimum elevation (in degrees) and the elevation of the pole star by thirteen observers at different points on the earth's surface. If the observations were exact, the distance could be determined from the observations of any two observers, and the $\binom{13}{2} = 78$ determinations made by pairing the observers in all possible ways would all give the same result. Since the observations are subject to error, 78 different distances of the star from the center of the earth are found, ranging from a value less than the radius of the earth to infinity and beyond. Both extremes are manifestly impossible. In the above statement of Salviati, one of the three participants in the dialogue, apropos this situation, we see the beginnings of the theory of errors, which attempts to determine the truth from inconsistent observations by minimizing various (non-decreasing) functions of the errors. Some of these functions, as we shall see later, are based on order statistics.

* Bernoulli, Nicolas I (1709). *Specimina Artis conjectandi, ad quaestiones Juris applicatae.* Basle. Reprinted in *Actorum Eruditorum quae Lipsiae Publicantur Supplementa* **4** (1711), 159-170.

**Summary:** Suppose there are b men who will all die within a year, and are equally likely to die at any instant within this time [uniform (rectangular) distribution of remaining life]. Find the probable duration of the life of the last survivor. Bernoulli views this problem as equivalent to the following: A line of length a is measured from a fixed origin; on this line b points are taken at random: determine the mean distance from the origin of the most distant point. He supposes that the line of length a is divided into an indefinitely large number n of equal parts, each of length c, so that nc = a; each of the b points may be at the distance c, or 2c, or 3c, . . . . up to nc; but no two or more at exactly the same distance. Bernoulli shows that the required mean distance is given by



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$$\lim_{n\to\infty} \sum_{x=b}^{n} xc \binom{x-1}{b-1} \Big/ \binom{n}{b} = \frac{ncb}{b+1} = \frac{ab}{b+1} .$$

The result obtained is precisely the expected value of the largest observation ($b^{th}$ order statistic) in a random sample of size b from a uniform distribution on the interval (0,a).

**Citations:** Todhunter (1865), Czuber (1899), Keynes (1921).

* Cotes, Roger (1722). Aestimatio errorum in mixta mathesi, per variationes partium trianguli plani et sphaerici. *Opera Miscellanea*, pp. 1-22. (Appended to his *Harmonia Mensurarum*). Cantabrigiae.

**Summary:** In the last paragraph (page 22), which is the only part of this work which is of special interest here, the author considers four observations p, q, r and s, which may not be equally reliable, of the position of a point. He proposes, as the most probable true position, a weighted average with weights P, Q, R and S which are inversely proportional to the spread of the errors to which the respective observations are subject.

**Comments:** This proposal of Cotes represents one of the earliest attempts to determine an average which uses all the observations but does not assign equal weights to all of them; later we shall encounter methods in which the weights vary according to the position of the observations in the ordered sample. Merriman (1877) has remarked (page 154): "Cotes's rule only agrees with modern methods when the observations are directly made upon one quantity". Eisenhart (1964), pages 25 and 26, has applied Cotes' suggestion to the determination of the slope of a line through the origin. For every observed point $(x_i,y_i)$ $(x_i \neq 0)$, $y_i/x_i$ is an observation on the slope. For the special case of equal weights, Cotes' suggestion leads to the condition of zero sum of residuals, so that the line must pass through the center of gravity $(\bar{x}, \bar{y})$ of the data, and hence must have slope $\bar{y}/\bar{x}$, provided $\bar{x}$ and $\bar{y}$ are not both equal to zero.

**Citations:** Lalande (1771), Laplace (1812), Ivory (1825), von Riese (1830), Jevons (1874), Merriman (1877), Pizzetti (1892), Czuber (1899).

* Euler, L. (1749). *Pièce qui a Remporté le Prix de l'Académie Royale des Sciences en 1748, sur les Inégalités du Mouvement de Saturne et de Jupiter*. Martin, Coignard et Guerin, Paris. Reprint, *Leonhardi Euleri Opera Omnia* **II 25** (*Commentationes Astronomicae* **I**), 45-157. Orell Füssli, Turici, 1960.

**Summary:** According to Eisenhart (1964), page 26, this work and the following one by Mayer (1750) give the results of independent investigations of the authors, who both in the same year (1748) developed what has come to be called the Method of Averages for fitting a linear equation to observed data. In this method the observational equations are divided into as many subsets as there are coefficients to be determined, the division being according to the values of (one of) the independent variable(s), those having the largest values of this variable grouped together, then the next largest in another group, etc. Then the equations in each group are added together, which is equivalent to applying to each subset the condition of zero sum of residuals inherent in the method of Cotes (1722) for the special case of equal uncertainties of the observations. The resulting equations, whose number is equal to the number of coefficients to be determined, are then solved simultaneously. [A drawback of this method is that the results depend on the way in which the observational equations are divided into subsets, and are therefore somewhat arbitrary and subjective.] Sheynin (1966), page 1004, credits Euler with being the first (in articles 122-123 of this work) to use a rudimentary form of the minimax principle (minimization of the maximum residual error) for solving a redundant system of linear equations.

**Citations:** Lalande (1771), von Zach (1805), Laplace (1812), Todhunter (1865), Merriman (1877).

* Mayer, Johann Tobias (1750). Abhandlung über die Umwälzung des Monds um seine Axe und die scheinbare Bewegung der Mondsflecken. *Kosmographische Nachrichten und Sammlungen* for 1748 **1**, 52-183.

**Summary:** After giving an account of his development [independent of that of Euler (1749)] of the Method of Averages, the author uses this method in an example. Having made twenty-seven observations on the position of a moon spot, he uses them to write twenty-seven equations each containing three unknown quantities. To solve these he adds together the nine equations in which the values of the coefficients of one of the unknown quantities are the greatest, then the nine in which they are least, and finally the remaining nine. He then solves simultaneously the resulting three equations in three unknowns in order to obtain the coefficients in the linear equation fitted to the observed data.

**Citations:** Lalande (1771), von Zach (1805), Laplace (1812), Hauber (1830-32), Todhunter (1865), Merriman (1877), Pizzetti (1892), Goedseels (1909), Whittaker & Robinson (1924), Bruen (1938), Deming (1943).

* Maire, Christopher; Boscovich, Roger Joseph (1755). *De Litteraria Expeditione per Pontificiam ditionem ad dimetiendas duas Meridiani gradus, et corrigendam mappam geographicam, jussu, et auspiciis Benedicti XIV Pont. Max. suscepta.* Romae. [French translation, *Voyage Astronomique et Géographique dans l'État de l'Église, entrepris par l'Ordre et sous les Auspices du Pape Benoît XIV, pour mesurer deux degrés du méridien, et corriger la Carte de l'État ecclésiastique*, Paris, 1770].

**Summary:** This volume is a report on the results of an expedition undertaken by the two authors (Jesuits) under the auspices of Pope Benedict XIV to measure two degrees of meridian and correct the map of the Papal State. The book is divided into five parts, of which the second and third are by Maire and the others by Boscovich. Our interest centers in a portion of the fifth part, especially pp. 499-501, where Boscovich attempts to determine the best value of the ellipticity of the earth from five measurements of degrees of meridian (the new one by Maire and himself reported earlier in the volume and four others) which he considers most reliable among a larger number of available measurements. If the earth were exactly an ellipsoid of revolution and if the measurements were perfectly accurate, any two measurements of degrees of meridian made at different latitudes would determine its ellipticity exactly. But because the measurements are subject to error, each of the $\binom{5}{2} = 10$ pairs of measurements yields a different value of the ellipticity, which is directly related to the excess of the polar degree over the equatorial. If the ellipticity is computed from the arithmetic mean of all ten excesses, the result is 1/255, but if the two most discrepant values of the excess (one of which is actually negative) are discarded and the ellipticity is computed from the arithmetic mean of the eight remaining ones, the result is 1/195. Boscovich gives both of these results, but is not satisfied with either.

**Comments:** This volume is of interest not only because of the material on the treatment of outliers (extreme order statistics), but also because of its relation to later publications by Boscovich (1757, 1760) in which he developed and applied a method, based on order statistics, to determine the straight line of best fit to three or more non-collinear points. The 1770 French translation includes a note which is a translation of the exposition of this method from Boscovich (1760). This and other added features make the translation more useful than the original, as Todhunter (1873), p. 332, has remarked.

**Citations:** Boscovich (1757, 1760), Lalande (1771), Bernoulli (1785), Laplace (1793,1799), von Lindenau (1806), Todhunter (1873), Merriman (1877), Gore (1889), Pizzetti (1892), Bruen (1938).

* Simpson, Thomas (1756). A letter to the Right Honourable George Earl of Macclesfield, President of the Royal Society, on the advantage of taking the mean of a number of observations, in practical astronomy. *Philosophical Transactions of the Royal Society of London* for 1755 **49** (1), 82-93.

**Summary:** The author points out that the practice of taking the mean of a number of observations,

while common among astronomers, has been questioned by some persons of considerable note who have maintained that a single observation, taken with due care, is as reliable as the mean of a great number. In order to refute that position, he determines the distributions of the mean errors of n independent observations from a discrete uniform (rectangular) population and from a discrete isosceles triangular population. He then compares these distributions with those of single observations from the same parent populations, and shows that the probability is less that the error of the mean of n observations equals or exceeds a given value than that the error of a single observation equals or exceeds the same value, the more so the greater the value of n.

**Comments:** Simpson's results, though quantitatively true only for the arithmetic mean, are qualitatively the same as those for other averages, including those (such as the median and the midrange) based on order statistics.

**Citations:** Merriman (1877), Czuber (1891a), Keynes (1921), Whittaker & Robinson (1924).

* Boscovich, Roger Joseph (1757). De litteraria expeditione per pontificiam ditionem, et synopsis amplioris operis, ac habentur plura ejus ex exemplaria etiam sensorum impressa. *Bononiensi Scientiarum et Artum Instituto Atque Academia Commentarii* **4**, 353-396.

**Summary:** The author summarizes the measurement of a meridian arc near Rome and reevaluates the data on this and previous measurements given by Maire and Boscovich (1755). He proposes for the first time two criteria for determining the best-fitting straight line $y = a+bx$ through three or more points: (1) The sums of the positive and negative residuals (in the y-direction) shall be numerically equal; and (2) the sum of the absolute values of the residuals shall be a minimum. He proceeds to apply these criteria to the data of Maire and Boscovich, though he does not give any indication as to the method of solution of the equation for the best value of the slope b resulting from the criteria.

**Comments:** Boscovich's first criterion requires that the best-fitting straight line pass through the centroid $(\bar{x},\bar{y})$ of the observations. The second criterion is then applied subject to the restriction imposed by the first. If this restriction were removed, the smallest possible minimum of the sum of the absolute values of the residuals would be attained. We shall see that this unrestricted minimum is precisely the solution proposed by some later authors.

**Reference:** Maire & Boscovich (1755).

**Citations:** Boscovich (1760), Bernoulli (1785), von Zach (1805), Gauss (1809), von Riese (1830), Todhunter (1873), Bruns (1882), Mansion (1913).

* Simpson, Thomas (1757). An attempt to shew the advantage arising by taking the mean of a number of observations, in practical astronomy. *Miscellaneous Tracts on Some Curious, and Very Interesting Subjects in Mechanics, Physical-Astronomy, and Speculative Mathematics*, pp. 64-75. J. Nourse, London.

**Summary:** This is essentially a revision of the author's earlier paper [Simpson (1756)]. Nothing of consequence in that paper has been omitted, but there are two notable additions. At the beginning, the author states explicitly for the first time the assumptions that the error distribution is (1) symmetric (positive and negative errors of equal magnitude are equally likely) and (2) limited in extent (with limits depending on the goodness of the instrument and the skill of the observer). Four pages of new material at the end are devoted to extension to a continuous isosceles triangular error distribution of the results previously given for the corresponding discrete distribution.

**Citations:** Lalande (1771), Todhunter (1865), Glaisher (1872), Merriman (1877), Czuber (1891a), Pizzetti (1892), Czuber (1899), Keynes (1921).

* Boscovich, R. J. (1760). De recentissimis graduum dimensionibus, et figura, ac magnitudine terrae inde derivanda. *Philosophiae Recentioris, a Benedicto Stay in Romano Archigynasis Publico Eloquentare Professore, versibus traditae, Libri X, cum adnotationibus et Supplementis P. Rogerii Josephi Boscovich S. J.*, Tomus II, pp. 406-426, esp. 420-425. Romae. [*French translation included in note appended to 1770 French edition of Maire and Boscovich (1755), pp. 501-512*].

**Summary:** This dissertation is a portion of Boscovich's commentary on the second of three volumes of Stay's treatise on natural philosophy in Latin hexameters. On page 420-425, Boscovich gives a geometric method of solving the equations resulting from the application of the two criteria stated in his earlier paper [Boscovich (1757)] to the problem of finding the straight line $y = a + bx$ of best fit to a number of points which are not collinear, and applies this method to the same five meridian arcs, obtaining the value 1/248 for the ellipticity of the earth. This method [see Eisenhart (1961), pp. 202-204, for details] is based on the ordered slopes $b_1 \geq b_2 \geq b_3 \geq b_4 \geq b_5$ of the lines connecting the five observational points $(x_i, y_i)$, $i = 1, 2, \ldots, 5$, to their centroid $(\bar{x}, \bar{y})$.

**References:** Maire & Boscovich (1755), Boscovich (1757).

**Citations:** Lalande (1771), Bernoulli (1785), von Lindenau (1806), Adrain (1818), Anonymous (1821), Cauchy (1831), Todhunter (1873), Merriman (1877), Gore (1889), Pizzetti (1892), Bruen (1938).

* Lambert, J. H. (1760). *Photometria, sive di Mensura et Gradibus Luminis, Colorum, et Umbrae.* Augustae Vindelicorum, Augsburg. (esp. Arts. 271-306). [Review, *Nova Acta Eruditorum* (1760), 564-578].

**Summary:** Sheynin (1966), p. 1003, has stated that Lambert should be given precedence over Gauss as the originator of the theory of errors. This is one of Lambert's three main works on the subject [see also Lambert (1765a,b)]. This work is available in the Ostwald's *Klassiker* series, but Sheynin states (p. 1004) that the translator not only omitted the relevant portion, but added a comment to the effect that the omitted portion contains nothing of interest to the reader. Similarly, no mention is made of this portion in the review cited above. According to Sheynin, the three works of Lambert contain: (a) The first general outline since Galilei (1632) of the properties of errors of observations; (b) A rule for estimating the precision of measurements by comparing the means taken with and without the most extreme observation. The latter of these is also mentioned by Merriman (1877), pp. 156-157, in his brief summary of the contents of this book. Again according to Sheynin, the specific contents of the present volume include use (in Arts. 299-303) of the principle of maximum likelihood, for which the author gives a graphical method of solution. Sheynin notes, however, that Lambert did not regard this principle as useful in practice, and never returned to it.

**Note:** The compiler has been unable to obtain a copy of this book, and has therefore had to rely on other authors [Sheynin (1966) and Merriman (1877), especially the former] for a summary of its contents.

**Citations:** Trembley (1804), Anonymous (1821), Glaisher (1872), Merriman (1877), Pizzetti (1892).

* Short, James (1763). Second paper concerning the parallax of the sun determined from the observations of the late transit of Venus, in which this subject is treated of more at length, and the quantity of the parallax more fully ascertained. *Philosophical Transactions of the Royal Society of London* **53**, 300-345.

**Summary:** The author endeavors to determine the parallax of the sun from observations, at various points, of a transit of Venus, on the basis of comparisons of (1) observed durations, (2) least distance at centers, and (3) internal contact at egress. He makes 116 comparisons of internal contacts observed at places north of the Equator, which he divides into two groups of 53 and 63 comparisons with means 8.61″ and 8.58″, respectively. By rejecting those results among the 53 which differ from the mean by more than one second (half a second), he finds the trimmed mean based on the remaining 45(37) results to be 8.55″

(8.57″). He takes as his result the mean of the untrimmed mean and the two trimmed means, 8.58″. By rejecting those results among the 63 which differ from the mean by more than 1″ (0.5″), he finds the trimmed mean based on the remaining 49(37) results to be 8.50″ (8.535″), and then takes as his result the mean of the untrimmed mean and the two trimmed means, 8.55″, which he then averages with the corresponding result for the 53 observations to obtain the value 8.565″, which he considers to be the best available from all 116 observations. Somewhat less variable (and more precise) results are obtained by comparison of observations at northern hemisphere points with those at the Cape of Good Hope and at Rodriguez (21 observations in each group) with means of 8.56″ and 8.57″ and trimmed means, after rejecting 2(8) values which differ from the mean by more than 0.2″, also 8.56″ and 8.57″. From 12 comparisons of total duration, he finds a mean parallax of 8.68″ and a trimmed mean, after rejecting 4 values which differ from the mean by more than 1″, of 8.61″, which he takes as the best value obtainable from these observations. He also finds means of 8.56″ and 8.53″ from 8 comparisons of least distances at centers for northern hemisphere points with that at Rodriguez and from 32 comparisons of least distances at centers computed from total durations. These two sets of observations are quite concordant, and he does not reject any of them. From the six means (8.565″, 8.56″, 8.57″, 8.61″, 8.56″, and 8.53″) obtained in different ways, the author finds the mean to be 8.566″, or if the mean (8.61″) based on total durations (which he says is least certain) is rejected, he finds the mean of the remaining five means to be 8.557″, so he concludes that the sun's parallax, on the day of the transit, was 8.56″.

* Lambert, J. H. (1765a). Anmerkungen und Zusätze sur practischen Geometrie. *Beyträge zum Gebrauche der Mathematik und deren Anwendung*, Vol. 1, pp. 1-313. Berlin. (Second edition, 1792).

**Summary:** A general statement as to the contents of the author's three main works dealing with the theory of errors [Lambert (1760, 1765a,b)] is given in the summary of the first. Specific contents of the present work include [see Sheynin (1966), p. 1004]: (a) A justification (Arts. 443-445) for preferring an arithmetical mean to a single observation; (b) A derivation (Arts. 429-430) of a semicircular probability density function for the distribution of errors; and (c) A statement (Art. 420) of the minimax principle (minimizing the maximum residual error). Lambert confesses that he does not know how to use this principle in a general and straightforward manner.

**Note:** The author has been unable to obtain a copy of this publication, and therefore has been forced to rely upon Sheynin (1966) for a summary of its contents.

**Comment:** It is not clear whether or not Lambert was aware of the use by Euler (1749) of a rudimentary form of the minimax principle.

**Citation:** Trembley (1804).

* Lambert, J. H. (1765b). Theorie der Zuverlässigkeit der Beobachtungen und Versuche. *Beyträge zum Gebrauche der Mathematik und deren Anwendung*, Vol. 1, pp. 428-488. Berlin. (Second edition, 1792).

**Summary:** See the summary of the book by Lambert (1760) for a general summary of the contents of his three main works [Lambert (1760, 1765a,b)] dealing with the theory of errors. On page 426 of the present work, the author repeats [see Merriman (1877), p. 157] the method given in the previous volume [Lambert (1760)] for judging the precision of the measurements which consists in comparing the mean of all observations with a new mean found after rejecting that observation deviating most from the first mean. He defines [see Sheynin (1966), p. 1004] the objectives of the theory of errors. In Art. 321 he states that these objectives are to find the relations between errors, their consequences, the conditions of observation and the accuracy of instruments. He also undertakes a study of the errors of functions of the observations, and endeavors to determine the 'true value' of the observed quantity and to estimate the accuracy of the observations. In Art. 20 he gives, according to Sheynin, rules for fitting straight lines and curves by dividing the observations into groups and taking their centers of gravity instead of the

original observations. It is not clear to the compiler whether these rules are the same as those to which Merriman refers when he states that this work contains a method of adjusting simple observations based on the principle that the algebraic sum of the errors shall be zero, and illustrates this method by determining empirical formulae for the length of the seconds pendulum, the declination of the magnetic needle, etc.

**Note:** The compiler has been unable to obtain a copy of this publication. He has therefore been forced to rely on other authors [Merriman (1877) and Sheynin (1966)] for an account of its contents.

**Citations:** Bernoulli (1785), Trembley (1804), von Zach (1805), van Beeck Calkoen (1816), Anonymous (1821), Muncke (1825), Hagen (1837), Merriman (1877), Pizzetti (1892), Czuber (1899).

* Lalande, Jérôme le Francois de (1771). *Astronomie*, Tome Troisième. Desaint, Paris. (Second edition). [Third edition, 1792].

**Summary:** The author reviews the literature on the theory of errors and its application to problems in astronomy and geodesy. He mentions the work of Cotes (1722) on weighted averages and that of Simpson (1757) on the advantages of the arithmetic mean, and states that he uses a simplification of Simpson's method in solving the problem of precession of equinoxes. He discusses in some detail the method of averages, due to Euler (1749) and Mayer (1750), and the method of minimizing the sum of absolute deviations, subject to the condition that the sums of positive and negative deviations be equal in magnitude, due to Boscovich (1760). He uses those two methods in the solution of systems of inconsistent equations resulting from measurements of degrees of longitude and of lengths of seconds pendulums, both of which are applied to the determination of the earth's ellipticity.

**References:** Cotes (1722), Euler (1749), Mayer (1750), [Maire &] Boscovich (1755) [1770], Simpson (1757), Boscovich (1760).

**Citation:** Gore (1889).

* Lagrange, J. L. (1774). Mémoire sur l'utilité de prendre le milieu entre les résultats de plusieurs observations; dans lequel on examine les avantages de cette méthode par le calcul des probabilités; et où l'on resoud différents problêmes relatifs à cette matière. *Miscellanea Taurinensia* for 1770-73 **5**, 167-232 (esp. Problème X, 225-229). [Reprinted in *Oeuvres de Lagrange*, Vol. 2, pp. 173-234. Gauthier-Villars, Paris, 1868].

**Summary:** After solving several problems concerning averages of observations having discrete error distributions, which are reminiscent of Simpson (1756, 1757), the author states his Problem X: "One supposes that each observation is subject to all possible errors between the two limits p and $-q$, and that the facility of each error x, that is, the number of cases in which it can occur, divided by the total number of cases, is represented by any function whatever of x designated by y; one requires the probability that the mean error of n observations shall be included between the limits r and $-s$." To solve this problem, he says, one first determines the probability [density] that the mean error shall be z, which is represented by a function of z, and then integrates from $z = r$ to $z = s$ [apparently he means from $z = -s$ to $z = r$]. To find the probability [density] that the mean error of n observations is z, consider the polynomial which is represented by the integral of $ya^x dx$ from $x = -q$ to $x = p$, raise this polynomial to the $n^{th}$ power, and find the coefficient of $a^z$. He says it is easy to see that this coefficient, which will be a function of z, expresses the probability that the mean error shall be z. Lagrange applies this procedure to two examples: (1) $y =$ K(a constant) [uniform or rectangular distribution of error]; (2) $y = K(p^2 - x^2)$, $(-p, p)$ [parabolic distribution of error ]. He remarks (p. 228) that the latter appears to be "the simplest and most natural which one can imagine." On pages 230-232, he considers a Problem XI, which is essentially a third example of Problem X with $y = K \cos x$, $(-\pi/2, \pi/2)$ [cosine distribution of error]. In each case the mean error of n observations has smaller dispersion (the more so the larger n) than the error of a single observation.

**Comments:** The results of this paper concern the advantages of averaging observations rather than particular methods of determining averages. The advantages found for the arithmetic mean are common to various averages, including those based on order statistics.

**Note:** Edwards (1974) has pointed out that the author (in his solution of Problem VI) uses the method of maximum likelihood, which was also employed by his contemporaries Lambert (1760) and Daniel Bernoulli (1778). Edwards is careful to distinguish between the method of maximum likelihood and the (Bayesian) method of maximum probability used by Gauss (1809), Laplace (1812) and Czuber (1890), as well as by Karl Pearson in some of his early writings. The method of maximum probability assumes a uniform prior distribution, obtaining the same result given by the method of maximum likelihood without the necessity of making any such assumption.

**Citations:** Laplace (1774, 1781), Bernoulli (1785), Trembley (1804), Anonymous (1821), Muncke (1825), Dirichlet (1836), Todhunter (1865), Glaisher (1872), Merriman (1877), Czuber (1891a), Pizzetti (1892), Czuber (1899), Keynes (1921), Kendall (1943).

* Laplace, P. S. (1774). Mémoire sur la probabilité des causes par les événemens. *Mémoires de Mathématique et de Physique Presentés . . . par Divers Savans* **6**, 621-657 (esp. Problème III: Déterminer le milieu que l'on doit prendre entre trois observations donnécs d'un même phénomène, 634-644). [Reprinted in *Oeuvres Complètes de Laplace*, Vol. 8, pp. 27-65. Gauthier-Villars, Paris, 1891. (JFM 23, 20 — listing only)].

**Summary:** In the introductory paragraph of section V, just before posing Problem III, Laplace states that he had presented two years earlier a method of determining the average which one should take among several observations of the same phenomenon, but had withdrawn it before publication because of the little use that could be made of it. Having read about, but not seen, the work of Lagrange (1774) and that of Daniel Bernoulli [then in manuscript form, but later published: Bernoulli (1778)], he has now had his interest revived and has decided to publish his own results after all. His Problem III is that of determining the average of three observations. He proposes two criteria: (1) the average should be such that it is equally likely to fall above or below the true value; and (2) the average should be such that the sum of the products of the errors and their respective probabilities is a minimum. He demonstrates that the two criteria lead to the same average. Let $x_1 \leq x_2 \leq x_3$ be the three observations, and let $p = x_2 - x_1$ and $q = x_3 - x_2$. Suppose the true value is $x_1 + x$; then the probability [density] that the three observations (assumed to be from a symmetric distribution) will fall at the points $x_1$, $x_2$, and $x_3$ will be $f(x)f(p-x)f(p+q-x)$, where $f(x)$ is the probability [density] that a single observation will fall at a distance x from the true value. Now construct a curve whose equation is $y = f(x)f(p-x)f(p+q-x)$. In order to satisfy Laplace's first criterion (and hence also his second) it is necessary to find the value of x such that an ordinate erected at the abscissa x (measured from $x_1$) divides the area under this curve equally. The solution depends, of course, on $f(x)$. Laplace takes $f(x) = (m/2)e^{-m|x|}$ [the density function of what we now call Laplace's first distribution] and finds the solution $x = p + (1/m)\ln[1 + (1/3)e^{-mp} - (1/3)e^{-mq}]$, which approaches the arithmetic mean $(2p+q)/3$ as $m \to 0$ and [see Eisenhart (1964)] the median as $m \to \infty$; for $0 < m < \infty$, the average found by Laplace's method lies between the arithmetic mean and the median. Laplace points out that the value of x is easily found if m is known, but observes that in practice m is often unknown. He devotes the five or six pages of section V to an attempt to solve the problem for m unknown by weighting the solution for each value of m by the conditional probability (given m) that the true value falls at a distance x from $x_1$, and then integrating from $m = 0$ to $m = \infty$.

**References:** Lagrange (1774), Bernoulli (1778).

**Citations:** Laplace (1781), Trembley (1804), Todhunter (1865), Glaisher (1872), Merriman (1877), Czuber (1891a,b), Pizzetti (1892), Czuber (1899), Keynes (1911, 1921), Wilson (1923), Bruen (1938).

* Bernoulli, Daniel (1778). Dijudicatio maxime probabilis plurium observationum discrepantium atque verisimillima inductio inde formanda. *Acta Academiae Scientiorum Petropolitanae* **1** (1), 3-23 (Memoirs). [English translation by C. G. Allen, *Biometrika* **48** (1961), 3-13].

**Summary:** Addressing himself primarily to astronomers, the author questions the practice common to astronomers of rejecting completely observations judged to be too wide of the truth, but assigning equal weights to all those retained. In discussing the treatment of outliers, he does not condemn the rejection of observations in every case, but advocates doing it only if an accident occurred which rendered an observation open to question, not simply because an observation made with care differs too much from other similar observations. The author proposes a semicircular distribution of error, and discusses the choice of diameter, offering the opinion that a diameter twice the distance between the two extreme observations is sufficient for three or four observations, while for more observations a diameter greater by half than the distance between the two extreme observations suffices. As limiting cases, the choice of an infinite diameter leads to taking the arithmetic mean as the average of the observations, while diminishing the diameter as much as possible without contradiction leads to taking the midrange, which he says he has found to be less often wrong than he thought before investigation. Next he proposes what has come to be known as the method of maximum likelihood to determine the average of a number of observations. For two observations the result, obtained by solving a cubic equation, is equal to the arithmetic mean. For three observations, $x_1 \leq x_2 \leq x_3$, it is necessary to solve an equation of the fifth degree containing twenty terms, and the result is greater than, equal to, or less than the arithmetic mean $(x_1 + x_2 + x_3)/3$ according as the median $x_2$ is less than, equal to, or greater than the midrange $(x_1 + x_3)/2$. The author presents arguments designed to show that this result is to be preferred to the arithmetic mean. He admits, however, that the procedure becomes unwieldy for a larger number of observations.

**Citations:** Laplace (1774), Euler (1778), Laplace (1781), Bernoulli (1785), Trembley (1804), von Zach (1805), Laplace (1810), Anonymous (1821), Todhunter (1865), Glaisher (1872), Merriman (1877), Czuber (1891a), Pizzetti (1892), Czuber (1899, 1903), Keynes (1921).

* Euler, L. (1778). Observationes in praecedentem dissertationem. *Acta Academiae Scientiorum Petropolitanae* **1** (1), 24-33 (Memoirs). [English translation by C. G. Allen, *Biometrika* **48** (1961), 13-18].

**Summary:** Instead of maximizing the product of the probability densities of the errors of the various observations, as Daniel Bernoulli (1778) had proposed in the paper commented upon, Euler proposes maximizing the sum of their fourth powers. He advances certain arguments for the use of his criterion instead of Bernoulli's, among these being (1) that it avoids the difficulty that would arise in the use of Bernoulli's criterion if one of the observations were so discrepant that it should be almost rejected, so that its probability density would be almost zero and (2) that it leads to simpler equations. He works out two examples based on real observations.

**Comments:** The compiler agrees with the following comments by earlier writers: "Euler seems to have objected to the wrong part of Daniel Bernoulli's method; the particular law of probability is really the arbitrary part, the principle of making the product of the probabilities a maximum is suggested by the Theory of Probability". [Todhunter (1865), p. 238]. "The commentary by Euler seems to me of less value [than Bernoulli's paper]". [Kendall (1961), p. 2]. Though this commentary does not contain anything of great significance in the theory of errors and the method proposed is not based on order statistics, it is included here because of its historical interest.

**Reference:** Bernoulli (1778).

**Citations:** Laplace (1781), von Zach (1805), Laplace (1810), Todhunter (1865), Merriman (1877), Czuber (1891a).

* Laplace, P. S. (1781). Mémoire sur les probabilités. *Mémoires de l'Académie royale des Sciences de Paris* **Année 1778**, 227-332 (esp. 322-332). [Reprinted in *Oeuvres Complètes de Laplace*, Vol. 9, pp. 383-485. Gauthier-Villars, Paris, 1893].

**Summary:** The author states that since his earlier paper [Laplace (1774)] on the determination of the average of several observations, three illustrious geometers, Lagrange (1774), Daniel Bernoulli (1778), and Euler (1778), have considered the subject. The fact that their principles differ from his has led him to take up the matter again and to present his results, he says, so as to leave no doubt as to their exactitude. He begins an extension (to any number of observations) and a generalization (to the case in which each observation may have a different law of facility of error) of the theory given in his earlier paper. Having proceeded as far as the curve of probability [joint probability density function] $y = f(x) \cdot f'(p-x) \cdot f''(p'-x) \ldots$, where x is the magnitude of the error of the smallest observation, the second, third, ... observations exceeding the first by p, p', ..., and f, f', f'', ... are the [symmetric] probability density functions of the errors of the respective observations, he pauses to discuss the choice of an average. He states that one can make infinitely many different choices according as one imposes various criteria, of which he enumerates four: (1) One may require that average such that the sum of the positive errors equal the sum of the negative errors; (2) one may require that the sum of the positive errors multiplied by their respective probabilities equal the sum of the negative errors multiplied by their respective probabilities; (3) one may require that the average be the most probable true value (Daniel Bernoulli's criterion); or (4) one may require that the error be a minimum, i.e., that the sum of the products of the errors (taken without regard to sign) and their respective probabilities be a minimum. He shows that criterion (4), which he regards as the fundamental one, is equivalent to criterion (2). If the functions f(x), f'(x), f''(x) are known, determination of the average value which divides equally the area under the curve of probability is a problem of pure analysis, but when these functions are unknown it becomes a problem in the calculus of probabilities. If $\pm a$, $\pm a'$, $\pm a''$, ... are the limits of the errors of the first, second, third, ... observations, one must assume, he says, $f(x) = (1/2a)\ln(a/x)$, $f'(x) = (1/2a')\ln(a'/x)$, $f''(x) = (1/2a'') \ln(a''/x)$, .... If $a = a' = a'' = \ldots = \infty$, criterion (4) leads to choice of the arithmetic mean, and thus agrees with criterion (1) as well as criterion (2). More generally, the author shows that it leads to the arithmetic mean when the following conditions are satisfied: (1) The law of facility of error is the same for all the observations; (2) positive and negative errors of the same magnitude are equally probable; and (3) errors can be infinite, but the probability of an error x tends to zero as $|x| \to \infty$. In the last section, the preceding theory is applied to the correction of instruments.

**References:** Lagrange (1774), Laplace (1774), D. Bernoulli (1778), Euler (1778).

**Citations:** Trembley (1804), Laplace (1810, 1811a), Todhunter (1865), Merriman (1877), Czuber (1891a), Pizzetti (1892), Czuber (1899), Keynes (1921), Wilson (1923).

* Bernoulli, Jean III (1785). Milieu. *Encyclopédie Méthodique*. Second edition (1789), entitled *Dictionnaire Encyclopédique des Mathématiques*, Vol. II, pp. 404-409. Hôtel de Thou, Paris.

**Summary:** Merriman (1877), p. 159, states that the author of this article refers to the methods of Boscovich (1757, 1760) and Lambert (1765b), and gives fuller accounts of the memoirs of Lagrange (1774) and Daniel Bernoulli (1778), the latter differing somewhat from the published memoir. The discrepancy is apparently accounted for by the fact [see Todhunter (1865), p. 442] that the summary given is based on a preliminary version which the present author states was communicated to him by its author in 1769. In this version [see Todhunter (1865), p. 443], a semicircular distribution of error is assumed as in the version published in 1778, but the method of maximum likelihood is not employed. Instead, the following iterative procedure is used: First take the mean of all the observations as the center of the semicircle and determine the center of gravity of the area corresponding to the observations; take this point as the center of a new semicircle, and repeat the operation until the center of gravity and the center of the semicircle coincide. With regard to the work of Boscovich, the author also

refers to the 1770 French translation of the book by Maire & Boscovich (1755). He states that the translater adds, at the end of his translation, a very interesting note in which he applies Boscovich's solution to a table of measured degrees more extensive than that given by Boscovich.

**References:** [Maire &] Boscovich (1755) [1770], Boscovich (1757, 1760), Lambert (1765b), Lagrange (1774), D. Bernoulli (1778).

**Citations:** von Riese (1830), Todhunter (1865), Merriman (1877), Czuber (1891a), Pizzetti (1892).

* Laplace, P. S. (1786). Mémoire sur la figure de la terre. *Mémoires de l'Académie royale des Sciences de Paris* **Année 1783**, 17-46. [Reprinted in *Oeuvres Complètes de Laplace*, Vol. 11, pp. 3-32. Gauthier-Villars, Paris, 1895. (JFM 26, 46)].

**Summary:** Given three or more non-collinear pairs of observations of two variables, x and y, the author proposes testing the adequacy of the linear relation $y=a+bx$ by first determining a and b so as to minimize the maximum absolute deviation from the fitted straight line, then deciding subjectively whether a deviation of this magnitude is consistent with the limits of the errors to which the observations are susceptible. He gives a procedure for determining the required values of a and b.

**Comments:** The above procedure for fitting a straight line $y=a+bx$ by minimizing the absolute value of the largest residual (actually, of the three largest residuals, two with one sign and one with the other) is optimal if the residuals have a uniform distribution. As a one-dimensional analogue, the choice of an average (the midrange) which minimizes the absolute value of the largest deviation (actually, of the two largest deviations, one positive and one negative) is optimal for a uniform distribution of errors.

**Citations:** Laplace (1793), Todhunter (1865, 1873), Gore (1889), Bruen (1938).

* Laplace, P. S. (1793). Sur quelques points du systême du monde. *Mémoires de l'Académie royale des Sciences de Paris* **Année 1789**, 1-87 (esp. Sur les degrés mesurés des méridiens, et sur les longeurs observées du pendule, 18-43). [Reprinted in *Oeuvres Complètes de Laplace*, Vol. 11, pp. 477-558. Gauthier-Villars, Paris, 1895. (JFM 26, 46)].

**Summary:** Commenting on efforts to determine the ellipticity of the earth by fitting a linear relation $y=a+bx$ to measurements of degrees of meridian, the author states that the resulting ellipses depart too far from the observations to be admissible, so that one concludes that the figure of the earth differs perceptibly from an ellipsoid of revolution. He insists, however, that before abandoning the elliptic figure entirely, it is necessary to determine the one such that the largest deviation is smaller than for any other elliptic figure and see whether this deviation is within the limits of the errors of observation. He refers to the procedure given in an earlier paper [Laplace (1786)] for solving this problem, and then goes on to give a procedure which he says is much simpler. In the course of presenting this procedure, he pauses to observe that when the absolute value of the largest deviation is made a minimum, there are actually three observations whose deviations, two with one sign and one with the other, have this same absolute value. After applying this procedure to nine measured lengths of degrees of meridian, he offers another method of treating the observations, based on the criteria that (1) the sum of deviations should be zero and (2) the sum of the absolute deviations should be a minimum. These criteria were first proposed by Boscovich (1757). Laplace develops an analytic procedure based on these criteria, while the procedure used by Boscovich (1760) was geometric. After applying this method to the nine measured lengths to which the first method has already been applied, he devotes the remainder of this portion of the memoir to observations on the lengths of the seconds pendulum, to which he applies both of the above methods.

**References:** Maire & Boscovich (1755) [1770 French edition], Laplace (1786).

**Citations:** Todhunter (1873), Merriman (1877), Gore (1889), Pizzetti (1892), Czuber (1899), Bruen (1938).

* Laplace, P. S. (1799). *Traité de Mécanique Céleste*, Vol. 2. J. B. M. Duprat, Paris [Reprinted as Vol. 2 of *Oeuvres Complètes de Laplace*. Gauthier-Villars, Paris, 1878.] Part I, Book III, ch. v, pp. 116-165 (esp. 134-165); English translation, with notes and commentary, by Nathaniel Bowditch. Hilliard, Gray, Little, and Wilkins, Boston, 1832.

**Summary:** The author proposes two methods, both based on order statistics, for determining the straight line $y = a + bx$ which best fits three or more points $(x, y)$ whose coordinates are pairs of related observations: (1) In section 39, pp. 134-143, he proposes minimizing the maximum residual, and seeing whether this deviation is within the limits of the errors of observations; (2) In section 40, pp. 143-147, he proposes minimizing the sum of the absolute residuals subject to the restriction that the sums of the positive and negative residuals shall be equal. In sections 41-43, pp. 147-165, he applies these methods to the problem of estimating the ellipticity of the earth from a number of measurements of degrees of meridian and observations of the length of a pendulum with period one second.

**Comments:** The first of the above methods, which was previously proposed by Laplace (1786), can be shown to be optimal when the distribution of residuals is uniform; its one-dimensional analogue, which leads to the use of the midrange as the average of a sample, is optimal for a uniform distribution of errors. The second method was first proposed by Boscovich (1757). A geometric solution of the equations which it yields for the slope b was given by Boscovich (1760) and an algebraic one by Laplace (1793). These solutions are based on order statistics, even though the arithmetic mean, which is the average required by the restriction that the sums of the positive and negative residuals shall be equal, is not.

**Reference:** Maire and Boscovich (1755) [1770].

**Citations:** Prony (1804), Puissant (1805), Delambre (1806-10), von Lindenau (1806), Gauss (1809), Laplace (1812), Mathieu (1813-14), van Beeck Calkoen (1816), Adrain (1818), von Riese (1830), Cauchy (1831), Poncelet (1835), Quetelet (1835), Hagen (1837), De Morgan (1847), von Andrae (1860), Boudin (1865), Todhunter (1865), Zachariae (1871), Todhunter (1873), Merriman (1877), Bruns (1882), Edgeworth (1886a, 1887b), Lehmann-Filhes (1887), Gore (1889), Estienne (1890), Czuber (1891b), Mansion (1906), Edgeworth (1911), de la Vallée Poussin (1911), Mansion (1913), Dodd (1922), Edgeworth (1923), Whittaker & Robinson (1924), Friedrich (1937), Bruen (1938).

* Prony, R. (1804). *Recherches Physico-Mathématiques sur la Théorie des Eaux Courantes*. L'Imprimerie Impériale, Paris (esp. pp. xvii-xxxii).

**Summary:** The author gives a geometric interpretation of the two methods of Laplace (1799), sections 39 and 40, for determining the coefficients a and b in the equation $y = a + bx$ of the line of best fit to a number of points $(x, y)$. Laplace's first method is based on the criterion that the largest deviation of any point from the line shall be a minimum and his second on the criterion that the sum of the absolute deviations shall be a minimum, subject to the restriction that the sums of the positive and negative deviations shall be equal in magnitude. Prony's geometric version of Laplace's first method demonstrates that, as previously pointed out by Laplace (1793), there are three points which deviate from the fitted line by the maximum amount, two in one direction and one in the other. His geometric solution by Laplace's second method is essentially that of Boscovich (1760), whom he does not mention. He gives three examples of the application of the two methods to actual data and compares the results with those obtained by a third method (his own) based on the idea that the deviation to be expected should be proportional to the independent variable x, or at least approximately so.

**Reference:** Laplace (1799).

**Citations:** Hagen (1837), Boudin (1865), Merriman (1877), Mansion (1913).

* Trembley, Jean (1804). Observations sur la méthode de prendre les milieux entre les observations. *Mémoires de l'Académie Royale des Sciences et Belles Lettres de Berlin, Classe de Mathématique* **Année 1801**, 29-58.

**Summary:** After brief mention of the work of Lambert, Laplace, and Bernoulli on the most advantageous method of taking averages of observations, the author turns to the work of Lagrange (1774) on the same problem. He states that his purpose is to use combinatorial theory to obtain the same results which Lagrange obtained by the use of integral calculus. He succeeds in using combinatorial theory to obtain results for discrete error distributions which Lagrange found with the aid of differential calculus and Simpson (1756, 1757) by series expansions. He does not treat the case of continuous error distributions, which is the only one for which Lagrange employed integral calculus.

**Note:** The author did not cite specific works of Lambert, Laplace, and Bernoulli; those listed below [see References] are the ones which he must have had in mind.

**Comment:** The compiler agrees with the statement by Todhunter (1865), p. 428: "Trembley's memoir appears to be of no value whatever."

**References:** Lambert (1760, 1765a,b), Lagrange (1774), Laplace (1774), Bernoulli (1778), Laplace (1781).

**Citations:** Todhunter (1865), Merriman (1877), Czuber (1891a), Pizzetti (1892), Czuber (1899), Keynes (1921).

* Legendre, A. M. (1805). *Nouvelles Méthodes pour la Détermination des Orbites des Comètes*. Courcier, Paris. (esp. Appendice sur la méthode des moindres quarrés, 72-80). [Second edition, with supplement, 1806]. For English translation of pp. 72-75 by Henry A. Ruger and Helen M. Walker, see Smith (1929), pp. 576-579.

**Summary:** This publication is noteworthy because it is apparently the first making use of the method of least squares, so named here by Legendre, though Gauss (1809) claims to have employed the method as early as 1795. The author starts with the linear form $E = a + bx + cy + \ldots$, where a,b,c, . . . are known coefficients which vary from one equation to another and x,y, . . . are unknowns which must be determined by the condition that the value of E reduces, for each equation, to zero or a very small number. He derives the normal equations without the explicit use of calculus by multiplying the linear form in the unknowns by the coefficient of each of the unknowns and summing over all the observations, then setting the sums equal to zero. If the results, when substituted in the normal equations, produce one or more errors judged too large to be admissible, he recommends rejecting the equations which produced them, and determining the unknowns from the remaining equations.

**Comments:** The method of least squares is not based on order statistics, but since several of its competitors are, this and other early works on the subject, notably those of Adrain, Gauss, and Laplace, will be included for comparative purposes. It has been stated by de la Vallée Poussin (1911) that Legendre, in this appendix, posed the problem of minimizing the maximum absolute error, but stated that he did not know an easy method of solving it, and proposed to substitute for it the method of least squares. This statement apparently rests on the flimsy evidence contained in the statement by Legendre (p. 72) that ". . . it is necessary above all to proceed so that the extreme errors, without regard to their signs, should be included within the narrowest possible limits." This statement by de la Vallée Poussin is all the more dubious in view of the fact that Legendre does not again mention the idea of minimizing the maximum error and of the claim for the superiority of the method of least squares which he makes in the next paragraph: "Of all the principles which one can propose for this object, I think that none is more general, more exact, or easier to apply than the one which we have used in the preceding research, which

consists in making the sum of the squares of the errors a minimum. By this means, a sort of equilibrium among the errors is established which, preventing the extremes from prevailing, is most proper to make known the state of the system nearest to the truth."

**Citations:** Puissant (1805), Delambre (1806-10), von Lindenau (1806), Gauss (1809), Laplace (1811a, 1812), Legendre (1814), Anonymous (1821), Gauss (1823), Ivory (1825), Muncke (1825), von Riese (1830), Hagen (1837), Gerling (1843), De Morgan (1847), Herschel (1850), von Andrae (1860), Chauvenet (1863), Glaisher (1872), Jevons (1874), Merriman (1877), Czuber (1891a,b), Pizzetti (1892), Czuber (1899, 1903), Mansion (1906), Goedseels (1909), de la Vallée Poussin (1911), Keynes (1921), Whittaker & Robinson (1924), Smith (1929), Bruen (1938), Boldrini (1942).

* Puissant, Louis (1805). *Traité de Géodésie, ou Exposition des Méthodes astronomiques et trigonométriques, applicable soit à la Mesure de la Terre, soit à la Confection des Canevas des Cartes et des Plans,* Volumes 1 and 2. Courcier, Paris. Second edition, 1819; third edition, 1842. Review, *Monatliche Correspondenz zur Beförderung der Erd- und Himmels-Kunde* **16** (1807), 443-473.

**Summary:** Most of the material in this two-volume treatise on geodesy is irrelevant to a study of order statistics; only the relevant parts will be summarized. The author (Vol. 2, pp. 245-246 of the second edition) gives a theoretical discussion of the method of least squares, followed (pp. 246-249) by an application of this method to the determination of the ellipticity of the earth from measures of degrees of meridian. He also mentions the method of conditional equations [method of averages] proposed by Mayer and the [Boscovich] method (preferred, he says, by Delambre) which gives "the least errors of latitude, half positive, half negative". He also (Vol. 2, pp. 337-343 of the second edition) applies the method of least squares to the determination of the ellipticity of the earth from the lengths of seconds pendulums, and compares the results with those obtained by Mathieu by minimizing the maximum discrepancy between observed and fitted values, as proposed by Laplace (1799).

**Comments:** The compiler has seen only the second edition, but the German review confirms that the material on the method of least squares was included in the first edition, published in the same year as Legendre's original work on the subject. The statement that Laplace (1812) later confirmed the exactitude of this method was, of course, added in the second edition; it is noteworthy that the author nevertheless elected to retain the material on the rival methods, two of which are based on order statistics.

**References:** Laplace (1799), Legendre (1805), {Delambre (1806-10), Laplace (1812), Delambre (1813)}.

**Citations:** Glaisher (1872), Todhunter (1873), Merriman (1877), Gore (1889), Pizzetti (1892).

* Svanberg, Jöns (editor) (1805). *Exposition des Opérations Faites en Lapponie, pour la Détermination d'un Arc du Méridien, en 1801, 1802 et 1803; par Messieurs Ofverbom, Svanberg, Holmquist et Palander.* L'Académie des Sciences, Stockholm.

**Summary:** Little of this book, which describes the measurement of a meridian arc in Lapland by the editor and three colleagues, is relevant to a study of order statistics. Two notable exceptions are found in the editor's preliminary discourse, where he compares the results obtained by applying the methods presented by Laplace [(1799), Livre III, Nos. 39 et 40] to the determination of the earth's ellipticity from fifteen measurements of the lengths of seconds pendulums and of degrees of meridian by various observers at different latitudes, apparently those given by Laplace [(1799), Livre III, No. 41], though he does not mention the source of either the methods or the data. On page viii he states: "The ellipticity which makes the largest of these errors [in the lengths of seconds pendulums] a minimum is = 1/322.48 [(?); the last digit is blurred in the compiler's copy] and the most likely ellipticity (that is to say, that which makes the sum of the positive errors equal to the sum of the negative errors, and the sum of all the

errors taken positively a minimum) is = 1/336.78 . . . ." On pp. xiii-xiv he states: ". . . in whatever manner one combines the different degrees [of meridian] that have been measured, it is impossible (always attributing to the earth the figure of an ellipsoid of revolution) to avoid an error of at least 189.4 m. in the measurements of Pennsylvania, of the Cape of Good Hope, and of Lapland [an earlier (1736) measurement]; also, the resulting ellipticity is = 1/277. . . . One regards as the most likely ellipticity that which makes the sum of the positive errors equal to the sum of the negative errors and the sum of all the errors taken positively a minimum, which gives 1/312 for the value of the ellipticity and 336.2 m. for that of the error corresponding to the degree of Lapland; this error appears much too large to be admitted, having passed the limit of those to which it seems that the observations should be susceptible, so that one finally begins to suspect that the earth cannot very well be an ellipsoid, or even some other solid of revolution." [Compiler's free translation].

**Citations:** Laplace (1812) [1818], Todhunter (1865, 1873), Gore (1889).

* von Zach, Franz Xaver (1805). Versuch einer auf Erfahrung gegrundeten Bestimmung terrestrischer Refractionen. *Monatliche Correspondenz zur Beförderung der Erd- und Himmels-Kunde* **11**, 389-415, 485-504.

**Summary:** In the second part of this paper, the author discusses the question of estimating the factor m, by which terrestrial refraction must be multiplied to correct for a unit increase in barometric pressure, from numerous observations (subject to error) on the two variables. He expresses the opinion that little reliance can be placed on the arithmetic mean when, as for the data in question, it does not stand equally far from the extremes. He reviews the writings of Lambert (1765b), Daniel Bernoulli (1778), and Euler (1778), which have not, he thinks, been used as much as they should have been. He credits Lambert with being the first, so far as he is aware, to point out that the arithmetic mean will yield the true value only when all observations are equally reliable, a situation which von Zach believes seldom or perhaps never exists for geodetic and astronomical observations. He also mentions Lambert's solution of the problem of finding the straight line from which a given set of points deviate as little as possible. However, he expresses a preference for the modification due to Euler (1778) of the procedure proposed by Daniel Bernoulli (1778). [It will be recalled that both of these formulations assume a semicircular distribution of error with center at the true value (which is, of course, unknown), but that they differ as to the method of estimating the true value. Both depend to some extent on the choice of the diameter of the semicircle, an infinite diameter leading to the arithmetic mean and a diameter equal to the distance between the two extreme observations to the midrange. A diameter smaller than the distance between the two extremes requires discarding one or more extreme observations.] The author considers the last possibility, but rejects it on the ground that the discarded observation(s) may be the best one(s) or the one(s) required to correct the others. He takes a diameter equal to twice the difference between the extreme observations, and proceeds to apply Euler's method to the data on terrestrial refraction and barometric pressure summarized in the first part of the paper, which contains nothing else of interest, except brief mention of the work of Euler, Mayer, Boscovich, and Lambert.

**References:** Euler (1749), Mayer (1750), Boscovich (1757?), Lambert (1765b), D. Bernoulli (1778), Euler (1778).

**Citation:** Merriman (1877).

* Delambre, J. B. (editor) (1806-10). *Base du Système Métrique Décimal, ou Mesure de l'Arc du Méridien Compris entre les Parallèles de Dunkerque et Barcelone, Executée en 1792 et Années Suivantes, par Mm. Méchain et Delambre.* Baudouin, Paris. Vol. 1, 1806; Vol. 2, 1807; Vol. 3, 1810.

**Summary:** In these three volumes the editor reports on a vast undertaking, carried out under the auspices of the Académie des Sciences with the support of the French government, to establish the base

of the metric system (1 meter = one ten-millionth of the distance from the Equator to the North Pole) by measuring the meridian arc between the parallels of Dunkerque and Barcelona (over 9°). Besides Delambre and Méchain, who were charged with the primary responsibility, many other eminent French scientists, including Lagrange, Laplace, Legendre and Prony, played important rôles, as did several less well-known scientists from other countries. The undertaking, like others of its type [see, for example, Maire & Boscovich (1755)], was complicated by the fact that the length of a degree of meridian is not constant, but increases as the latitude increases, because of the flattening of the earth at the poles. This important scientific investigation established the length of the meter, the basic unit of distance in the metric system now in use throughout the world (unfortunately not in the United States, except in scientific circles). Our interest in it, however, will be restricted to matters directly or indirectly related to the use of order statistics, especially the treatment of outlying observations and the solution of n inconsistent equations in m unknowns (n>m), where the inconsistencies are due to errors in the observations, so as to minimize some function of the errors. On page 117 of the first volume, the editor places himself squarely on the side of those who never suppress an observation or assign it a smaller weight simply because it deviates from other observations of the same kind. On pages 92 and 110 of the third volume, he compares values of the earth's eccentricity calculated from the observations of Delambre and Méchain by Laplace (1799), by Legendre (1805), and by himself. Laplace [by minimizing the maximum deviation] obtained the value 1/150; Legendre [by the method of least squares], 1/148; and the editor [by an unspecified method, probably that of Delambre (1813)], 1/139. However, by combining the observations of Delambre and Méchain with those made by Bouguer in Peru about 60 years earlier, the task force obtained the value 1/334, which agrees much better with results obtained from measurements of the length of a pendulum of known period and with those predicted by the theory of nutation and precession (see pages 432 and 620-621 of the third volume). This latter value was used in determining the length of the standard meter.

**References:** Laplace (1799), Legendre (1805).

**Citations:** Puissant (1805) [1819], Delambre (1813), Fourier (1824c), Stampfer (1839), Todhunter (1873), Gore (1889).

* von Lindenau, Bernhard August (1806). Über den Gebrauch der Gradmessungen zur Bestimmung der Gestalt der Erde. *Monatliche Correspondenz zur Beförderung der Erd- und Himmels-Kunde* **14**, 113-158.

**Summary:** The author states Laplace's (1799) analytic form of the method of Boscovich (1760), as well as Legendre's (1805) method of least squares, and applies both in the determination of the elliptic meridian. He does not comment as to the relative merits of the two methods. He reports that, in at least one instance, they yield very nearly the same results.

**References:** [Maire &] Boscovich (1755) [1770], Boscovich (1760), Laplace (1799), Legendre (1805).

**Citations:** Merriman (1877), Gore (1889), Pizzetti (1892).

* Adrain, Robert (1808). Research concerning the probabilities of the errors which happen in making observations. *Analyst* (Philadelphia) **1**, 93-109; reprinted (in part) in *American Journal of Science* **1** (1871), 412-414.

**Summary:** The author develops the method of least squares and uses it to solve the following problems: (1) Supposing a,b,c,d, . . . to be the observed measures of any quantity x, the most probable value of x is required [Ans. the arithmetic mean of the observations]; (2) Given the observed positions of a point in space, to find the most probable position of the point [Ans. the center of gravity of the observed positions]; (3) To correct the dead reckoning at sea, by an observation of the latitude [the answer differs from all rules previously used, which he hopes will be abandoned]; (4) To correct a survey. The author mentions

that he has also used the same principle to determine the most probable value of the earth's ellipticity. [These last results were not published until ten years later.]

**Comments:** While the method of least squares had been published by Legendre (1805), we have here a report of the independent discovery of this method by Adrain, who was apparently unaware of Legendre's work and could hardly have known of that of Gauss, who, though he had used this method since 1795, did not publish it until 1809.

**Citations:** Adrain (1818), Glaisher (1872), Merriman (1877), Czuber (1891a), Pizzetti (1892), Czuber (1899, 1903), Keynes (1921), Goodrich (1927).

* Gauss, Carolo Friderico [Carl Friedrich] (1809). *Theoria Motus Corporum Coelestium in Sectionibus Conicis Solem Ambientium*. Frid. Perthes et I.H. Besser, Hamburgi [Reprinted in *Werke*, Band VII, Königlichen Gesellschaft der Wissenschaften, Göttingen, pp. 1-280. (JFM 37, 12-14)]; German summary, *Göttingische Gelehrte Anzeigen*, 1809, Junius 17 [Reprinted 1874 in *Werke*, Band VI, pp. 53-60]; English translation of paragraphs 175-186 by Hale F. Trotter in Technical Report No. 5, Statistical Techniques Research Group, Princeton University, 1957, pp. 127-147.

**Summary:** The author deduces the normal (Gaussian) law of error from the postulate that when any number of equally good direct observations of an unknown magnitude x are given, the most probable value is their arithmetic mean. He proceeds to discuss direct measurements of a single quantity and indirect observations; this discussion includes a consideration of the case in which the observations are assigned different weights. Gauss shows that the method of least squares, used by him since 1795, but named by Legendre (1805), follows as a consequence of the Gaussian law of error. If one does not assume this law, he might minimize the sum of the $2n^{th}$ powers of the errors for $n = 1,2,3, \ldots$, but the author points out that minimizing the sum of squares ($n=1$) is simplest. Letting $n \to \infty$ is equivalent to minimizing, as proposed by Laplace, the maximum errors (one positive and one negative, equal in magnitude). Gauss also mentions Laplace's other principle, first proposed by Boscovich, of making the sum of the absolute values of the deviations a minimum.

**Comments:** Gauss was apparently unaware that Boscovich proposed to minimize the sum of the absolute values of the deviations subject to the restriction that the sums of the positive and negative deviations shall be equal, since he speaks of this restriction as one added by Laplace. He does not mention the fact, though he may have been aware of it, that this restriction results in minimizing the sum of the absolute deviations from the arithmetic mean instead of from the median. The same is true of the fact that minimizing the sum of the $2n^{th}$ powers for $n \to \infty$ results in the choice of the midrange as an average instead of the arithmetic mean.

**References:** Boscovich (1757?), Laplace (1799), Legendre (1805).

**Citations:** Laplace (1810, 1811a, 1812), Gauss (1816), Anonymous (1821), Gauss (1823), Ivory (1825), Muncke (1825), Hauber (1830, 1830-32), von Riese (1830), Encke (1832-34), Hagen (1837), Gerling (1843), Ellis (1844), De Morgan (1847), Herschel (1850), Petzval (1857), von Andrae (1860), Chauvenet (1863), Zachariae (1871), Glaisher (1872), Jevons (1874), Laurent (1875), Merriman (1877), Bruns (1882), Schreiber (1882), Merriman (1884), Lehmann-Filhes (1887), Bertrand (1888a), Faye (1888), Czuber (1890), Estienne (1890), Czuber (1891a,b), Pizzetti (1892), Fechner (1897), Czuber (1899, 1903), Mansion (1906), Goedseels (1909), Keynes (1911), Mansion (1913), Brunt (1917), Keynes (1921), Whittaker & Robinsion (1924), Coolidge (1925), Estienne (1926-27), Goodrich (1927), Smith (1929), von Mises (1931), Bartlett (1933), Friedrich (1937), Bruen (1938), Boldrini (1942), Deming (1943), Cramér (1946).

* Laplace, P. S. (1810). Mémoire sur les approximations des formules qui sont fonctions de très-grands nombres, et sur leur application aux probabilités. *Mémoires de la Classe des Sciences Mathématiques et Physiques de l'Institut de France* **Année 1809**, 353-415; supplement, 559-565. [Reprinted in *Oeuvres Complètes de Laplace*, Vol. 12, pp. 301-353. Gauthier-Villars, Paris, 1898. (JFM 29, 7-8)].

**Summary:** The author shows, in article VI (pp. 382-389), that if random samples of size n are drawn from a distribution with mean $\mu$ (the abscissa of the center of gravity of the area under the probability [density] curve) and known dispersion, then the distribution of sample means has mean $\mu$ and dispersion $1/\sqrt{n}$ times that of the parent distribution; moreover, under very general conditions [which Laplace does not state explicitly] on the parent distribution, the distribution of sample means tends to normality as the sample size n increases. In the supplement, he shows that when the law of error is the normal law, his own "most advantageous method" [Laplace (1781)], the method of maximum likelihood first used by Bernoulli (1778) and afterward by Euler [see note below] and Gauss (1809), and the method of least squares [which he introduces without referring either to Legendre (1805) or to Gauss (1809)], are all equivalent and lead to the choice of the arithmetic mean as the average of a number of observations.

**Comments:** Eisenhart (1964) has pointed out (pp. 30-31) that the results given in article VI greatly strengthened the justification given by Gauss (1809), who based his argument on the postulate that the arithmetic mean is the best average of a number of observations, for the use of the method of least squares, especially when dealing with a large number of observations.

**Note:** It is not clear what paper by Euler Laplace had in mind. The only paper by Euler on this subject that has come to the attention of the compiler is the one [Euler (1778)] immediately following that of Bernoulli, commenting adversely on it and proposing the maximization of the sum of the fourth powers of the probabilities [densities] instead of the product of the probabilities [densities].

**References:** Bernoulli (1778), Euler (1778?), Laplace (1781), Gauss (1809).

**Citations:** Laplace (1811a), von Riese (1830), Todhunter (1865), Merriman (1877), Pizzetti (1892), Czuber (1899), Keynes (1921).

* Laplace, P. S. (1811a). Mémoire sur les intégrales définies, et leur application aux probabilités, et specialment à la recherche du milieu qu'il faut choisir entre les résultats des observations. *Mémoires de la Classe des Sciences Mathematiques et Physiques de l'Institut de France* **Année 1810**, 279-347. [Reprinted in *Oeuvres Complètes de Laplace*, Vol. 12, pp. 357-412. Gauthier-Villars, Paris, 1898. (JFM 29, 7-8)]. [Review by J.B.J. Delambre, *Bulletin de la Société Philomathique de Paris* **2** (1811), 262-266; English translation, *Philosophical Magazine* (1) **39** (1812), 240-244].

**Summary:** In articles VI and VII (pp. 317-334), the author considers the problem of choosing the average to take of n observations in order to correct an element already known approximately. He finds that the normal (Gaussian) law of error is the only one of the form $f(x) = K\, e^{-g(x^2)}$, where $g(x^2)$ is continuous, for which the arithmetic mean of the n observations is the "most advantageous" in the sense of Laplace (1781). However, because of the rudimentary form of the central limit theorem given by Laplace (1810), choice of the arithmetic mean is advantageous when the number of observations is large or when one is taking the average of results each based on a large number of observations, and hence in these cases one may use the method of least squares, which Gauss (1809) developed from the postulate that the arithmetic mean is the best average of a number of observations. In article VII (pp. 334-347) he extends these results to the case of correcting two unknown elements [regression coefficients]. His analysis is already quite laborious for this case, but he indicates that the results hold for any number of unknown elements whatever.

**Comments:** In his review, Delambre gives a brief summary of this work and relates it to the earlier

work of Laplace (1799) based on the Boscovich criterion and to that of Legendre (1805) and Gauss (1809) on the method of least squares.

**References:** Laplace (1781), Legendre (1805), Gauss (1809), Laplace (1810).

**Citations:** von Riese (1830), Todhunter (1865), Merriman (1877), Gore (1889), Pizzetti (1892), Czuber (1899), Keynes (1921), Eddington (1933).

* Laplace, P. S. (1811b). Du milieu qu'il faut choisir entre les résultats d'un grand nombre des observations. *Connaissance des Tems* **Année 1813**, 213-223; German translation, *Monatliche Correspondenz zur Beförderung der Erd- und Himmels-Kunde* **25** (1812), 105-120. [See also *Oeuvres Complètes de Laplace*, Vol. 12, 401-412. Gauthier-Villars, Paris, 1898; Vol. 13, p. 78. Gauthier-Villars, Paris, 1904. (JFM 36, 10-11)].

**Note:** The compiler had difficulty in obtaining the original French version of this paper. The German translation and the comments of Todhunter (1865) indicate that it is simply a reprint of article VIII (pp. 334-347) of the author's earlier paper [Laplace (1811a)], of which a summary has already been given and need not be repeated here. This indication is confirmed on p. 78 of the *Oeuvres*, Vol. 13, and by a reading of the belatedly received original French version.

**Citations:** Muncke (1825), Todhunter (1865), Merriman (1877), Gore (1889), Pizzetti (1892).

* Laplace, P. S. (1812). *Théorie Analytique des Probabilités*. Courcier, Paris. Book II, ch. iv, pp. 304-348; second supplement (Application du calcul des probabilités aux opérations géodésiques), 1818. [Third edition (1820), containing three supplements, reprinted 1886, together with a fourth supplement added in 1825, as *Oeuvres Complètes de Laplace*, Vol. 7. Gauthier-Villars, Paris. Book II, ch. iv, pp. 309-354; second supplement, pp. 531-580]. For English translation of pp. 304-327 by Julian L.C.A. Gÿs, see Smith (1929), pp. 588-604.

**Summary:** A large part of the material in Book II, ch. iv, which is entitled "Of the probability of errors of the mean results of a large number of observations, and of the most advantageous mean results," appeared previously in memoirs by the author. Articles 18 and 19 (pp. 304-312) are not relevant to our purposes. Articles 20 and 21 (pp. 312-329), which deal respectively with the correction of one and two elements, already known approximately, by the aggregate of a large number of observations, and which contain Laplace's "proof" of the method of least squares, follow closely the treatment of Laplace (1811a,b). Article 22 (pp. 329-333), which deals with the case in which the facility of positive errors is not the same as that of negative ones [the distribution of errors is not symmetric] follows Laplace (1810). Article 23 (pp. 333-343), unlike the preceding ones, deals with the case in which the observations have already been made. The idea of the "most advantageous" average as the abscissa corresponding to the ordinate which divides equally the area under the [joint] probability [density] curve [likelihood curve] of the observations goes back to two of Laplace's earliest memoirs [Laplace (1774, 1781)], which contain solutions of the problem of finding such an average for three observations and for any number of observations, respectively. The results given on pp. 333-335 and pp. 340-343 are those of the supplement of Laplace (1810), while the intervening pages contain a more straightforward proof than that given by Laplace (1811a) of the fact that the normal law of error is the only one of the form $f(x)=Ke^{-g(x^2)}$ for which the arithmetic mean is most advantageous. In article 24 (pp. 343-348), the author mentions various other methods of averaging observations, including the one proposed by Cotes (1722) and applied by Euler (1749) and Mayer (1750) and the one based on minimizing the sum of the $2n^{th}$ powers of the deviations, which for $n \to \infty$ is equivalent to minimizing the maximum deviation, as proposed by Laplace (1786, 1799). He concludes that the best choice of method depends on the law of error when the number of observations is small, but that the method of least squares proposed by Legendre (1805) and Gauss (1809) is best whenever the number of observations is large. [The latter conclusion is unwarranted; see

Dirichlet (1836).] In the second supplement, Laplace examines the method proposed by Boscovich (1757, 1760) [see also Laplace (1793, 1799)] based on minimizing the sum of the absolute values of the deviations, to which he gives the name "method of situation." For an odd number of observations on a single variable, this method leads to the median as the best average, while his own "most advantageous method" leads to the arithmetic mean. By finding the respective probabilities that the two averages are in error by a given amount, he determines a condition (on the law of error) under which the median is preferable to the arithmetic mean and, for a given law of error, explores the possibility of finding a weighted average of the two which is more precise than either.

**References:** Cotes (1722), Euler (1749), Mayer (1750), Laplace (1799), Legendre (1805), Svanberg (1805), Gauss (1809).

**Citations:** Puissant (1805) [1819], Legendre (1814), Gauss (1816, 1823), Ivory (1825), Muncke (1825), Hauber (1830, 1830-32), von Riese (1830), Encke (1832-34), Quetelet (1835), Dirichlet (1836), Ellis (1844), De Morgan (1847), Herschel (1850), von Andrae (1860), Airy (1861), Chauvenet (1863), Todhunter (1865), Zachariae (1871), Glaisher (1872), Fechner (1874), Jevons (1874), Laurent (1875), Merriman (1877), Edgeworth (1883b, 1885, 1886a,b, 1887b,c,d,e, 1887-90, 1888), Gore (1889), Pizzetti (1889), Czuber (1891a), Pizzetti (1892), Czuber (1899, 1903), Kapteyn (1903), Edgeworth (1905), Charlier (1906), Mansion (1906), Charlier (1910), Edgeworth (1911), Keynes (1911), Gini (1912), Edgeworth (1913), Mansion (1913), Contarino (1914), Julin (1921), Keynes (1921), Dodd (1922), Edgeworth (1923), Rietz (1924), Whittaker & Robinson (1924), Bowley (1928), Smith (1929), Craig (1932a), Gumbel (1936c), Thompson (1936), Bruen (1938), Gini (1939), Simon (1941), Boldrini (1942), Cramér (1946), Kendall (1946).

* Delambre, J. B. (1813). *Abrégé d'Astronomie, ou Leçons Élémentaires d'Astronomie Théorique et Pratique*. Courcier, Paris.

**Summary:** Lesson XXIII (pp. 581-623) deals with the size and the figure of the earth. The author returns to the question [see Delambre (1806-10)] of determining the eccentricity of the earth from inconsistent observations on the lengths of meridian arcs. On page 608, he advocates a method, which is probably the one he used in the earlier work, in the following words: "It seems that one should seek neither the least sum of errors nor the least sum of squares, but the least errors, half negative, half positive." (Compiler's translation). On pp. 607-608, he applies this method to the determination of the earth's eccentricity from the Delambre-Méchain observations.

**Comments:** Since the least sum of absolute deviations is achieved when the deviations are taken from the median, in which case half the deviations are positive and half negative, it appears that the author, perhaps without realizing it, is advocating the Boscovich-Laplace method without the restriction that the sums of the positive and negative deviations be equal in magnitude, which requires that deviations be taken from the arithmetic mean rather than the median.

**Reference:** Delambre (1806-10).

**Citations:** Puissant (1805) [1819], Gore (1889).

* Mathieu, Claude Louis (1813-14). Sur les expériences du pendule, faites par les navigateurs espagnols, en différens points du globe. *Connaissance des Tems* **Année 1816**, 314-332.

**Summary:** The author uses 16 measurements made by a Spanish expedition, composed of two frigates, at various points on the globe, on the length of a seconds pendulum, to compute, by the method of least squares, the eccentricity of the earth. He obtains the values 1/323.2, 1/311.5 and 1/323.3 from the 9 points in the northern hemisphere, the 7 points in the southern hemisphere, and all 16 points, respectively. He also computes the eccentricity from 15 similar observations given by Laplace (1799), from

which Laplace found the values 1/321.5 and 1/335.8 by minimizing respectively the maximum error and the sum of the absolute values of the errors (subject to the restriction that the sums of positive and negative errors be equal in magnitude). Mathieu finds 1/323.3 by the former method and 1/319.0 by the method of least squares. The maximum residual of 0.132 millimeter (well within the limits of error to which measurements of the length of the pendulum are susceptible) in the former case is attained at three points (the Equator and Lapland with negative signs and the Cape of Good Hope with positive sign) as required by theory. For the method of least squares, the maximum residual is +0.174 mm. at the Cape of Good Hope. The author remarks that there is a very small difference between the eccentricities obtained by the two methods.

**Reference:** Laplace (1799).

**Citation:** Gore (1889).

* Legendre, A. M. (1814). Méthode des moindres quarrés, pour trouver le milieu le plus probable entre les résultats de différentes observations. *Mémoires de la Classe des Sciences Mathématiques et Physiques de l'Institut de France* **Année 1810** (2), 149-154.

**Summary:** All of this paper except the first and last sentences is quoted from pages 72-75 of the author's earlier work [Legendre (1805)]. In the first sentence he sets the stage by stating that Laplace (1812?) has found by considerations based on the calculus of probabilities that the method of least squares should be used in preference to all others to find the most exact average value of one or of several unknown elements among all those which are given by different observations. In the last sentence he calls attention to the application, in his own earlier work, of this method to the determination of the length of the 45$^{th}$ degree of meridian and of the amount of flattening of the earth.

**Comments:** In his first sentence, the author overstated the generality of what Laplace had proved about the method of least squares, as Laplace himself and many later writers have done. For a clarification, see Dirichlet (1836).

**References:** Legendre (1805), Laplace (1812?).

**Citations:** Muncke (1825), Merriman (1877), Gore (1889), Pizzetti (1892), Czuber (1899), Keynes (1921).

* van Beeck Calkoen, J. F. (1816). Over de Theorie der Gemiddelde Waardij uit een Zeker Aantal Grootheden of Afzonderlijke Waarnemingen. *Verhandelingen der K. Nederlandsch Instituut van Wetenschappen* **2**, 1-19.

**Summary:** This paper deals with the average value of a certain number of quantities or of separate observations. For several observations of a single quantity, the author advocates the use of the arithmetic mean, from which the sums of the positive and negative deviations are numerically equal. If one of the observations differs from the mean by an amount greater than the assumed limit of error, that observation is discarded, and the arithmetic mean of the remaining ones is taken. For observations on two related quantities, the author proposes two methods of determining the best fitting straight line. The first, which he attributes to Lambert [(1765b)], involves dividing the points representing the pairs of observed (x,y) values into two groups (as nearly as possible equal in number), one containing the points with the smallest abscissa and the other those with the largest abscissa, and joining the centers of gravity of the two sets of points. The other is based on the use of the Boscovich criteria, which the author attributes to Laplace (1799). It will be recalled that these involve minimizing the sum of the absolute deviations subject to the condition that the sums of the positive and negative deviations be numerically equal, which requires ordering the slopes of the lines joining the observed points to their centroid $(\bar{x},\bar{y})$.

By taking $x^2$ or $\sqrt{x}$ rather than x as the independent variable, the author obtains curvilinear regression equations of the forms $y=\alpha+\beta x^2$ and $y=\alpha+\beta\sqrt{x}$ as well as the linear regression equation of the form $y=\alpha+\beta x$. He advocates using that power of x which gives the best fit in the sense that the sum of the absolute deviations of the observed points from the fitted curve is smallest, subject to the condition that the algebraic sum be zero.

**Comments:** It is interesting to note that the author makes no mention of the method of least squares, although the work of Legendre (1805), Gauss (1809), and Laplace (1812) was already widely known at the time this paper was published.

**References:** Lambert [(1765b)], Laplace (1799).

**Citations:** Merriman (1877), Pizzetti (1892).

* Gauss, Carl Friedrich (1816). Bestimmung der Genauigkeit der Beobachtungen. *Zeitschrift für Astronomie und verwandte Wissenschaften* **1**, 185-196. [Reprinted 1880 in *Werke*, Band IV, Königlichen Gesellschaft der Wissenschaften, Göttingen, pp. 109-117; English translation by Hale F. Trotter in Technical Report No. 5, Statistical Techniques Research Group, Princeton University, 1957, pp. 156-167].

**Summary:** The author points out that it is not necessary to know the precision h [$= 1/\sigma\sqrt{2}$] of the observations in order to apply the method of least squares, and that the relation of the precision of the results to the precision of the observations is independent of h, but that the value of h is itself interesting and instructive. He then proceeds to give various methods of determining h, including methods based on the $n^{th}$ root of the sum of the $n^{th}$ powers of the absolute errors (deviations from the true value) for $n=1,2,3,4,5,6$, and an alternate method based on the median M of the absolute values of the errors. Gauss shows that the method based on $n=2$ gives the greatest precision, at least for samples from a normal population, 100 observations for $n=2$ yielding the same precision as 114 for $n=1$, 109 for $n=3$, 133 for $n=4$, 178 for $n=5$, 251 for $n=6$, or 249 [actually 272—see comments below] for the alternate method based on M, but notes that the last method and the one based on $n=1$ are arithmetically more convenient.

**Comments:** Although he gave the correct mathematical expression for the probable error of the median absolute error M, Gauss made a mistake in calculating the value of the numerical coefficient. Several later authors, including Hauber (1830), Encke (1832-34), and Jordan (1869), have given the correct value, but it is interesting to note that the first two, writing during the lifetime of Gauss, did so without mentioning his mistake, which remains uncorrected in his collected works.

**References:** Gauss (1809), Laplace (1812).

**Citations:** Gauss (1823), Muncke (1825), Hauber (1830), Encke (1832-34), Gerling (1843), Chauvenet (1863), Jordan (1869), von Andrae (1872), Helmert (1872, 1875b), Mees(1875), Helmert (1876b), Mees (1876), Merriman (1877), Lehman-Filhes (1887), Gore (1889), Czuber (1891a), Pizzetti (1892), Czuber (1899, 1903), Whittaker & Robinson (1924), Münzner (1934), Cramér (1946).

* Adrain, Robert (1818). Investigation of the figure of the earth, and of the gravity in different latitudes. *Transactions of the American Philosophical Society* (New Series) **1**, 119-135.

**Summary:** The author calculates the earth's ellipticity by the method of least squares from data on the lengths of pendulums vibrating seconds at different latitudes given by Laplace (1799). He compares the results not only with those obtained by Laplace, based on the criteria of Boscovich (1760), but also with the results obtained by that method after correcting two errors made by Laplace. He finds that most of

the discrepancy between Laplace's results and his own is due to those errors. The corrected results of applying the Boscovich-Laplace method, based on minimizing the sum of the absolute values of the residuals subject to the restriction that the algebraic sum of the residuals shall be zero, differ by less than 1% from those obtained by the method of least squares.

**References:** Boscovich (1760), Laplace (1799), Adrain (1808).

**Citations:** Glaisher (1872), Todhunter (1873), Merriman (1877), Gore (1889), Pizzetti (1892).

* Anonymous (1821). Dissertation sur la recherche du milieu le plus probable, entre les résultats de plusieurs observations ou expériences. *Annales de Mathématiques Pures et Appliquées* **12**, 181-204.

**Summary:** The author, a subscriber writing from Lyon, gives a discussion, which is as much philosophical as mathematical, of the problem of finding the best average of a number of observations. He distinguishes between two cases, one in which the observations are all made on the same identical object and thus differ only because of errors of observation and the other in which observations are made on a quantity which is itself variable. He traces the history of the problem from the time when the arithmetic mean was used without question, through the period in which students of the theory of probability (among whom he mentions Boscovich, Bernoulli, Lambert and Lagrange) questioned its use, to the time when wide acceptance of the method of least squares developed by Legendre (1805) and Gauss (1809) led to the belief that the arithmetic mean is indeed the most probable value. He pleads for further examination of the question, raising objections to the use of the arithmetic mean when the observations are not closely bunched, especially if they are so asymmetric that there are many more on one side of the arithmetic mean than on the other, or when there is reason to believe that they are not all equally reliable. He mentions a number of other possible averages, including several based on order statistics, such as the median, the midrange, and the arithmetic mean of those remaining after discarding the (one or more) largest and smallest observations. He concludes that the problem of the best average depends on the law of facility of error and hence has no general solution. Nevertheless, at the end of the paper he proposes an iterative procedure which starts from the arithmetic mean (or some other reasonable value), then takes the reciprocals of the residuals (or their squares) as weights of the corresponding observations and thus obtains a second approximation, which gives new residuals, after which the process is repeated, and so on. The editor (J. D. Gergonne) appends a note in which he points out that this iterative procedure converges to one of the observed values, as one can quite easily verify [except for the case of an even number of symmetrically placed observations].

**Comments:** Czuber (1891a, 1899) attributes the authorship of this paper to G. (Jöns?) Svanberg.

**References:** Boscovich (1760?), Lambert (1760, 1765b), Lagrange (1774), D. Bernoulli (1778), Legendre (1805), Gauss (1809).

**Citations:** Merriman (1877), Wright (1884), Czuber (1891a, 1899), Wellisch (1909).

* Fourier, J. B. J. (1823). Solution d'une question particulière au calcul des inégalités, premier extrait. *Histoire de l'Académie des Sciences* pour 1823, p. xxix. Reprinted in *Oeuvres de Fourier* (ed. Gaston Darboux), Vol. 2, pp. 321-324. Gauthier-Villars, Paris, 1890.

**Summary:** The author discusses the solution of a system of inequalities. Regardless of the relation between the number of unknowns and the number of inequalities, such systems do not, in general, have a unique solution, and the problem is to find all possible values of the unknowns which satisfy the inequalities. Without specifying the procedure, the author asserts that one of the applications of his theory is to find, given a set of observational equations, the values of the unknowns such that the largest error, taken without regard to sign, shall be the least possible, or such that the average error (i.e., the

sum of the errors taken without regard to sign, divided by their number), shall be the least possible.

**Citations:** Fourier (1824b, 1826b).

* Gauss, Carolo Friderico [Carl Friedrich] (1823). Theoria combinationis observationum erroribus minimis obnoxiae. *Commentationes Societatis Regiae Scientiarum Gottingensis Recentiores* **5**; German summary, *Göttingische Gelehrte Anzeigen* (1821), 321-327 (pars prior) and (1823), 313-318 (pars posterior). [Reprinted 1880 in *Werke*, Band IV, Königlichen Gesellschaft der Wissenschaften, Göttingen, pp. 3-53, 95-104; English translation by Hale F. Trotter in Technical Report No. 5, Statistical Techniques Research Group, Princeton University, 1957, pp. 1-82].

**Summary:** The author compares his earlier formulation of the method of least squares [Gauss (1809)] with that of Laplace (1812), and concludes that neither is entirely satisfactory. The former is based on the assumption that the errors of observation follow a normal (Gaussian) distribution, which follows from his postulate that the best average of the observations is their arithmetic mean. Laplace showed, under very general conditions which he did not state explicitly, that, when the number of observations is sufficiently large, the distribution of their arithmetic mean tends to a normal distribution with mean equal to the population mean. Hence the method of least squares yields a result which is consistent, but not asymptotically most advantageous, as Laplace in effect claimed. That left a gap, which the author now proposes to fill, for the case of a small or moderate number of observations whose errors are not normally distributed. Gauss begins by comparing the situation to a game in which there is no gain to hope for, but a loss to fear, the problem being how to minimize the loss, which is assumed to be the same for positive and negative errors of equal magnitude. This assumption can be met by choosing a loss function proportional to the sum of the absolute values of the errors, as Laplace did, or to the sum of their $n^{th}$ powers, n being a positive even integer. In the German summary, but not in the Latin text, Gauss points out, as Laplace had already done, that the larger the exponent n becomes, the nearer one comes to the situation where the most extreme errors alone serve as a measure of precision. Gauss chooses $n=2$, which besides being the simplest of its type also possesses certain other desirable properties [see Gauss (1816)], and on the basis of this choice he justifies the use of the method of least squares, whatever the number of observations and whatever the distribution of their errors.

**Comments:** Gauss' second exposition seems to the compiler to be no more satisfactory than his first. In each case he starts from a postulate, plausible but not universally valid, which leads inexorably to the foregone conclusion. Nevertheless, his argument apparently convinced his contemporaries, since the literature of the next few decades includes many writings on least squares but only a few on rival methods. This bibliography will include most of the latter but only a few of the former.

**References:** Legendre (1805), Gauss (1809), Laplace (1812), Gauss (1816).

**Citations:** Muncke (1825), Hauber (1830, 1830-32), von Riese (1830), Encke (1832-34), Hagen (1837), Gerling (1843), Ellis (1844), von Andrae (1860), Chauvenet (1863), Jordan (1869), Zachariae (1871), von Andrae (1872), Glaisher (1872), Jordan (1872), Fechner (1874), Merriman (1877), Schreiber (1882), Edgeworth (1883b, 1886a), Newcomb (1886), Lehmann-Filhes (1887), Gore (1889), Pizzetti (1892), Fechner (1897), Czuber (1899, 1903), Mansion (1906), Goedseels (1909), Edgeworth (1911), Contarino (1914), Brunt (1917), Keynes (1921), Whittaker & Robinson (1924), Estienne (1926-27), von Mises (1931), Snedecor (1937), Deming (1943), Cramér (1946).

* Cauchy, Augustin-Louis (1824). Sur le système des valeurs qu'il faut attribuer à deux éléments déterminés par un grand nombre d'observations, pour que la plus grande de toutes les erreurs, abstraction faite du signe, devienne un minimum. *Bulletin de la Société Philomathique de Paris* (3) **11**, 92-99.

**Summary:** Given a large number of observations of two variables, x and y (points in the plane), the

author seeks to determine the values of two elements (coefficients in the linear regression equation) such that the absolute value of the largest residual is a minimum. He accomplishes this minimization by means of an iterative scheme. He shows that a line in the plane may be such that one, two, or three of the given points deviate from it by the maximum amount, but that for the line which is the unique solution of the problem there are three such points with the maximum residual, two residuals of one sign and one of the other. He proves four theorems concerning the possible system of values of the elements, and gives a geometric interpretation of each in terms of the number of faces, edges, and vertices of a convex polyhedron.

**Note:** This is a condensation of a memoir presented to the Institut in 1814; the entire memoir was published later [see Cauchy (1831)].

**Citation:** Merriman (1877).

* Fourier, J. B. J. (1824a). Règle usuelle pour la recherche des résultats moyens d'un grand nombre d'observations. *Bulletin des Sciences Mathématiques, Astronomiques, Physiques et Chimiques* **2**, 88-90.

**Summary:** After defining a measure of precision of a large number of measurements of the same quantity which is equal to $\sqrt{2}$ times their standard deviation (the square root of the difference between their mean square and the square of their mean, divided by *half* their number), the author writes (pp. 88-89): "This square root gives the exact measure of precision sought. By multiplying it by a constant number g, one finds the mean error [probable error], and by multiplying this same square root by another constant number h, one finds the upper limit of the error. One should in practice regard as entirely impossible that the error of the mean result should surpass or even attain its upper limit; the chances are more than 50,000 to 1 that this does not happen. . . . The number h is equal to 3, and the number g is approximately 1/2 (its value is 0.477) . . . ."

**Comments:** The author seems to have confused the probable error and the upper limit (which would be exceeded less than 1 time in 50,000) of the mean with those of the individual observations; what he says of the former is actually true of the latter. Nevertheless, this early attempt to determine the maximum error is noteworthy, preceding by more than half a century the next serious attempts by Jordan and by Helmert.

**Citations:** Merriman (1877), Czuber (1891a), Pizzetti (1892), Czuber (1899), von Bortkiewicz (1922b).

* Fourier, J. B. J. (1824b). Solution d'une question particulière au calcul des inégalités, second extrait. *Histoire de l'Académie des Sciences* pour 1824, pp. xlvii. Reprinted in *Oeuvres de Fourier* (ed. Gaston Darboux), Vol. 2, pp. 325-328. Gauthier-Villars, Paris, 1890.

**Summary:** The author observes at the outset (*Oeuvres*, p. 325): "Consider linear functions of several unknowns x, y, z, . . .; the numerical coefficients which enter into the functions are known quantities. If the number of functions is not greater than that of the unknowns, one can find for x, y, z, . . . a system of numerical values such that the simultaneous substitution of these values in the functions gives a zero result for each. But one can not, in general, satisfy this condition when the number of the functions exceeds that of the unknowns." (Compiler's translation). He asserts again, as in his earlier memoir [Fourier (1823)], that, by the analysis of inequalities, whatever be the number of unknowns, one can find either the set of values of the unknowns which makes the largest error (deviation from zero) as small as possible or the set which makes the average error (deviation from zero) as small as possible. He gives the following geometric solution of the first problem (*Oeuvres*, pp. 325-326): "The following construction represents clearly the method which must be followed in order to arrive without unnecessary calculation at the values of x, y, z, . . . which give the largest deviation its smallest value. Although this construction is fitted to the case of two variables, it suffices to make well known the general procedure. x and y are, in

the horizontal plane, the coordinates of any point whatever. The vertical coordinate z measures the value of the function; each inequality is represented by a plane whose position is given. In the question under consideration, the number of these planes is double the number of functions, because it is necessary to attribute to each value the sign + and the sign −. One considers only the parts of the planes which are situated above the horizontal xy plane; and these upper parts of the planes are extended indefinitely. It is principally necessary to remark that the system of all these planes forms a vessel . . . [or] polyhedron of which the convexity is turned toward the horizontal plane. The lowest point of the vessel or polyhedron has for coordinates X, Y, Z which are the object of the question; i.e. Z is the least possible value of the largest deviation and X and Y are the proper values to give this minimum, taken without regard to sign. In order to reach quickly the lowest point of the vessel, one erects a vertical ordinate at any point whatever of the horizontal plane, e.g. the origin of x and y, up to its intersection with the highest plane. . . . Let $m_1$ be this point of intersection situated on the extreme plane. One descends on this same plane from the point $m_1$ to a point $m_2$ on an edge of the polyhedron, and, following this edge, one then descends from the point $m_2$ to the vertex common to three extreme planes. From the point $m_3$, one continues to descend along a second edge to a new vertex $m_4$; and one continues the application of the same procedure, following always that one of the two edges which leads to a lower vertex. One thus arrives very soon at the lowest point of the polyhedron. Now this construction represents exactly the series of numerical operations which the analytic rule prescribes; it makes very sensible the course of the method, which consists of passing successively from one extreme to another, and diminishing more and more the value of the largest deviation." (Compiler's translation.)

**Comments:** The author (over a century before George Dantzig) formulates the problem as one in linear programming: minimize an objective function (the largest deviation) subject to constraints in the form of linear inequalities. His method of solution has, for obvious reasons, become known as Fourier's method of descents.

**Reference:** Fourier (1823).

**Citations:** Fourier (1826b, 1831).

* Fourier, Joseph [J. B. J.] (1824c). Éloge de Delambre. *Memoires de l'Académie des Sciences* (Paris) **4**, cciv-ccxxiv; translation by C. A. Alexander, *Smithsonian Institution Annual Report* (1864), pp. 125-134. Washington, D. C.

**Summary:** This memoir does not contain anything directly relevant to our study, but it does contain useful background information. Of particular interest is the following statement (p. 130) concerning the analysis of the data on the Dunkerque-Barcelona meridian arc: "The results obtained were calculated according to different methods, several of which were proposed by Delambre. A remarkable theorem of Legendre's was also employed in these calculations and evinced its peculiar adaptation to the uses of geodetic mensuration." Apparently the references are to the work of Legendre (1805) on the method of least squares, to the proposal by Delambre (1813) of the Boscovich-Laplace method minus the restriction that the sums of the absolute values of the positive and negative deviations be equal, and perhaps to other methods proposed by Delambre.

**Reference:** Delambre (1806-10).

* Ivory, James (1825). On the method of the least squares. *Philosophical Magazine* **65**, 1-10, 81-88, 161-168.

**Summary:** This paper is divided into three parts. In the first part, the author gives two demonstrations of the method of least squares, neither of which is based on the theory of probability, which he considers irrelevant. In the second part, he discusses the probability of errors, failing to recognize that the

probability of any definite error for a continuous distribution must be an infinitesimal, and making no distinction between true errors and residuals. In the third part, he attempts to show that the method of least squares cannot give the most advantageous or probable results unless the law of facility of error is the normal law $\phi(x) = c\, e^{-h^2x^2}$. On page 165, he makes the following statement concerning the demonstration of Laplace (1812), Book II, ch. iv, Art. 20: ". . . whatever merit it may have in other respects, [it] is neither more nor less general than the other solutions of the problem."

**Comments:** Later authors have regarded Ivory's demonstrations as unsatisfactory, and the compiler shares this opinion. Glaisher (1872) analyzed Ivory's criticism of Laplace, which he regarded as a result of Ivory's failure to understand the demonstration of Laplace. It appears to the compiler that Glaisher was guilty of the same fault. In modern terminology, what Laplace actually proved in the article cited by Ivory [see also Laplace (1810, 1811a)] is that the method of least squares is *asymptotically* advantageous [good] for any error distribution which is well enough behaved so that its mean is asymptotically normally distributed. He claimed to have shown that it is most advantageous [best] for a very large number of observations from any such error distribution, but this conclusion is unwarranted— see Dirichlet (1836) for a clarification.

**References:** Cotes (1722), Legendre (1805), Gauss (1809), Laplace (1812).

**Citations:** Ellis (1844), Glaisher (1872), Merriman (1877), Gore (1889), Pizzetti (1892), Czuber (1899).

* Muncke, Georg Wilhelm (1825). Beobachtung. *Physikalisches Wörterbuch* (ed. Johann Samuel Traugott Gehler), Second edition. Vol. I, pp. 884-912.

**Summary:** The author discusses the use of observations subject to error in determining the best value of a single quantity or the best relation between two or more quantities. As the best average of a number of observations on a single quantity, he advocates taking the arithmetic mean of the observations remaining after discarding those which deviate farthest from the mean of all the observations, as suggested by Lambert (1765b) and practiced by Gauss (1809), when the deviations are greater than the limits of the error which one may commit in the observations. He points out that the practice of taking the arithmetic mean is nothing but a simple application of the method of least squares, which he advocates for the determination of the best relation between two or more quantities when the number of observations exceeds the number of unknown constants to be determined. He gives an exposition of the method of least squares which follows that of Gauss (1823), though he also mentions the work of Legendre (1805, 1814) and Laplace (1811b, 1812), as well as the earlier work of Gauss (1809). In addition, he refers to the work of Lagrange (1774) on the advantage of taking averages and to that of Gauss (1816) on the precision of observations.

**References:** Lambert (1765b), Lagrange (1774), Legendre (1805), Gauss (1809), Laplace (1811b, 1812), Legendre (1814), Gauss (1816, 1823).

**Citations:** von Riese (1830), Quetelet (1846), Merriman (1877), Wellisch (1909).

* Fourier, J. B. J. (1826a). Mémoire sur les résultats moyens déduits d'un grand nombre d'observations. *Recherches Statistiques sur la Ville de Paris et sur le Département de la Seine* **3**, ix-xxx. Reprinted in *Oeuvres de Fourier* (ed. Gaston Darboux), Vol. 2, pp. 525-545. Gauthier-Villars, Paris, 1890.

**Summary:** "As for the general conclusion, one can express it thus: when one has found a mean result A, by adding a large number m of particular values and dividing the sum of these values by their number m, it remains to evaluate the degree of approximation. It is necessary for that to square each of the particular values; form the mean value of these squares by dividing their sum by their number, which is m; one subtracts from this mean value of the squares the square $A^2$ of the mean result; one divides twice

the difference by the number m, and one extracts the square root of the quotient; thus one finds a quantity g which serves to measure the degree of approximation: the precision is in inverse ratio to the number g. The error of the result will be positive or negative; but one should regard it as certain in practice that the absolute value of this error is less than three times g. One also finds this same quantity g by dividing by m the square root of twice the sum of the squares of the differences between the mean result and each of the particular values." (Compiler's translation of author's summary, page xxviii). The author also considers the probable error (erreur moyenne), which he defines as the value such that among the absolute values of the observed errors, half fall above and half below it. Assuming a normal distribution of the errors, he states that the probable error is 0.47708g, where g is defined as above.

**Comments:** The author's g is equal to $\sqrt{2}$ times the standard deviation. This definition conforms with the nineteenth-century practice of using the modulus $c = \sigma\sqrt{2}$ instead of the standard deviation $\sigma$.

**Citations:** Fourier (1829), Pizzetti (1892).

* Fourier, J. B. J. (1826b). Solution d'une question particulière du calcul des inégalités. *Nouveau Bulletin des Sciences par la Société Philomathique de Paris* (3) **13**, 99-100. Reprinted in *Oeuvres de Fourier* (ed. Gaston Darboux), Vol. 2, pp. 317-319; note by editor, 320-321; premier extrait [Fourier (1823)], 321-324; second extrait [Fourier (1824b)], 325-328. Gauthier-Villars, Paris, 1890.

**Summary:** The author considers the problem of dividing a line segment of unit length into three parts such that the length of the longest does not exceed (1+r) times the length of the shortest. He gives a geometric construction which determines an irregular hexagon such that if one chooses any point $\mu$ of its area and drops a perpendicular to the given segment mm′, meeting it at point $\alpha$, and then describes an arc of radius $\alpha\mu$ with center $\alpha$, meeting mm′ at point $\mu'$ on the opposite side of $\alpha$ from m, then the segments m$\alpha$, $\alpha\mu'$ and $\mu'$m′ satisfy the conditions of the problem. For r=0, the hexagon reduces to a single point; as $r\to\infty$, the hexagon tends to an isosceles right triangle with legs of length 1 (and hence with area 1/2). If one divides the given line segment at random into three parts, the probability that they will satisfy the conditions of the problem is given by the ratio of the area of the hexagon for the given value of r to that of the triangle. In the *Oeuvres*, the editor (Darboux) appends a note giving an analytic solution equivalent to the author's geometric solution and two related papers by the author [Fourier (1823, 1824b)].

**References:** Fourier (1823, 1824b).

* Fourier, J. B. J. (1829). Second mémoire sur les résultats moyens, et sur les erreurs des mesures. *Recherches Statistiques sur la Ville de Paris et sur le Département de la Seine* **4**, ix-xliv. Reprinted in *Oeuvres de Fourier* (ed. Gaston Darboux), Vol. 2, pp. 549-590. Gauthier-Villars, Paris, 1890.

**Summary:** The author extends the results of his earlier memoir [Fourier (1826a)] on maximum errors and probable errors to the case of quantities which are not directly measurable, but are known functions of measurable quantities.

**Reference:** Fourier (1826a).

* Hauber, C. Fr. (1830). Über die Bestimmung der Genauigkeit der Beobachtungen. *Zeitschrift für Physik und Mathematik* **7**, 286-314.

**Summary:** The author extends the work of Gauss (1816, 1823) on the estimation of the precision of observations to the case of s observations arising from populations having (possibly) different dispersions. The situation in which all come from the same population is included as a special case. He

considers estimators based on the square root of the mean of the squares of the errors, the mean absolute error, and the median absolute error. He compares the precision of these estimators when the law of facility of error is the normal (Gaussian) law. The mathematical expressions which he obtains agree with those given by Gauss (1816), as do the numerical results for the root-mean-square error and the mean absolute error. For the probable error of the median absolute error M he gives $0.78671w/\sqrt{s}$ (where w is the true value of M), which he approximates by $0.78671M/\sqrt{s}$, without mentioning that Gauss (1816) incorrectly calculated the numerical coefficient (for which he gave the correct mathematical expression $\sqrt{\pi/8}\ e^{\rho^2}$, where $\rho = .4769363$) to be 0.7520974. [Hauber's value is correct to within a unit in the fifth decimal place.] Hauber states as an advantage of the median absolute error that it is independent of the law of facility of error.

**Comments:** The advantage claimed by Hauber for the median absolute error is actually enjoyed by that estimator in the sense that it is unbiased for any law of facility of error. As for most so-called "distribution-free" estimators, however, its precision and its efficiency relative to other estimators do depend on the law of facility of error.

**References:** Gauss (1809), Laplace (1812), Gauss (1816, 1823).

**Citations:** Merriman (1877), Gore (1889), Pizzetti (1892), Czuber (1899).

* Hauber, C(arl) Fr(iedrich) (1830-32). Theorie der mittleren Werthe. *Zeitschrift für Physik und Mathematik* **8**, 25-56, 147-179, 295-316, 443-455; **9**, 302-322; **10**, 425-457.

**Summary:** In the first three of the six parts of this article, the author discusses arithmetic means, both population means (expected values) and sample means, along with the root-mean-square error and its probable error, and various applications. In the fourth, he deals with cases in which values of the quantities of interest are not given by the observations, but functions of these quantities. In the fifth and sixth parts, he deals with methods of solving the set of observational equations, greater in number than the number of quantities to be determined. He gives three methods: (1) the method of averages, which he attributes to Tobias Mayer; (2) the method of least squares; and (3) a hybrid method in which the number of equations is reduced to a manageable number (still greater than the number of quantities to be determined) by combining subsets of them as in the method of averages, and the resulting equations are then solved by the method of least squares.

**Comments:** None of the three methods discussed is based on order statistics, but this article is summarized to facilitate comparison of these methods with alternatives to the method of least squares other than the method of averages, most of which are based on order statistics.

**References:** Mayer [(1750)], Gauss (1809), Laplace (1812), Gauss (1823).

**Citations:** Merriman (1877), Gore (1889), Pizzetti (1892), Czuber (1899).

* von Riese, C. (1830). I. Theoria Combinationis observationum erroribus minimis obnoxiae, auctore Carolo Friderico Gauss. . . . Gottingae 1823. II. Supplementum theoriae combinationis observationum erroribus minimis obnoxiae, auct. C. F. Gauss etc. Gottingae 1828. [Eine Recension]. *Jahrbücher für Wissenschaftliche Kritik* **1**, 269-284.

**Summary:** The author summarizes the results of the paper by Gauss (1823), which contains Gauss' second "proof" of the method of least squares, and the 1828 supplement thereto. He mentions the earlier "proof" by Gauss (1809) based on the postulate, which the author attributes to Cotes (1722?), that the arithmetic mean is the best average of a number of equally precise observations. He also refers to the work of Legendre (1805) and Laplace (1810, 1811a, 1812) on the method of least squares and to that of

Laplace (1812) on the rival method based on minimizing the sum of the absolute errors, as well as the earlier work of Boscovich (1757?) on the latter method and the articles of J. Bernoulli (1785) on various averages and of Muncke (1825) on the method of least squares.

**References:** Cotes (1722?), Boscovich (1757?), J. Bernoulli (1785), Laplace (1799), Legendre (1805), Gauss (1809), Laplace (1810, 1811a, 1812), Gauss (1823), Muncke (1825).

**Citation:** Merriman (1877).

* Cauchy, A. L. (1831). Mémoire sur le système de valeurs qu'il faut attribuer à divers élémens déterminés par un grand nombre d'observations, pour que la plus grande de toute les erreurs, abstraction faite du signe, soit un minimum. *Journal de l'École Polytechnique* **13** (20), 175-221. [Lith. MS, 1814]. [Reprinted in *Oeuvres d'Augustin Cauchy*, Series 2, Vol. 1, pp. 358-403. Gauthier-Villars, Paris, 1905. (JFM 36,12)].

**Summary:** The problem considered is that of minimizing the absolute value of the largest residual. The author starts by observing that Boscovich [(1760?)] solved the problem for the case in which only one element must be considered, and that Laplace [(1799)] solved it for the case of two elements. [See also Laplace (1786, 1793, 1812).] He then proceeds to generalize the theory in two directions, considering the case in which the function of the elements which represents the errors is a power series and the number of elements exceeds two. He shows that the number of residuals whose absolute value is equal to the maximum always exceeds by at least one the number of variable elements.

**Note:** This paper bears a notation, immediately under the title, to the effect that it was presented to the first class of the Institute 28 February 1814. There is evidence that the lithographed manuscript was widely circulated prior to publication. See Cauchy (1824) for a condensation covering the special case of a linear function of two elements.

**References:** Boscovich (1760?), Laplace (1799).

**Citations:** Merriman (1877), Pizzetti (1892), Wölffing (1899), Keynes (1921).

* Fourier, J. B. J. (1831). *Analyse des Équations Déterminées, Première Partie.* Didot Frères, Paris.

**Summary:** Section 21 (pp. 81-84), apparently the only part of this book relevant to our study, is essentially a reprint of the portions of two of the author's papers [Fourier (1823, 1824b)] dealing with the problems of fitting a linear equation in n variables to a set of m observed points ($m > n$) so as to minimize (1) the maximum absolute deviation or (2) the average absolute deviation. The author gives, for the case $n = 2$, a geometric solution of the first problem which is equivalent to the analytic solution of a system of inequalities. The latter can easily be extended to larger values of n. He states that the second problem can be solved in an analogous manner.

**Reference:** Fourier (1824b).

**Citation:** Poncelet (1835).

* Encke, J. F. (1832-34). Über die Methode der kleinsten Quadrate. *Berliner Astronomisches Jahrbuch* **1834**, 249-312; **1835**, 253-320; **1836**, 253-308. Reprinted in *Encke's Astronomische Abhandlungen* (Berlin, 1866), Vol. 1, Nos. 12, 13, 14; also as Vol. 2 of *Gesammelte Mathematische Abhandlungen* (ed. Gravelius) (Ferd. Dümmler's Verlag, Berlin, 1888). (JFM 20, 1242).

**Summary:** The first of the three parts contains the proof of the method of least squares by Gauss (1809), an attempt by Encke to demonstrate that the arithmetic mean is necessarily the best average of a

number of observations, a discussion of weights and probable errors, and two tables of the probability integral $(2/\sqrt{\pi})\int e^{-t^2}\,dt$. The second part contains Gauss' algorithm for the solution of normal equations, his method of determining weights, etc. The third deals with conditioned observations. The only portion which is directly relevant to a study of order statistics is found on pages 294-299 of the first part, where the author gives a proof, which he attributes to his colleague Dirichlet, of the expression for the probable error of the median absolute deviation, which Gauss (1816) had given without proof. He also gives the value of the numerical coefficient in this expression as 0.786716 [correct to six decimal places], as compared with the values 0.78671 [Hauber (1830)] and 0.7520974 [Gauss (1816)], neither of which he mentions, as well as a numerical example of application of this method to actual data.

**References:** Gauss (1809), Laplace (1812?), Gauss (1816, 1823).

**Citations:** Gerling (1843), Ellis (1844), De Morgan (1847), Chauvenet (1863), Glaisher (1872), Peirce (1873), Fechner (1874), Jevons (1874), Jordan (1877a,b), Merriman (1877), Czuber (1891a), Pizzetti (1892), Fechner (1897), Czuber (1899), Contarino (1914), Keynes (1921), Whittaker & Robinson (1924).

Poncelet, J. V. (1835). Sur la valeur approchée linéaire et rationnelle des radicaux de la forme $\sqrt{(a^2+b^2)}$, $\sqrt{(a^2-b^2)}$ etc. *Journal für die Reine und Angewandte Mathematik* **13**, 277-291.

**Summary:** The author considers the problem of approximating the radicals $\sqrt{a^2+b^2}$ and $\sqrt{a^2-b^2}$, between certain limits on $k=a/b$, by linear expressions of the form $\alpha a+\beta b$. He defines the best approximation to be the one for which the maximum error is a minimum, noting (p. 282) that Laplace (1799) and Fourier (1831) used an analogous definition of the best fit of a linear function to a finite set of points.

**Comments:** The problem considered by the author is not directly relevant to our study, but it is interesting to see how it relates to the relevant work of Laplace and Fourier. Later we shall see how the idea proposed here by Poncelet was used by Chebyshev in developing his method of fitting a polynomial or other approximation to a continuous function over an interval. Approximating to a finite set of points may be thought of as a special case involving minimization of the largest of a finite rather than an infinite set of errors or deviations.

**References:** Laplace (1799), Fourier (1831).

**Citation:** Tchebychef (1854).

* Quetelet, Ad. (1835). *Sur l'Homme et sur le Développement de ses Facultés, ou Essai de Physique Sociale* (2 volumes). Bachelier, Paris. [English translation, *A Treatise on Man and the Development of his Faculties*, William and Robert Chambers, Edinburgh, 1842; reprinted 1968 by Burt Franklin, New York]. Second edition, *Physique Sociale ou Essai sur le Développement des Facultés de l'Homme*, Muquardt, Bruxelles; J.-B. Baillière et Fils, Paris; Jacques Issakoff, St. Petersbourg, 1869.

**Summary:** In Book 2, Chapters I and II, the author gives the extreme values (maximum and minimum) of the height and weight of groups of individuals; in one case (page 58 of the 1842 English translation), he gives the midrange, though he does not use that term. Having tabulated the length of a child at birth as from 0.487 metres (minimum) to 0.541 metres (maximum), he states, "The medium length of the child at birth would then be 0.514 metres." A French translation of the essay by Herschel (1850) on a later volume by Quetelet (1846) is included as an introduction to the second edition. In this essay, as we shall see, Herschel discusses the use of the median as a measure of central tendency. Quetelet himself, on page 58 of Volume 2 of the second edition, uses the interquartile distance (or the quartile deviation, which is half the interquartile distance) as a measure of dispersion (probable error).

**References:** Laplace (1799, 1812), {Quetelet (1846), De Morgan (1847), Herschel (1850)}.

**Citations:** Quetelet (1846), Herschel (1850), Venn (1866), Edgeworth (1885), Fechner (1897), Bowley (1901), Edgeworth (1911), Gini (1917), Niceforo (1919), Julin (1921), Jordan (1927), Boldrini (1942).

* Dirichlet, Peter Gustav Lejeune (1836). Über die Frage, in wiefern die Methode der kleinsten Quadrate bei sehr zahlreichen Beobachtungen unter allen linearen Verbindungen der Bedingungsgleichen als das vortheilhafteste Mittel zur Bestimmung unbekannter Elemente zu betrachten sei (extract). *Abhandlungen der Akademie der Wissenschaften* (Berlin) **33**, 67-68. Reprinted in Lejeune Dirichlet's *Werke*, Vol. I, pp. 281-282 (extract); Vol. II, pp. 347-351 (complete.) G. Reimer, Berlin, 1889 and 1897; reprinted by Chelsea Publishing Co., New York, 1969.

**Summary:** The author quotes from Laplace (1812), p. 348, to the effect that the best method of correcting one or more elements on the basis of linear observational equations greater in number than the elements depends, for a small number of observations, upon the underlying law of error, but that this dependence disappears when the number of observations is very large, in which case the method of least squares is to be preferred to all others. The author points out that Laplace's conclusion concerning the superiority of the method of least squares for very large numbers of observations is not only unwarranted by the evidence presented by Laplace but is actually incorrect. In the case of a single element, for example, Laplace's conclusion does not follow from the fact that the arithmetic mean tends in probability to the true value as the number of observations increases, since the same is true of other measures of central tendency, e.g. the median. Whatever the sample size, the author points out, the question of whether or not the arithmetic mean is superior to the median depends upon the ratio of two constants [their standard errors], one of which [the standard error of the arithmetic mean] depends on an integral over the whole range of the error curve, while the other [the standard error of the median] depends only on the maximum ordinate of the curve.

**References:** Lagrange (1774), Laplace (1812).

* Cauchy, Augustin (1837). Mémoire sur l'interpolation. (Lithographed manuscript, 1835). *Journal de Mathématiques Pures et Appliquées* (1) **2**, 193-205; English translation: On a new formula for solving the problem of interpolation in a manner applicable to physical investigations. *Philosophical Magazine* (3) **8** (1836), 459-468.

**Summary:** The author states the problem as follows (pp. 460-461 of English translation): "... I suppose that a function of x represented by y is developed in a converging series arranged according to the ascending or descending powers of x, or according to the sines and cosines of an arc x, or, more generally, according to other functions of x which I shall represent by $\phi(x)=u$, $\chi(x)=v$, $\psi(x)=w$; so that we have (1) $y = au + bv + cw + \ldots$ where a,b,c... are constant coefficients. Now the question is, 1st, how many terms of the second member of the equation (1) are to be employed, in order that the difference between it and the exact value may be very small, and capable of being compared with the errors to which the observations are liable; 2ndly, to determine in numbers the coefficients of the terms retained, or, in other words, to find the approximate value just mentioned." The data consist of n values of y represented by $y_i$ ($i=1,\ldots,n$) and the corresponding values of $x_i$ (and hence of $u_i,v_i,w_i, \ldots$) related by n equations (2) $y_i=au_i+bv_i+cw_i+\ldots$. The author proposes successive approximations based on neglecting all but one, two ... terms on the right-hand side of equations (2), the process continuing until the residuals are comparable with the inevitable errors of observation.

**Comments:** This paper is included here, even though the method proposed is not based on order statistics, because the method is one of the alternatives, along with several which are based on order statistics, to the method of least squares.

**Citations:** Merriman (1877), Pizzetti (1892), Czuber (1899), Goedseels (1909), Bruen (1938).

* Hagen, G. (1837). *Grundzüge der Wahrscheinlichkeitsrechnung*. Ernst & Korn, Berlin. [Third edition, 1882. (JFM 14, 153-155)].

**Summary:** In Chapter II (pp. 28-46) of the third edition, the author gives a demonstration of the normal law of error based on the assumption that the error in an observation results from the composition of a large number of small elementary errors. In Chapter III (pp. 47-72) of the third edition, he advocates the use of the method of least squares [Legendre (1805), Gauss (1809, 1823)], which he explains in considerable detail. He does mention, however, the use by Prony (1804), before the method of least squares was known, of the method of Laplace (1799) based on the criteria of Boscovich, as well as the work of Lambert (1765b). Of special interest is his strong opinion concerning the treatment of outlying observations: "The deception that one commits through the suppression of measurements is just as inexcusable as if he were to falsify or invent measurements.—If one is persuaded, during the measurement, of great uncertainty of individual measurements, one may leave these unconsidered; one may not do this simply because he later notices that they deviate considerably from the remaining ones (this is identical with assuming an infinitesimally small weight)." [First edition, as quoted by Wellisch (1909), p. 151, and translated by the compiler]. During the next 45 years, the author did not change his mind on this point, though he did soften his language somewhat, as can be seen by the following translation of his remarks on the same subject from the third edition (1882), pp. 70-71: "The exact determination of . . . the weights of individual observations is unquestionably very difficult and requires in any case great care and complete lack of prejudice. One must especially beware of inferring from the agreement of several observations among themselves that they are especially sharp or on the other hand of denying any value or of assigning no weight at all to those which show large deviations. In this case one would in no way present the most probable result, but only one which agrees with those observations which happen to agree with each other. With the heterogeneous behavior of errors, it can easily happen that exactly those measurements which deviate most strongly from the others really contribute to the correction of the result. One must therefore follow absolutely the principle that no measurement or observation may be disregarded if one did not already notice, when it was made, that it was not right or at least very doubtful."

**Note:** The compiler has seen only the third edition, and has therefore had to rely on other authors, especially Wellisch (1909), for information about the contents of the first edition.

**References:** Lambert (1765b), Laplace (1799), Prony (1804), Legendre (1805), Gauss (1809, 1823).

**Citations:** Quetelet (1846), Merriman (1877, 1884), Gore (1889), Pizzetti (1892), Blümcke (1897), Czuber (1899), Kozák (1907), Vogeler (1907), Kozák (1908-10), Wellisch (1909), Charlier (1910), Mansion (1913), Contarino (1914), Keynes (1921), Coolidge (1925), Goodrich (1927), Bond (1935).

* Bessel, F. W.; Baeyer, J. J. (1838). *Gradmessung in Ostpreussen und ihre Verbindung mit Preussischen und Russischen Dreiecksketten*. Berlin. [Reprinted in part in *Abhandlungen von Friedrich Wilhelm Bessel* (edited by Rudolf Engelmann), Vol. 3, pp. 62-138. Wilhelm Engelmann, Leipzig, 1876. (JFM 8, 737)].

**Summary:** In this account of a geodetic survey in East Prussia, the authors make the following comment with regard to the treatment of outlying observations: "We have adopted and followed without exception the rule of regarding the making of an observation as recognition of sufficiently favorable external conditions; i.e. we have allowed every observation made, and indeed all with equal weights, to contribute to the result, without allowing the possible coincidence of unfavorable conditions with the large deviation of an observation to be considered as a ground for its rejection. We have believed that only through strict observance of this rule could we remove arbitrariness from our results." [*Abhandlungen*, Vol. 3, p. 88, as translated by the compiler].

**Comments:** The authors joined Hagen (1837) in taking a firm stand against the rejection of outlying observations, which had been advocated and practiced by such earlier authors as Boscovich (1755, 1757), Lambert (1760, 1765b), and Legendre (1805).

**References:** {von Andrae (1869), Jordan (1869)}.

**Citations:** Chauvenet (1863), Zachariae (1871), Jordan (1872), Merriman (1877), Gore (1889), Pizzetti (1892), Czuber (1899), Wellisch (1909), Rider (1933).

* Dove, Heinrich Wilhelm (1838). Über die geographische Verbreitung gleichartiger Witterungserscheinungen. Erste Abhandlung. Über die nicht periodischen Änderungen der Temperaturvertheilung auf der Oberfläche der Erde. *Abhandlungen der Königliche Akademie der Wissenschaften (Berlin)* **35**, 285-415 (Physikalische Abhandlungen).

**Summary:** No attempt will be made to summarize the entire article. The only relevant part occurs on page 351, where the author, in discussing the variability of the mean monthly temperatures for a given month over a period of years, points out that a single unusually severe winter or a very hot summer can possibly double the extreme deviation from the average temperature for the month found from data for a long series of preceding years. Therefore a true quantitative comparison must not only consider the absolute extremes but must span all of them and hence must necessarily extend over a very long series of years.

**Citation:** Fechner (1897).

* Stampfer, S. (1839). Ueber das Verhältniss der Wiener Klafter zum Meter. *Jahrbücher des K. K. Polytechnisches Institutes* (Vienna) **20**, 145-176.

**Summary:** The author gives the following results of nine determinations, by various methods, of the ratio of the lengths of the Vienna fathom and the meter: 1.896614, 1.896666, 1.896663, 1.896647, 1.896562, 1.896522, 1.896677, 1.896680, 1.896658. He notes that the two smallest of these results, 1.896522 and 1.896562, were both obtained by comparisons with the French standard half toise, and he therefore rejects these two values on the grounds that the large deviations of these values from the others appear to point to a constant error in the standard half toise. He notes that the remaining seven values agree well and give the mean value of one Vienna fathom as 1.8966580 meters, with the probable error ±0.000006 meter. But he is still not satisfied with this result, and proceeds to discard also the smallest of the remaining values, 1.896614, apparently for no other reason than its discrepancy from the six remaining ones, from which he obtains his final mean value, 1.8966657, with probable error ±0.0000037.

**Reference:** Delambre (1806-10).

**Citations:** Gore (1889), Wellisch (1913).

* Cournot, A. A. (1843). *Exposition de la Théorie des Chances et des Probabilités*. Librairie de L. Hachette, Paris. German translation by H. Schnuse, Braunschweig, 1849.

**Summary:** Chapter IX (pp. 181-208) deals with statistics in general and the experimental determination of chances. In Section 114 (pp. 197-199) the author distinguishes between the probability that the proportion of male births in one of the 86 departments of France chosen at random will differ by more than $\delta$ from the proportion for France as a whole and the corresponding probability for the department chosen because its proportion deviates most from that of France as a whole. The same distinction, he says, does not apply if the statistician chooses in advance, before seeing the data, a particular department (say, the department of the Seine) because he has reason to believe that it is subject to exceptional conditions which can have a very palpable influence on the chance of male births. But, he inquires, does the same principle apply if one chooses the department of the Corse, or the department of the North, or any one of several others which seem *a priori* to be subject to exceptional conditions? Evidently, he says,

there enters into this estimation an element which is variable and resistant to mathematical determination. Chapter X (pp. 209-224) deals with the experimental determination of mean values and the formation of probability tables. In section 122 (pp. 210-213), the author raises the question as to whether other averages may be better than the arithmetic mean for certain types of data (e.g., economic time series) in which perturbations may lead to extreme values which have an undue influence on the arithmetic mean. He suggests the possible use of the median or a discard average, e.g., the average of the middle 10 of 14 observations arranged in order of magnitude (or the middle 1000 of 1400). In Chapter XI (pp. 225-260), which is devoted to means among measurements and observations, the author discusses weighted means and the method of least squares. Without giving any specific references, he alludes to the work of Cotes (1722?) on the former and that of Legendre (1805?), Gauss (1809?), and Laplace (1812?) on the latter.

**Comments:** The question raised by the author in section 114 will be encountered again much later in connection with the rejection of outlying observations and with tests of significance in harmonic analysis and in multiple comparisons.

**Citations:** Lexis (1875), Edgeworth (1887a), Gore (1889), Pizzetti (1892), Czuber (1899), Edgeworth (1913), Niceforo (1919), Keynes (1921), Jordan (1927).

* Gerling, Christian Ludwig (1843). *Die Ausgleichungs-Rechnungen der Practischen Geometrie, oder die Methode der Kleinsten Quadrate mit ihren Anwendungen für Geodätische Aufgaben*. Friedrich und Andreas Perthes, Hamburg und Gotha.

**Summary:** This book gives an excellent treatment of the method of least squares, but its interest to us lies mainly in the author's pronouncements concerning the treatment of outliers. Wellisch (1909), p. 151, quotes the author as saying (p. 68): "Every observation that does not have definite recorded ground for suspicion against it I have to regard as a witness for the truth, and I may no more disregard his testimony because it deviates from that of the others than I may twist the witness until he says what I would want to have said." [Compiler's translation]. In the next paragraph, however, Gerling says: "But if we make the rule only to be very cautious in discarding or reducing weight and to suppress doubt rather than follow it when we can still think that one of the rarer unknown error sources has acted by chance in an isolated case, still there remain observations which we must discard after the fact, because we hold it to be more probable that a gross blunder has occurred than that an unavoidable error can produce such a large deviation." [Compiler's translation]. In those rare instances in which such drastic action is necessary, Gerling opines (p. 67) that it is better to discard an observation entirely than merely to reduce its weight.

**References:** Legendre (1805), Gauss (1809, 1816, 1823), Encke (1832-34).

**Citations:** Jordan (1869), Merriman (1877), Gore (1889), Pizzetti (1892), Czuber (1899), Kozák (1907, 1908-10), Wellisch (1909), Whittaker & Robinson (1924).

* Ellis, R. L. (1844). On the method of least squares. *Transactions of the Cambridge Philosophical Society* **8**, 204-219. [Summary, *Proceedings of the Cambridge Philosophical Society* **1** (1), 5-6].

**Summary:** The author examines in detail the demonstrations of the method of least squares by Gauss (1809), Laplace (1812), Gauss (1823), and Ivory (1825). He concludes that Laplace's objection to Gauss' first demonstration based on the postulate that the arithmetic mean is the best average to take of a number of observations is justified. He regards Laplace's demonstration and Gauss' second as somewhat more satisfactory, but endeavors to show that none of the three tends to prove that the results of the method of least squares are the most probable of all possible results. He finds Ivory's demonstrations, which are not based on the theory of probabilities, not at all conclusive.

**References:** Gauss (1809), Laplace (1812), Gauss (1823), Ivory (1825), Encke (1832-34).

**Citations:** Glaisher (1872), Merriman (1877), Edgeworth (1886a), Gore (1889), Czuber (1891a), Pizzetti (1892), Czuber (1899), Keynes (1921).

* Quetelet, A. (1846). *Lettres à S. A. R. le Duc Regnant de Saxe-Cobourg et Gotha, sur la Théorie des Probabilités, Appliquée aux Sciences Morales et Politiques*. Hayez, Bruxelles. [English translation by O. G. Downes, London, 1849].

**Summary:** "These letters . . . contain . . . a valuable popular exposition of the theory of means and of the laws of error." [Merriman (1877), p. 190]. Of particular interest is the material on the interquartile distance, which the author (page 122) advocates as a measure of the probable error. On subsequent pages he uses this method of estimating the probable error of the right ascension of the North Star (p. 132) [repeated measurements of the same quantity] and of the chest measures of Scottish soldiers (p. 136) [measurements of related quantities].

**References:** Muncke (1825), Quetelet (1835), Hagen (1837).

**Citations:** Quetelet (1835) [1869], Herschel (1850), Galton (1869), Glaisher (1872), Jevons (1874), Lexis (1875), Merriman (1877), Edgeworth (1883a, 1885, 1886a), Czuber (1891a), Pizzetti (1892), Fechner (1897), Czuber (1899), Sheppard (1899a), Bowley (1901), Kapteyn (1903), Edgeworth (1911), Niceforo (1919), Julin (1921), Keynes (1921), von Bortkiewicz (1922b), Rietz (1924), Boldrini (1942).

* De Morgan, Augustus (1847). Theory of probabilities. *Encyclopaedia of Pure Mathematics (Encyclopaedia Metropolitana)*, Part II, pp. 393-490.

**Summary:** This article contains an extensive treatment (pp. 440-456) of the theory of errors and the method of least squares which consists largely of a translation of and comments on the treatment of Laplace (1812). In cases in which the relative precision of the observations is in doubt, the author (page 456) proposes an iterative procedure: "Assuming the weights as nearly as they can be found, ascertain the most probable result, from which find the weights of the equations. If these agree with the assumed weights, the process is finished; if not, repeat the process with the new weights, and so on, until a result is obtained for which the assumed and deduced weights of the equations are sufficiently near to equality." On the same page, he alludes to, but does not explain, Laplace's "method of situation".

**Comments:** This article is only marginally relevant to a study of order statistics, but the above quotation is of interest because, as we shall see, Glaisher (1872, 1873) used it as the basis for an alternative to the rejection of anomalous observations suggested, but not worked out in detail, by De Morgan.

**References:** Laplace (1799), Legendre (1805), Gauss (1809), Laplace (1812), Encke (1832-34).

**Citations:** Quetelet (1835) [1869], Herschel (1850), Glaisher (1872, 1873), Jevons (1874), Merriman (1877), Edgeworth (1883a,b), Doolittle (1884), Wright (1884), Edgeworth (1886a, 1887a), Czuber (1891a), Pizzetti (1892), Czuber (1899), Sheppard (1899a), Saunder (1903), Wellisch (1909), Edgeworth (1911, 1913), Keynes (1921), Ogrodnikoff (1928), Lidstone (1930).

* Herschel, Sir John F. W. (1850). [Review of Quetelet's 1846 work on Probabilities]. *Edinburgh Review* **92** (185), 1-30; reprinted in Herschel's *Essays from the Edinburgh and Quarterly Reviews* (Longman, Brown and Co., London, 1857), pp. 365-465. [French translation included (as introduction) in Quetelet, Ad. (1869), *Physique Sociale ou Essai sur le Développement des Facultés de l'Homme*, Vol. 1 (C. Muquardt, Bruxelles; J. B. Baillière et Fils, Paris; Jacques Issakoff, St. Pétersbourg)].

**Summary:** This paper contains a demonstration of the method of least squares, similar to that given by Adrain (1808), which is based on the assumption that the components of error in two orthogonal directions are independent. Of greater interest in the present study are the following comments (page 15) on Quetelet's work: "We are . . . left at a loss to decide by what numerical process his mean results, where stated, have been arrived at in some of the examples set down. For instance, in that of the [chest measures of the] Scotch soldiers, where all the groups are regular and all stated, we find it merely mentioned incidentally that the mean is 'a little more than 40 inches' [Quetelet (1846), page 136], whereas the really most probable mean is 39.830, while that which the course of the figures in the tabulated working of the example would appear to indicate as resulting from an equipartition of the numbers of cases in excess and defect is 39.525. Again, in the example of the [French] conscripts, where the extreme groups are massed undistinguishably, the rule of equipartition, according to its simplest and most obvious application to the tabulated figures, would place the mean [height] at 63.939 inches, whereas we find it indicated, rather than stated, as follows: '*If it be observed* that the mean height is about 63.947 inches' [ibid., 402]. The difference, it is true, is trifling in itself, but becomes of consequence when the object is from the figures set down to discover by what process they were obtained."

**Note:** The review was originally published anonymously, and it was not until publication of the *Essays* in 1857 that its authorship became known, not only to the public, but to Quetelet himself (see footnote on page 1 of the French translation cited above).

**Comments:** The "rule of equipartition" is, of course, a rule for determining the sample median, though Herschel did not call the result by that name. Incidentally, his arithmetic was faulty. The correct values of the medians of the two sets of data given by Quetelet (1846), pages 400 and 401, are 39.816 inches and 63.618 inches, respectively.

**References:** Legendre (1805), Gauss (1809), Laplace (1812), Quetelet (1835, 1846), De Morgan (1847).

**Citations:** Quetelet (1835) [1869], Venn (1866), Glaisher (1872), Jevons (1874), Merriman (1877), Edgeworth (1886a), Czuber (1891a), Pizzetti (1892), Czuber (1899), Edgeworth (1911), Keynes (1921).

* Peirce, Benjamin (1852). Criterion for the rejection of doubtful observations. *Astronomical Journal* **2**, 161-163.

**Summary:** The author proposes a criterion for the rejection of outlying observations. The criterion is an iterative one in the sense that it is not to be used to reject N observations unless it has already rejected N-1. The underlying principle is that "the proposed observations should be rejected when the probability of the system of errors obtained by retaining them is less than that of the system of errors obtained by their rejection multiplied by the probability of making so many, and no more, abnormal observations" [page 161]. The author cautions: "In determining the probability of the two systems of errors, it must be carefully observed that, because observations are rejected in the second system, the corresponding observations of the first system must be regarded, not as being limited to their actual values, but only as surpassing the limits of rejection." Let $n$ = the total number of observations; $N$ = the number rejected; $n' = n - N$ = the number retained; $\sigma$ = the standard deviation of all the observations; $\sigma'$ = the standard deviation of the retained observations; $y$ = the probability (assumed unknown) that a disturbing influence was in operation, causing an error large enough for the observation to be rejected; and $\kappa\sigma$ = the rejection limit (deviation from the arithmetic mean). The value of $\kappa$ for the rejection of N observations (the N−1 deviating farthest from the mean already having been rejected) is found by solving the equation $(\sigma'/\sigma)^{n'}[R(\kappa)]^n = T^N$, where $T^N = y^N(1-y)^{n'} = N^N n'^{n'}/n^n$ [since $y = N/n$, $1-y = n'/n$], $R(\kappa) = e^{-(\kappa^2-1)}2\psi(\kappa)$, where $\psi(\kappa) = 2\int_{\kappa\sigma}^{\infty}\phi(x)dx$ and $\phi(x) = e^{-x^2/2\sigma^2}/\sigma\sqrt{2\pi}$. Peirce gives an approximate solution and tables found by iteration, and works out two examples of the use of his criterion.

**Comments:** The notation used in the above summary is that of Rider (1933), who states (page 8): "Peirce's criterion has been discussed, and its merits and demerits debated by a number of writers, but the real objection to it seems to have been pointed out by Stewart [(1920a)]." We shall consider the various contributions to this discussion, as well as other criteria that have been proposed, in their proper chronological order.

**Citations:** Gould (1855), Airy (1856), Winlock (1856), Chauvenet (1863), Stone (1868), Glaisher (1872, 1873), Peirce (1873), Merriman (1877), Peirce (1878), Schott (1878), Edgeworth (1883b), Doolittle (1884), Merriman (1884), Wright (1884), Edgeworth (1886a), Newcomb (1886), Edgeworth (1887a), Lehmann-Filhes (1887), Czuber (1891a), Pizzetti (1892), Czuber (1899), Saunder (1903), Wellisch (1909), Brunt (1917), Stewart (1920a), Archibald (1925), Coolidge (1925), Ogronikoff (1928), Rider (1933).

* Tchebychef, P. L. [Chebyshev, P. L.]. Théorie des mécanismes connus sous le nom de parallélogrammes. *Mémoires Présentés à l'Académie Impériale des Sciences de St. Pétersbourg par Divers Savants* **7**, 539-568. Reprinted in *Oeuvres de P. L. Tchbeychef* (ed. A. Markoff & N. Sonin), Vol. I (1899), pp. 111-143. Imprimerie de l'Académie Impériale des Sciences, St. Pétersbourg.

**Summary:** The author observes that if one wants the best polynomial approximation (of a given degree) to a continuous function $f(x)$ in the neighborhood of $x=a$, one should use the sum of the appropriate terms in the Taylor series expansion in powers of $x-a$, but that if one wishes to find the best such approximation in the interval $(a-h, a+h)$, one should prefer another polynomial, whose maximum deviation from $f(x)$ in the given interval is less than for any other polynomial of the same degree, as proposed by Poncelet (1835). If U is a polynomial of degree n with $[n+1]$ arbitrary coefficients [including the constant term], and if one chooses these coefficients so that the difference $f(x)-U$, from $x=a-h$ to $x=a+h$, remains within these limits the closest to 0, he shows that the difference $f(x)-U$ has the property that, among the largest and smallest values of the difference $f(x)-U$ between the limits $x=a-h$ and $x=a+h$, one finds the same numerical value at least $n+2$ times. He gives a method of finding the abscissas of the $n+2$ points where the absolute error should take its maximum value, which serves as the basis for his theory of approximation.

**Comments:** The property that the maximum absolute deviation occurs at least $n+2$ times when one fits a polynomial of degree n so as to minimize the magnitude of that deviation was, in the special case of fitting to a finite set of points, known already to Laplace. However, the analogy ends there. For reasons that should be obvious, the author's method of finding the abscissas of the points where the absolute error should take its maximum value and the theory of approximation based upon it are not applicable to the case of a finite set of points.

**Reference:** Poncelet (1835).

**Citation:** Kirchberger (1903).

* Gould, B. A., Jr. (1855). On Peirce's criterion for the rejection of doubtful observations, with tables for facilitating its application. *Astronomical Journal* **4**, 81-87. [Also published, under a different title, in *Report of the U. S. Coast Survey* for **1854**, pp. 131-138].

**Summary:** Gould gives tables which are more extensive than those of Peirce (1852) and include two more significant figures. He reworks the two examples of Peirce, and in one case concludes that only one observation should be rejected, whereas Peirce had rejected two. [However, Rider (1933) has pointed out (page 7) that neither result is trustworthy, since Gould uses Peirce's incorrect value for the standard deviation.]

**Reference:** Peirce (1852).

**Citations:** Airy (1856), Winlock (1856), Chauvenet (1863), Jevons (1874), Merriman (1877), Peirce (1878), Doolittle (1884), Wright (1884), Gore (1889), Czuber (1891a), Pizzetti (1892), Czuber (1899), Saunder (1903), Wellisch (1909), Brunt (1917), Archibald (1925), Rider (1933).

* Lloyd, Humphrey (1855). On the mean results of observations. *Transactions of the Royal Irish Academy* **22**, 61-73; abstract, *Proceedings of the Royal Irish Academy* **4**, 180-183.

**Summary:** Most of this paper, devoted to the determination of mean daily values of magnetical and meteorological elements from two, three, or more observations equally spaced in time, is irrelevant to a study of order statistics. The following passage (abstract, pp. 182-183) is, however, of interest: "In choosing the particular hours for a continuous system of observations, we should select those which correspond nearly to the maxima and minima of the observed elements, so as to obtain also the *daily range*. This condition is fulfilled, in the case of the magnetic declination, very nearly, by the hours 6 A.M., 2 P.M., 10 P.M.; and if we add the intermediate hours 10 A.M., 6 P.M., we shall have, nearly, the principal maxima and minima of the other magnetical elements, the maximum of temperature, and the two maxima of pressure. The author accordingly proposes, as the best hours of observation in a limited system, 6 A.M., 10, 2 P.M., 6, 10."

**Citations:** Merriman (1877), Pizzetti (1892).

* Airy, G. B. (1856). Letter from Professor Airy, Astronomer Royal, to the Editor. *Astronomical Journal* **4**, 137-138.

**Summary:** After a study of the papers of Peirce (1852) and Gould (1855) on Peirce's criterion, the author writes (page 137): "In an attempt to master these, I have been compelled to consider the subject from its first principles. And I have, not without surprise to myself, been led to think, that the whole theory is defective in its foundation, and illusory in its results; that no rule for the exclusion of observations can be obtained by any process founded purely upon a consideration of the discordance of those observations; and that the only rule (imperfect as it may be) by which such exclusion can be justified, must be based upon considerations of a totally different kind. . . . In seeking for truth or accuracy in a matter of chance, we are bound to take all the evidence that we can find; every individual observation contributes its evidence; and if we have no *a priori* reason for preferring one observation to another while we are yet in the dark as to the result, and no cogent reason for supposing that unusual causes of error must have intervened in special observations, we are bound to admit all on the same terms, as giving equally valid evidence. It appears to me that, in the whole process to which I am alluding, there is a fault of reasoning of a vital character; namely, the assumption that we can apply the reasoning of antecedent probability to the discussion of observations which are recorded as having been actually made. In ordinary life, instances perpetually occur . . . which, considered beforehand, are in the highest degree improbable, but which, as matters of history, are absolutely certain." He summarizes his conclusions as follows (page 138): "1. The mathematical theory of probabilities fails in all questions applying to errors of extreme magnitude. 2. No consideration of the magnitude of residual errors *per se* will justify us in rejecting a result. 3. We are justified in rejecting a result only when, from the best estimate that we can form of the extent of action of the various causes which can produce error, we find that the combination of those causes of error cannot possibly produce the discordance in question;—4. And when we perceive that other causes may have intervened, whose nature is such that they cannot be recognized as occurring in the ordinary series of observations."

**References:** Peirce (1852), Gould (1855).

**Citations:** Winlock (1856), Chauvenet (1863), Stone (1868), Merriman (1877), Edgeworth (1883b), Doolittle (1884), Wright (1884), Edgeworth (1886a, 1887a), Czuber (1891a), Pizzetti (1892), Czuber (1899), Wellisch (1909), Brunt (1917), Archibald (1925), Coolidge (1925), Irwin (1925b), Rider (1933).

* Winlock, Joseph (1856). On Professor Airy's objections to Peirce's criterion. *Astronomical Journal* **4**, 145-147.

**Summary:** The author answers Airy's criticisms of Peirce's criterion. He strongly refutes Airy's contention that we cannot "apply the reasoning of antecedent probability to the discussion of observations which are recorded as having been actually made." [Airy (1856), p. 137]. On this point Winlock writes (page 145): "If this could be urged as an objection of a vital character against the criterion, it would prove equally fatal to any theory of combination of discordant observations. The Professor certainly accepts the Method of Least Squares; but, in using that, is he not applying a 'theory of antecedent probability' to a set of observations for the purpose of obtaining a result which he receives as the most probable, although it may be at variance with the evidence of each and every one of the observations which are recorded as having been actually made? Why should there be, *a priori*, any greater objection to approaching a series of observations with a theory of antecedent probability for the purpose of detecting abnormality in some of them, than for the purpose of ascertaining any other peculiarity or property that they may possess, whose existence or magnitude has to be received or rejected by us on an estimate of the probabilities for or against it?" In answer to Airy's objection that the criterion fails [by rejecting a good observation] in a particular case, Winlock states (page 146): "He seems to hold the opinion that it is just as injurious to reject a good observation as it is to retain a bad one, provided the latter is within the limit of the greatest possible chance error. . . . But we are not required to use the same care to avoid the rejection of a good observation (that is, one affected only by ordinary causes of error) as the retention of a bad one. When we reject a good observation, we do not necessarily introduce error, we simply decline to use all the evidence we have, and our conclusions are of proportionally less value; but retaining one which is abnormal vitiates the result." Winlock sums up the case in favor of Peirce's criterion as follows (page 147): "Regarding the probability of an error as a function of its magnitude, we are enabled to find the probability of any system of residual errors, and by the comparison of the systems of errors before and after rejection in accordance with the rule of the criterion, we can decide, within safe limits, whether the probability of our final result is lessened by retaining the doubtful observations."

**References:** Peirce (1852), Gould (1855), Airy (1856).

**Citations:** Chauvenet (1863), Merriman (1877), Doolittle (1884), Wright (1884), Czuber (1891a), Pizzetti (1892), Czuber (1899), Wellisch (1909), Archibald (1925), Coolidge (1925), Rider (1933).

* Petzval, Joseph (1857). Fortsetzung des Berichtes über optische Untersuchungen. *Sitzungsberichte der Mathematisch-Naturwissenschaftliche Klasse, Königliche Akademie der Wissenschaften, Vienna* **24**, 129-144.

**Summary:** The author holds that the method of least squares is not applicable in optics, because of failure of the underlying assumptions that positive and negative errors of equal magnitude are equally probable and that all observations are made under equally favorable conditions by equally skilled observers with equally good instruments. He questions especially the second of these assumptions. He proposes instead what he calls "the method of numerically equal maxima and minima" in which the maximum of the absolute values of the residuals is minimized. He points out that this is equivalent to minimizing the sum of the 2m powers of the residuals, where m is an integer which tends to infinity as a limit. He proposes to devote a substantial part of the third volume of his three volume treatise on optics to this method.

**Comments:** Minimization of the maximum absolute residual had been proposed previously by Euler (1749), Lambert (1765a), Laplace (1786, 1793, 1799, 1812), and Gauss (1809). The equivalence of this minimization to that of the sum of the 2m powers of the residuals (m an integer tending to infinity) had been pointed out, not only by Gauss (1809) as mentioned by the author, but also by Laplace (1799, 1812).

**Reference:** Gauss (1809).

**Citations:** Merriman (1877), Pizzetti (1892).

* von Andrae, C. G. (1860). Udvidelse af en af Laplace i Mécanique Céleste angivet Methode for Bestemmelsen af en unbekjendt Størrelse ved givne umiddelbare Iagttagelser. *Oversigt af Kgl. Danske Videnskabernes Selskabs over . . . Forhandlingen* (Copenhagen) **1860**, 198-225.

**Summary:** The author studies the problem of choosing one, two, three, . . . of a series of n equally reliable observations to be used instead of all n observations in determining the most advantageous value of the measured quantity. In choosing a single observation, he uses the principle [Laplace (1799), Livre III, No. 40] of minimizing the sum of the absolute values of the errors. Hence he chooses the median, which he defines, for a sample of size n, as the $m^{th}$ ordered observation, where m = n/2. By analogy, if s observations are to be used, he chooses the $m_i^{th}$ ordered observations (i = 1,2, . . ., s), where $m_i = ni/(s+1)$.

**Comments:** The author does well to drop the Boscovich condition that the sums of positive and negative errors be equal in magnitude. His definition of the median, while asymptotically correct, is unsatisfactory for small or even moderate samples. We shall see later that analogy is a poor guide in choosing s observations (s>1), and that the optimum choice depends upon the underlying distribution.

**References:** Laplace (1799), Legendre (1805), Gauss (1809), Laplace (1812) [1818], Gauss (1823).

**Citations:** Zachariae (1871), Pizzetti (1892), Czuber (1899).

* Airy, George Biddell (1861). *On the Algebraical and Numerical Theory of Errors of Observations and the Combination of Observations*. Macmillan and Co., Cambridge and London.

**Summary:** The author (page 25) makes the following remarks concerning the treatment of outlying observations: "In collecting the results of a number of observations, it will frequently be found that, while the results of the greater number of observations are very accordant, the result of some one single observation gives an error, or a discordance, of large magnitude. There is, under these circumstances, a strong temptation to erase the discordant observation, as having been manifestly affected by some extraordinary cause of error. Yet a consideration of the law of Frequency of Error . . . or a consideration of the formation of a complex error by the addition of numerous simple errors . . . will show that such large errors may fairly occur; and if so, they must be retained. We may perhaps think that where a cause of unfair error *may* exist (as in omission of clamping a zenith-distance-circle), and where we know by certain evidence that in some instances that unfair cause *has* actually come into play, there is sufficient reason to presume that it has come into play in an instance before us. Such an explanation, however, can only be admitted with the utmost caution." The author gives a discussion of the method of least squares, which he calls "the method of minimum squares"; he does not mention any of its competitors.

**Reference:** Laplace (1812).

**Citations:** Todhunter (1865), Venn (1866), Jevons (1874), Merriman (1877), Edgeworth (1887b, d), Gore (1889), Pizzetti (1892), Czuber (1899), Edgeworth (1911, 1923).

* Chauvenet, William (1863). *A Manual of Spherical and Practical Astronomy*, Volume 2. J. B. Lippincott & Co., Philadelphia. (esp. Appendix, pp. 469-566, and Tables IX and X, pp. 593-599, reprinted 1868 as *A Treatise on the Method of Least Squares*).

**Summary:** Sections 1-56 of the Appendix (pages 469-558) and Table IX (pages 593-595) are concerned

with the method of least squares *per se*. Sections 57-60 (pages 558-566) deal with Peirce's criterion for the rejection of doubtful observations and with Chauvenet's own criterion for the rejection of a single observation. The latter is based on the principle that, since the number of errors numerically greater than $\kappa\sigma$ that may be expected to occur in n observations is $2n\int_{\kappa}^{\infty} \phi(t)\,dt = n\,\psi(\kappa)$, where $\phi(t) = e^{-t^2/2}/\sqrt{2\pi}$, an observation deviating from the mean by an amount greater than $\kappa\sigma$ should be rejected if the quantity $n\,\psi(\kappa)$ exceeds 1/2, since such an error "will have a greater probability against it than for it" (page 565). After correcting an error in the standard deviation calculated by Peirce (1852) and assumed correct by Gould (1855), the author applies Peirce's criterion and a modification involving two successive applications of his own criterion to data on the vertical semidiameter of Venus (15 observations) previously analyzed by Peirce and Gould. Both lead to the conclusion that two observations should be rejected. Table X (pages 596-599) is a table (adapted from Gould's) for Peirce's criterion. The author states (page 565) that he has prepared a table for his own criterion "which agrees so nearly with Table X that, for practical purposes, it may be regarded as identical with that table."

**Comments:** Rider (1933), page 8, has pointed out that Chauvenet's argument is wrong, since $n\,\psi(\kappa)$ is not a probability but a frequency. He goes on to say (pages 8-9): "In essence his [Chauvenet's] criterion rejects errors numerically greater than $\kappa\sigma$ where $\kappa$ is such that only half an error is to be expected greater than this limit. It is tantamount to assuming that a mistake is as likely as not to occur in n observations, that is, that a mistake occurs, on the average, once in 2n observations, where n is the number of observations in the set under consideration."

**References:** Legendre (1805), Gauss (1809), Laplace (1812), Gauss (1816, 1823), Encke (1832-34), Bessel & Baeyer (1838), Peirce (1852), Gould (1855), Airy (1856), Winlock (1856).

**Citations:** Stone (1868), Glaisher (1872, 1873), Peirce (1873), Stone (1873b), Jevons (1874), Merriman (1877), Chaplin (1880, 1882), Doolittle (1884), Merriman (1884), Wright (1884), Edgeworth (1887a,b,d,e), Lehmann-Filhes (1887), Gore (1889), Czuber (1891a), Pizzetti (1892), Vallier (1894), Czuber (1899), Sheppard (1899b), Bowley (1901), Saunder (1903), Wellisch (1909), Fisher (1912), Contarino (1914), Walker (1914), Becker (1917), Brunt (1917), Stewart (1920a), Edgeworth (1923), Rietz (1924), Archibald (1925), Coolidge (1925), Irwin (1925b), Ogrodnikoff (1928), Scarborough (1930), Pearson (1931), Jeffreys (1932a), Rider (1933), von Eberhard (1938), Jeffreys (1939), Smirnoff (1941).

* De Morgan, Augustus (1864). On the theory of errors of observation. *Transactions of the Cambridge Philosophical Society* **10** (2), 409-427.

**Summary:** Glaisher (1872) writes (p. 90): "Perhaps the most important contribution that has been made to the theory of the arithmetic mean is contained in a paper by De Morgan [(1864)]. It is there pointed out [p. 416] that the mean value of all the given values is also the *mean supposition of all possible suppositions* as to the mode of obtaining value. The investigation is too long to reproduce, but the result may be stated thus. If $\phi(x_1, x_2, \ldots, x_n)$ is the most probable result of the discordant observations, $x_1, x_2, \ldots, x_n$ then $\phi(x_1, x_2, \ldots, x_n) = (x_1 + x_2 + \ldots + x_n)/n + Q\Sigma\epsilon_i^2 + R\Sigma\epsilon_i\epsilon_j + S\Sigma\epsilon_i^3 + T\Sigma\epsilon_i^2\epsilon_j + U\Sigma\epsilon_i\epsilon_j\epsilon_k + \ldots$ the $\epsilon$'s being small quantities, and Q, R, S . . . dependent on $\phi$; 'hence $\phi(x_i\, x_2, \ldots, x_n)$ is $\Sigma x/n$ augmented by terms of which we have no knowledge whatever, either as to sign or value, and no means of getting any: we are therefore wholly without reason for supposing that the value of $\phi(x_1, \ldots)$ lies on one side of the average [arithmetic mean] rather than the other, and must take this average as the most probable value a priori' [p. 417]." The author continues, however, with the following statement not quoted by Glaisher: "The average, then, is the most probable result so long as we know *nothing* of the law of facility of error: but this is only so long as the observations are either not made, or not disclosed. So soon as we see the *second* observation, we have some information about the law of error: not much, but *some*."

**Citations:** Glaisher (1872), Jevons (1874), Merriman (1877), Gore (1889), Pizzetti (1892), Czuber (1899).

* Boudin, E.J. (1865). *Leçons de Calcul des Probabilités*. Hoste et Cie. (Maison d'éditions et d'impressions), Gand. Reprinted 1874, 1886, 1889. Revised edition, with notes and additions by Paul Mansion, Gauthier-Villars et Cie., Paris, 1916; review, *Bulletin des Sciences Mathématiques* (2) **43** (1919), 129-133.

**Summary:** This book is divided into two parts, the first part dealing with the theory of probability and the second with applications, including the theory of errors. The author presents both the method of minimum approximation (minimizing the maximum deviation) and the method of situation (minimizing the sum of the absolute deviations subject to the restriction that the algebraic sum of the deviations be zero) [Laplace (1799)]. Boudin follows the geometric exposition of Prony (1804). In the 1916 edition, Mansion adds a comparison of the method of least squares with the above methods of Laplace, a discussion of what he calls the surest method (reducing all the given equations to the form $x=a$), and a supplementary chapter on the purely algebraic theory of least squares. He states that the method of minimum approximation should most often be rejected because it leads to the use of only those observations which one has the greatest reason to believe inexact.

**Note:** The compiler has been unable to obtain a copy of any edition of this book, so in writing the above summary he has relied on information given in the 1919 review of the 1916 edition and in papers by Mansion (1913) and Alliaume (1927a).

**References:** Laplace (1799), Prony (1804).

**Citations:** Mansion (1913), Julin (1921), Alliaume (1927a).

* Todhunter, Isaac (1865). *A History of the Mathematical Theory of Probability from the Time of Pascal to that of Laplace*. Macmillan and Co., London. [Reprinted 1949 by Chelsea Publishing Co., New York].

**Summary:** This is simply what the name implies—a history—and does not contain any original contributions, though the author's treatment is often more understandable than the original sources. Included are summaries of the work of Nicolas Bernoulli (1709) on the expected value of the largest of n observations which are uniformly distributed with known range and of various writers on the theory of errors, including Simpson (1757), Lagrange (1774), Daniel Bernoulli (1778), Euler (1778), Jean Bernoulli (1785), Trembley (1804), and Svanberg (1805), as well as numerous writings of Laplace (1774, 1781, 1786, 1799, 1810, 1811a,b, 1812). The last four of these deal primarily with the method of least squares, but it is clear from Todhunter's pages 588 and 612 that Laplace never entirely abandoned some of his earlier methods based on order statistics in favor of the method of least squares.

**Comments:** The compiler has found this book to be of great value in preparation of the present work. In several cases he first became aware of the existence of relevant publications through reading it, and in one or two cases (in which the compiler has been unable to obtain copies of the original sources), it has served as the chief source of information concerning their contents.

**References:** N. Bernoulli (1709), Euler (1749), Mayer (1750), Simpson (1757), Lagrange (1774), Laplace (1774), D. Bernoulli (1778), Euler (1778), Laplace (1781), J. Bernoulli (1785), Laplace (1786, 1799), Trembley (1804), Svanberg (1805), Laplace (1810, 1811a,b, 1812), Airy (1861).

**Citations:** Venn (1866), Glaisher (1872), Todhunter (1873), Jevons (1874), Merriman (1877), Edgeworth (1885, 1886a, 1887a,e), Pizzetti (1892), Czuber (1899), Bowley (1901), Edgeworth (1905, 1911), Brunt (1917), Keynes (1921), Rietz (1924), Thompson (1936), Boldrini (1942), Kendall (1946).

* Clifford, W. K. (1866). Solution to problem 1878. *Educational Times* **6**, 83-87. [Reprinted 1882 in Clifford's *Mathematical Papers* (edited by Robert Tucker), Macmillan and Co., London, pp. 601-607].

**Summary:** Problem 1878, which was proposed in the January 1866 issue of *Educational Times*, is the following: "A line of length a is broken up into n pieces at random; prove that (1) the chance that they cannot be made into a polygon of n sides is $n\,2^{1-n}$; and (2) the chance, that the sum of the squares on them does not exceed $a^2/(n-1)$, is

$$\left(\frac{\pi}{n^2-n}\right)^{\frac{1}{2}(n-1)} \cdot \frac{\Gamma(n)}{\Gamma\{\frac{1}{2}(n+1)\}} \cdot \frac{1}{n^{1/2}} \cdot"$$

Clifford's solution, given in the November 1866 issue, does not admit a brief summarization.

**Comments:** This problem is relevant to a study of order statistics because the lengths of the n segments are the successive differences of $(n+1)$ numbers [0, the order statistics of a sample of size $(n-1)$ from R(0,a), and a], where R(0,a) represents a uniform or rectangular distribution between the limits 0 and a.

**Citation:** Moran (1947).

* Venn, John (1866). *The Logic of Chance*. Macmillan and Co., London-New York. Second edition 1876; third edition, 1888; reprinted 1962, Chelsea Publishing Co., New York.

**Summary:** The first edition apparently contains nothing directly relevant to our subject. Chapters on the laws of error (Ch. II) and on the method of least squares were added in the second edition, but the latter was dropped in the third edition. Chapters XVIII and XIX, added in the third edition, deal with averages. Concerning laws of error, the author writes (pp. xix-xx): "Is there necessarily but one such law, applicable to widely distinct classes of things? This cannot be proved directly by statistics, which in certain cases show actual asymmetry; nor deductively; nor by the Method of Least Squares." In two sections (pp. 33-35) added in the third edition, he notes that at the time of the first edition writers on the subject of statistics were still heavily influenced by Quetelet, who apparently believed the normal law of error to be universal, but cites the work of Fechner, Galton, and Edgeworth to show that many distributions found in nature are not even symmetric, much less normal. On page 41 he states: "The Method of Least Squares is usually applied, no doubt, to the most familiar form of the Law of Error, namely the exponential [normal] form . . . . But other forms of laws of error may exist, and, if they did, the method in question might equally well be applied to them. I am not asserting that it would necessarily be the best method in every case, but it would be a possible one; indeed we may go further and say, as will be shown in a future chapter [apparently Chapter XIII of the second edition, which is not found in the third edition], that it would be a good method in almost every case." In Chapter XVIII (pp. 435-464) the author discusses the arithmetic, geometric and harmonic means, the median and the mode, and the conditions under which each should be used. He states that, for most ordinary purposes, the arithmetic mean is the simplest and best, but he mentions the following advantages of the median: (1) in most cases it is the simplest average to calculate; (2) the probable error is a by-product in calculating it; and (3) it is disturbed very little by a few exceptional and extreme values. He also discusses various measures of dispersion, including the mean error (average deviation), median error (probable error or quartile deviation), and the root-mean-square error. Chapter XIX (pp. 465-503), which deals with the theory of the average as a means of approximation to the truth, appears to contain little or nothing that is relevant to our study.

**Note:** Unless otherwise stated, all references in the above summary to specific chapters and pages are to those in the third edition, which is the only one the compiler has seen. Some information about the contents of earlier editions has been gleaned from their prefaces, which are reprinted in the third edition.

**References:** Quetelet (1835) [1869], Herschel (1850) [1857], Airy (1861), Todhunter (1865), {Fechner (1874), Galton (1875), Merriman (1877), Galton et al. (1881), Merriman (1884), Edgeworth (1885, 1886a, 1887-90)}.

**Citations:** Jevons (1874), Edgeworth (1887-90), Venn (1891), Pizzetti (1892), Czuber (1899), Edgeworth (1911, 1913), Julin (1921), Keynes (1921), Rietz (1924), Bowley (1928), Bartlett (1933), Kendall (1943, 1946).

* Whitworth, W. A. (1867). *Choice and Chance*. Cambridge University Press, London. Second edition, 1870; third edition, 1878; fourth edition, 1887. Fifth edition, Deighton Bell and Co., Cambridge, 1901 (reprinted by G. E. Stechert & Co., New York, 1934).

**Summary:** The author (pp. 200-201 of the fifth edition) states and proves the following proposition [LI]: "If an event happen at random on an average once in time t, the chance of its not happening in a given period $\tau$ is $e^{-\tau/t}$. [Proof:] Divide each unit of time into an indefinitely large number n of small instants. Then in the periods t, $\tau$, there will be nt, $n\tau$ respectively of such instants. By making n large enough we may make the instants so short that the event may be said to happen at one such instant. Since it happens on an average once in nt instants, the chance of its happening at any particular instant is 1/nt, and the chance of its not happening in the given period of $n\tau$ instants is $(1-1/nt)^{n\tau}=[(1-1/nt)^{nt}]^{\tau/t}$. But when n is indefinitely large, $(1-1/nt)^{nt}$ ultimately = 1/e. Therefore the required chance is $(1/e)^{\tau/t}$, or $e^{-\tau/t}$. Q.E.D. Example. If an earthquake happens on an average once a year, the chance that in a given year there should not be an earthquake is 1/e." The following exercises are given on page 331 and their answers on page 342: "666. A line of length c is divided into n segments by n−1 random points. Find the chance that no segment is less than a given length a, where c>na. (Say c−na=ma.) [Answer: $m^{n-1} \div (m+n)^{n-1}$.] 667. In the last question find the chance that r of the segments shall be less than a and n−r greater than a. [Answer: $\{m^{n-1}-C_1^r(m+1)^{n-1}+C_2^r(m+2)^{n-1}-$ &c. $\pm(m+r)^{n-1}\}/\pm(m+n)^{n-1}$.]" Hints as to the solution of the exercises are given by Whitworth (1897).

**Comments:** The intervals between occurrences in a Poisson process considered in Proposition LI are seen to follow a one-parameter negative exponential distribution, a fact which will prove useful later. The questions in exercises 666 and 667 fall into a class dealing with the random division of an interval, and have obvious interpretations in terms of the order statistics of samples from a uniform (rectangular) distribution.

**Citations:** Jevons (1874), Whitworth (1897), Bowley (1901), Edgeworth (1911), Keynes (1921), Morant (1921), Neyman & Pearson (1928), Scarborough (1930), Pearson & Pearson (1932), Greenwood (1946), Kendall (1946).

* Stone, E. J. (1868). On the rejection of discordant observations. *Monthly Notices of the Royal Astronomical Society* **28**, 165-168.

**Summary:** Stone defines what he calls the *modulus of carelessness*, m, which, for a given observer and a given class of observations, expresses the average number of observations which that person makes with one mistake. If $\kappa$ is the value such that $\psi(\kappa) = 1/m$, where $\psi(\kappa) = 2\int_{\kappa}^{\infty} \phi(t)dt$ and $\phi(t) = e^{-t^2/2}/\sqrt{2\pi}$, then all deviations greater in absolute value than $\kappa\sigma$ are more likely to have arisen from mistakes than from ordinary errors, and the corresponding observations are to be discarded.

**Comments:** Stone's criterion with m = 2n, where n is the number of observations, is equivalent to Chauvenet's. Rider (1933), page 9, gives a table of rejection limits, according to Stone's criterion, and remarks that they "can also, of course, be interpreted as giving the rejection limit when an observation is to be discarded if the probability of the corresponding error is less than 1/m, without reference to Stone's hypothesis that the observer makes, on the average, one mistake in m observations." As Rider points out (page 10, Stone's criterion is subject to the objection that if one takes m=100, all observations that deviate from the mean by more than $2.58\sigma$ will be discarded, but if one makes 200 observations, one observation would be expected to deviate by more than $2.81\sigma$, and such an observation would be rejected.

**References:** Peirce (1852), Airy (1856), Chauvenet (1863).

**Citations:** Glaisher (1873), Stone (1873b, 1874), Merriman (1877), Doolittle (1884), Wright (1884), Edgeworth (1887a), Czuber (1891a), Pizzetti (1892), Vallier (1894), Czuber (1899), Archibald (1925), Coolidge (1925), Ogrodnikoff (1928), Rider (1933).

* von Andrae, C. G. (1869). Schreiben . . . an den Herausgeber. (Danish). *Astronomische Nachrichten* **74**, 283-284.

**Summary:** With regard to the use of the $m(m-1)/2$ differences of m observations instead of the observations themselves to estimate the probable error, as proposed by Jordan (1869), the author raises the objection that the differences are not independent of each other. Therefore, he contends, the entire development is incorrect [see, however, the paper by Jordan (1872)], the only exception being the expression for the probable error (but not the expression for its probable uncertainty) for $n=2$. This expression is identical with the expression based on the observations themselves, since the mean of the squares of the differences is twice the mean of the squares of the deviations of the observations from their arithmetic mean.

**Reference:** Jordan (1869).

**Citations:** Bessel & Baeyer (1838) [1876], von Andrae (1872), Jordan (1872), Merriman (1877), Czuber (1891a), Pizzetti (1892), Czuber (1899, 1903), Wellisch (1909), Gini (1912), Tucker (1927), de Finetti (1931a), Gini (1939).

* Galton, Francis (1869). *Hereditary Genius*. Publisher unknown, London (?). Second American edition, D. Appleton and Company, New York, 1887.

**Summary:** The author shows how to use the median and any other quantile in fitting a theoretical normal distribution to observed data. On pp. 27-28 he writes: "Suppose a million . . . men to stand in turns, with their backs against a vertical board of sufficient height, and their heights to be dotted off upon it. . . . The line of average height [the median] is that which divides the dots into two equal parts. . . . Next let a hundred dots be counted from above downwards, and let a line be drawn below them [a quantile]. . . . Using the data afforded by these two lines [or any two quantiles], it is possible, by the help of the [normal] law of deviation from the average, to reproduce, with extraordinary closeness, the entire system of dots on the board." Subsequently, he discusses at some length data given by Quetelet (1846) on the chest measures of 5,738 Scotch soldiers and the heights of 100,000 French conscripts, in addition to an example of his own dealing with marks made on the entrance examination by candidates for admission to the Royal Military College at Sandhurst.

**Note:** The compiler has seen only the second American edition, but this presumably does not differ materially from the original British edition.

**Reference:** Quetelet (1846).

**Citations:** Jevons (1874), Galton (1875, 1889), Gini (1909, 1936), Davis (1941).

* Jordan, W. (1869). Ueber die Bestimmung der Genauigkeit mehrfach wiederholter Beobachtungen einer Unbekannten. *Astronomische Nachrichten* **74**, 209-226. (JFM 2, 841-842).

**Summary:** The author reviews the work of Gauss (1816), who explored the use of the $n^{th}$ root of the mean value of the $n^{th}$ powers of the absolute values of the deviations from the true value as a measure of the precision of repeated observations of the same quantity, concluding that $n=2$ gives the best results, at least for the normal (Gaussian) distribution. He notes that Gauss (1823) pointed out that if the true value of the observed quantity is unknown, and is replaced by a least-squares estimate, the resulting

measure of precision will be too small, and must be multiplied by the correction factor $\sqrt{m/(m-p)}$, where m is the sample size and p is the number of parameters estimated [p = 1 if only the true mean is estimated (by the arithmetic mean of the m observations)]. The author extends Gauss' table of factors for computing the probable error and its probable uncertainty from the $n^{th}$ root of the mean of the $n^{th}$ powers of the absolute values of deviations from the true value up through n = 10 and corrects Gauss' factors for the median, which he shows to give a slightly less (rather than more) precise estimate than the other method with n = 6. The author then proposes the use of the m(m − 1)/2 differences of the m sample values instead of the sample values themselves, pointing out that the probable error of a difference is $\sqrt{2}$ times that of a single observation. He adapts the formulas used with his table of factors to the computation of the probable error from the $n^{th}$ root of the mean of the $n^{th}$ powers of the absolute values of the differences [n = 1(1)10] and from the median of the differences, and professes to show that the probable uncertainties of these probable errors are substantially less than for those obtained from the individual observations [see, however, the paper by von Andrae (1869)].

**References:** Gauss (1816, 1823), Gerling (1843).

**Citations:** Bessel & Baeyer (1838) [1876], von Andrae (1869, 1872), Jordan (1872), Helmert (1876a), Merriman (1877), Gore (1889), Czuber (1891a), Pizzetti (1892), Czuber (1899), Gini (1912), Weinberg (1916), Bartels (1928), von Bortkiewicz (1931), de Finetti (1931a), Münzner (1934), Gini (1939).

* Zachariae, Georg Carl Christian (1871). *De Mindste Kvadraters Methode*. Boghandler v. Schönemanns Forlag, Nyborg. Second edition, Copenhagen, 1887.

**Summary:** This book, as the title implies, deals primarily with the method of least squares, of which the author gives an excellent textbook treatment. In Section 31 (pp. 95-101) he gives two examples. In the first of these he follows the suggestion of von Andrae (1860), based on the work of Laplace (1799), concerning the use of the median. For a grouped frequency distribution, he interpolates between class boundaries to find the median and the two quartiles, corresponding to cumulative frequencies equal to 1/2, 1/4, and 3/4 of the total frequency (1000). He takes half the difference between the two quartiles (the quartile deviation or semi-interquartile range) as an estimate of the probable error.

**References:** Laplace (1799), Gauss (1809), Laplace (1812), Gauss (1823), Bessel & Baeyer (1838), von Andrae (1860).

**Citations:** Helmert (1877b), Merriman (1877), Gore (1889), Pizzetti (1892), Czuber (1899).

* von Andrae, C. G. (1872). Ueber die Bestimmung des wahrscheinlichen Fehlers durch die gegebenen Differenzen von m gleich genauen Beobachtungen einer Unbekannten. *Astronomische Nachrichten* **79**, 257-272. (JFM 4, 577-578).

**Summary:** After reviewing the two papers by Jordan (1869, 1872) and his own earlier paper on the subject [von Andrae (1869)], the author reiterates that the problem of estimating the probable error from the $n^{th}$ root of the mean of the absolute values of the $n^{th}$ powers of the m(m − 1)/2 differences of m observations has already been solved for the case n = 2 and that the results are identical with those obtained from the square root of the mean of the squares of the deviations of the observations from their arithmetic mean. He then turns to the solution of the problem for n = 1, and obtains expressions for the estimate of the probable error and for its probable uncertainty. In the development of these expressions, he arranges the m observations in order of their magnitude, starting with the smallest, denoting the ordered observations by $a_1, a_2, \ldots, a_m$, and the corresponding deviations from the true value by $\Delta_1, \Delta_2, \ldots, \Delta_m$, also arranged in the order of their algebraic size. The m(m − 1)/2 positive differences can then be represented in the form of a triangular table:

$a_2 - a_1$

$a_3 - a_1 \quad a_3 - a_2$

$a_4 - a_1 \quad a_4 - a_2 \quad a_4 - a_3$

$a_5 - a_1 \quad a_5 - a_2 \quad a_5 - a_3 \quad a_5 - a_4$

$\ldots \quad \ldots \quad \ldots \quad \ldots$

$a_m - a_1 \quad a_m - a_2 \quad a_m - a_3 \quad a_m - a_4 \ldots \quad a_m - a_{m-1}$

The sum [d] of the differences is then $[d] = (m-1)(a_m - a_1) + (m-3)(a_m - a_2) + (m-5)(a_m - a_3) + \ldots$, or after introduction of the true errors, $[d] = (m-1)(\triangle_m - \triangle_1) + (m-3)(\triangle_m - \triangle_2) + (m-5)(\triangle_m - \triangle_3) + \ldots$, where the two series for m odd ($=2n+1$) end with the term $2(a_{n+2} - a_n)$ or $2(\triangle_{n+2} - \triangle_n)$, respectively, but for m even ($=2n$), with $(a_{n+1} - a_n)$ or $(\triangle_{n+1} - \triangle_n)$, respectively.

**References:** Gauss (1816, 1823), von Andrae (1869), Jordan (1869, 1872).

**Citations:** Helmert (1876a), Merriman (1877), Gore (1889), Czuber (1891a), Pizzetti (1892), Czuber (1899), Gini (1912), Tucker (1927), de Finetti (1931a), Gini (1939).

* Glaisher, J. W. L. (1872). On the law of facility of errors of observations, and on the method of least squares. *Memoirs of the Royal Astronomical Society* **39**, 75-124; abstract, *Monthly Notices of the Royal Astronomical Society* **32**, 241-242. (JFM 4, 92-94).

**Summary:** This paper is devoted primarily to a history of the method of least squares, including an account and a critical evaluation of the contributions of Legendre, Adrain, Gauss, Laplace, Ivory, Ellis, De Morgan and others. The author also proposes (pp. 103-104) an alternative to the rejection of anomalous observations, concerning which he writes (pp. 120-121): "The method described on pp. 103-104, of returning to the observations again after finding the most probable result by the rule of least squares, and assigning weights, the process being repeated as often as necessary, distinctly appears to be the proper and philosophical method of treating observations; and, as a consequence, Professor Peirce's [(1852)] criterion for the rejection of doubtful observations seems to me to be destitute of scientific precision. If an observation has been made as carefully as the rest, it ought on no account to be rejected entirely. It may be, and no doubt is, true that in many cases it is better to reject it than to retain it, *giving it an equal weight with the best observation*, but the true principle is to weight the observation as the method itself indicates, when an abnormal observation would receive a very small weight. It appears quite evident that under no circumstances have we a right to say an observation has no weight, though it may be better to give it none than to give it as much as the best. The fact of such a criterion having been proposed is a strong argument in favour of the necessity of the completion of the method of least squares as indicated; attention to this necessity was first insisted on by De Morgan [(1847), p. 456]." Last, but not least, the author proves (pp. 121-122) that if errors are distributed according to Laplace's first law [$f(x) = (m/2)e^{-m|x|}$], the median of the observations is the most probable true value. He does this by showing that the probability (density) of the true value x is proportional to exp [$-m$(the sum of the absolute values of the deviations of the observations from x)] and that that sum is a minimum when taken about the median [the middle one of an odd number of observations or any value between the middle two of an even number of observations].

**References:** Simpson (1757), Lambert (1760), Lagrange (1774), Laplace (1774), D. Bernoulli (1778), Legendre (1805), Puissant (1805), Adrain (1808), Gauss (1809), Laplace (1812), Adrain (1818), Gauss (1823), Ivory (1825), Encke (1832-34), Ellis (1844), Quetelet (1846), De Morgan (1847), Herschel (1850), Peirce (1852), Chauvenet (1863), De Morgan (1864), Todhunter (1865).

**Citations:** Glaisher (1873), Peirce (1873), Jevons (1874), Merriman (1877), Wilson (1877), Edgeworth (1883a,b), Wright (1884), Edgeworth (1885, 1886a, 1887a,b,e), Gore (1889), Czuber (1891a,b), Pizzetti (1892), Czuber (1899), Edgeworth (1905, 1911), Brunt (1917), Keynes (1921), Rietz (1924), Bruen (1938).

* Helmert, F. R. (1872). *Die Ausgleichungsrechnung nach der Methode der kleinsten Quadrate mit Anwendung auf die Geodäsie und die Theorie der Messinstrumente*. B. G. Teubner, Leipzig-Berlin. (JFM 5, 593-594); second edition, 1907. (JFM 37, 276-277); third edition (with supplement by H. Hohenner), 1924; English translation, USAF Aeronautical Chart and Information Center, St. Louis, 1958.

**Summary:** The first edition of this book on the adjustment computation by the method of least squares contains (pp. 21-26) a proof, following that of Gauss (1816), that the probable error can be determined more precisely from the mean of the squares of the errors of a number of observations than from the mean of the absolute values of the errors. The author does not consider in detail the use of higher powers, mentioning them only in passing. In later editions, a section is added on the exclusion of individual observations and the maximum error, in which the author states: "If in the course of an adjustment, it becomes apparent that an observation requires correction $\lambda$ which by many times exceeds the mean error $\mu$, then the question arises whether or not an exclusion is warranted so that results could be obtained which approach the actual picture closer than without exclusion. In order to establish a criterion, a number of more or less involved probability considerations have been analyzed. . . . According to the stipulations of the Gauss' law of errors, with 20 observations, one observation will have an error $>2\mu$ [where $\mu$ is the standard deviation ($\sigma$ in modern notation)]; with 400 observations, one will have an error $>3\mu$; with 16,000 observations, one will have an error $>4\mu$; with $2 \cdot 10^6$ observations, one will have an error $>5\mu$. Thus the anticipated maximum error with 10 observations equals about $2\mu$, with 200 observations equals about $3\mu$, with 8000 observations equals about $4\mu$, with $1 \cdot 10^6$ observations equals about $5\mu$. . . . Using [this] as a criterion, we can establish whether a stronger deviation of observation should be rejected." [pp. 364-365 of third edition (pp. 480-482 of English translation)].

**References:** Gauss (1816), {Helmert (1877b), Jordan (1877b) [1904], Newcomb (1886), Czuber (1891a)}.

**Citations:** Helmert (1875b), Mees (1875), Helmert (1876b), Mees (1876), Helmert (1877b), Merriman (1877), Gore (1889), Pizzetti (1892), Czuber (1899, 1903), Kozák (1907), Vogeler (1907), Kozák (1908-10), Brunt (1917), Bartels (1928), von Mises (1931), Friedrich (1937), Arley & Buch (1940), Gaede (1942).

* Jordan, W. (1872). Ueber die Bestimmung des mittleren Fehlers durch Wiederholung der Beobachtungen. *Astronomische Nachrichten* **79**, 219-222. (JFM 4, 577).

**Summary:** In reply to the objection raised by von Andrae (1869) to the method proposed in his earlier paper [Jordan (1869)], the author admits that the fact that the $m(m-1)/2$ differences of m observations are not independent of each other invalidates his expressions for the probable uncertainties of his estimates of probable error. He insists, however, that the expressions for the estimates themselves are correct, so that all that is necessary is to correct the expressions for the probable uncertainties, taking due account of the fact that the differences are not independent of each other. He endeavors to solve this problem by analogy to that of the measurement of the length of a straight line PAP′ by measuring the two segments PA and AP′.

**References:** Gauss (1823), Bessel & Baeyer (1838), von Andrae (1869), Jordan (1869).

**Citations:** von Andrae (1872), Merriman (1877), Gore (1889), Pizzetti (1892), Czuber (1899), Gini (1912), de Finetti (1931a), Gini (1939).

* Glaisher, J. W. L. (1873). On the rejection of discordant observations. *Monthly Notices of the Royal Astronomical Society* **33**, 391-402. (JFM 5, 121).

**Summary:** The author elaborates on the alternative to the rejection of discordant observations, based on a suggestion of De Morgan [(1847), p. 456], which he proposed in an earlier paper [Glaisher (1872), pp. 103-104]. He states the problem as follows (p.391): "Given a number of direct observations of the same

quantity, determine the most probable value of that quantity." He assumes that the observations $x_i (i = 1, \ldots, n)$ are normally distributed, with law of facility of error $(h/\sqrt{\pi})e^{-h^2x^2}$, and assumes initially that the precision constant h is the same for all observations. This leads to the arithmetic mean a of the observations as the most probable true value. Then he finds the probability $p_i = e^{-nh^2(x_i-a)^2}$ that $x_i$ is the true value (relative to the probability that a is the true value, which probability he takes as unity), and takes $p_i^2$ as the weight of $x_i$, which results in the weighted mean $b = \Sigma_{i=1}^n p_i^2 x_i / \Sigma_{i=1}^n p_i^2$ as a second approximation to the true value. Then, as before, he obtains new weights $p_i'^2$, which lead to a third approximation c to the true value. He continues the process until successive sets of weights (and hence successive approximations to the true value) agree to within some (unspecified) tolerance. He also examines the criterion proposed by Stone (1868) for the rejection of outliers, and criticizes it on two grounds: (1) Any rejection criterion based on the supposition of the validity of the arithmetic mean is inconsistent; (2) Even among such criteria Stone's is not the most desirable one and is impractical because of the practical impossibility of determining the value of n, it being assumed that the observer makes one mistake in n observations.

**Comments:** Rider [(1933), p. 15] has pointed out that Glaisher's proof is based upon a questionable application of Bayes' theorem [what Glaisher calls the probability of $x_i$ is actually the ratio of the probability density at $x_i$ to the maximum probability density at a, given that a is the true value]. Rider also writes [(1933), p. 16]: "It may be argued that Glaisher's method may never lead to a limit, or that the limit may be reached very slowly and tediously." On this point, Glaisher's method has the advantage that it does not always converge to one of the observations, as does a somewhat similar method proposed half a century earlier [Anonymous (1821)] which uses the reciprocals of the residuals (or of their squares) as weights.

**References:** De Morgan (1847), Peirce (1852), Chauvenet (1863), Stone (1868), Glaisher (1872).

**Citations:** Stone (1873b), Glaisher (1874), Stone (1874), Merriman (1877), Edgeworth (1883a), Doolittle (1884), Wright (1884), Edgeworth (1886a), Gore (1889), Czuber (1891a), Pizzetti (1892), Czuber (1899), Saunder (1903), Wellisch (1909), Keynes (1921), Archibald (1925), Coolidge (1925), Ogrodnikoff (1928), Rider (1933).

* Halphen, G. (1873). Sur un problème de probabilités. *Bulletin de la Société Mathématique de France* **1**, 221-224.

**Summary:** The author generalizes the result of Lemoine (1873) for the special case n = 3 by showing that if a straight line is divided at random into n segments, the probability that a closed n-gon can be formed from the segments is $1 - n/2^{n-1}$. He gives two proofs, of which the simpler is as follows: Let a be the length of the straight line, which may, without loss of generality, be taken as the interval (0,a). Let $B_1$, $B_2, \ldots, B_{n-1}$ be the breaking points, $0 < B_i < a$, $i = 1,2, \ldots, n-1$. The required probability is the probability that the length of the longest of the n segments is less than a/2. The probability that the first segment is greater than a/2 in length is the probability that $B_1, B_2, \ldots, B_{n-1}$ are all greater than a/2, which is $1/2^{n-1}$, since the $n-1$ breaking points are independent and the probability that any one of them exceeds a/2 is 1/2. The probability that the $2^{nd}$, $3^{rd}, \ldots, n^{th}$ segment is greater than a/2 in length is also $1/2^{n-1}$. Hence the probability that one of the n segments is greater than a/2 in length (n mutually exclusive events, each with probability $1/2^{n-1}$) is $n/2^{n-1}$, and the required probability is $1 - n/2^{n-1}$. The second proof makes use of integral calculus.

**Reference:** Lemoine (1873).

**Citations:** van den Berg (1891), Wölffing (1899), Mantel (1915).

* Lemoine, E. (1873). Sur une question de probabilités. *Bulletin de la Société Mathématique de France* **1**, 39-40.

**Summary:** The author states and solves analytically the following problem: "A rod is broken into three pieces; what is the probability that, with these three pieces, one can form a triangle?" His solution is as follows: "Let us divide the rod into 2m equal parts, and suppose that the three parts contain respectively x, y and z of these parts; we will have (1)$x+y+z = 2m$. In order for the triangle to be possible, it is necessary that one have $x \leq y+z$, $y \leq x+z$, $z \leq x+y$. Eliminating z with the aid of equation (1), these inequalities become $x \leq m, y \leq m$, $x+y \geq m$. We seek the number of favorable cases: $x=0$ allows for y the value m; $x=1$ allows for y the values m, $m-1$;. . . ; $x=m$ allows for y the values m, $m-1$, $m-2$, . . . ,0. There are thus in all $1+2+3+\ldots+(m+1)=(m+1)(m+2)/2$ favorable cases. We seek the number of possible cases: $x=0$ allows for y the values $2m, 2m-1, \ldots, 2,1,0$; $x=1$ allows for y the values $2m-1, \ldots, 2,1,0$; . . . ; $x=2m$ allows for y the value 0. There are thus in all $1+2+3+\ldots+(2m+1)=(2m+1)(2m+2)/2$ possible cases. The ratio of the number of favorable cases to the number of possible cases is $(m+1)(m+2)/(2m+1)(2m+2) = (m+2)/2(2m+1)$, which, for $m=\infty$, gives 1/4. Thus, the probability sought is 1/4." [Compiler's translation].

**Comments:** The probability sought is equivalent to the probability that the length of the longest piece is less than $\ell/2$, where $\ell$ is the length of the rod. Hence the problem can be restated in terms of order statistics.

**Citations:** Halphen (1873), Lalanne (1879), van den Berg (1891), Wölffing (1899), Keynes (1921).

* Peirce, C. S. (1873). On the theory of errors of observations. *Report of the Superintendent of the United States Coast Survey* (for the year ending November 1, 1870), Appendix No. 21, pp. 200-224 and Plate No. 27. U. S. Government Printing Office, Washington.

**Summary:** The author begins (p. 200) with the statement: "The object of this paper is to give a general account of the theory of errors of observations, with the design of showing what the limitations to the applicability of the method of least squares are, and what course is to be pursued when that method fails." He shows that the method of least squares, with equal weights for all observations, is valid only when the precision constant $h(=1/\sigma\sqrt{2}$, where $\sigma$ is the standard deviation) is the same for all the observations; otherwise the weights assigned to the observations must be proportional to the squares of their respective precision constants. But since the true precision constants are almost always unknown, the weights must be estimated from the observations themselves; however, the author points out (p. 210) that "it must never be forgotten that h is a statistical quantity; not one which belongs to a single observation, but one which belongs to an infinite series of observations". Since the method of least squares also depends on the errors of observation being normally distributed, the author reports the results of an experiment performed to test the validity of this assumption. In this experiment the time required for a previously untrained observer to respond to a signal by pressing a telegraph key was measured (in thousandths of a second) about 500 times each day for 24 days. The author concludes that the response times on a given day are approximately normally distributed, though there is evidence of an increase in precision from day to day, especially at the outset (learning effect). On the question of treatment of outlying observations, the author writes (p. 210): "It is entirely in accordance with the method of least squares to reject discordant observations, and this has always been done, even by those who object to an exact criterion for determining what observations should be rejected. For example, Mr. Glaisher says that no observation should be rejected excepting obvious mistakes, thereby admitting that it is proper sometimes to reject observations, and nobody is more opposed to the rejection of observations than he. But no line of demarkation can be drawn between mistakes which are obvious and mistakes which are not obvious. In some cases it may be obvious that 53 has been written by the recorder instead of 35. In other cases it may be doubtful whether it ought to be called an obvious mistake or whether there may be some doubt hanging over it, and there is every grade of probability, from the greatest to the least; and when we examine into the facts of observation, and do not attempt to make our way through a vacuous space of pure theory, it will be found that the occasional rejection of observations is justified from every point of view; and if observations are to be rejected, exact criteria are necessary to determine

upon principles of probabilities in what cases they should be rejected. The criterion of Professor Peirce [father of the author] is the only one which conforms rigidly to those principles. . . ."

**References:** Encke (1832-34), B. Peirce (1852), Chauvenet (1863), Glaisher (1872?).

**Citations:** Gore (1889), Pizzetti (1892), Czuber (1899), Wilson & Hilferty (1929).

* Stone, E. J. (1873a). On the most probable result which can be derived from a number of direct determinations of assumed equal value. *Monthly Notices of the Royal Astronomical Society* **33**, 570-572. (JFM 5, 122—listing only).

**Summary:** The author assumes as an axiom that since all the direct measures are by assumption of equal value, or equally good, the most probable value which can be adopted is that to which each individual measure contributes equally. He states that this condition of equal contribution of the independent measures to the most probable result appears to him necessary and sufficient. Let $u = \phi(x_1, x_2, \ldots, x_n)$ be the value adopted as the most probable. The author uses partial differential coefficients to prove that u must be a function of the arithmetic mean. When there are only two independent measures, $x_1$ and $x_2$, the most probable result is known to be $(x_1+x_2)/2$ since there is no reason why we should adopt a value nearer to $x_1$ than to $x_2$ (or vice versa). The author uses mathematical induction to show that if it is true that the arithmetic mean is the most probable result for n observations, it is true also for n+1, and since it is true for n=2, it must be true in general. He closes with the following statements (p. 572): "That is, the most probable result which can be deduced from the n independent direct measures $x_1\, x_2 \ldots x_n$ is the arithmetic mean provided *we assume that each of these measures is equally probable*. If we give up or change that assumption then the proposition is no longer necessarily true. The [normal] law of frequency can of course be at once deduced from the above result."

**Comments:** This paper is not directly relevant to a study of order statistics, but it is included here because of its bearing on the choice of averages and the treatment of outlying observations.

**Citations:** Glaisher (1874), Merriman (1877), Gore (1889), Czuber (1891a), Pizzetti (1892), Czuber (1899), Contarino (1914).

* Stone, E. J. (1873b). On the rejection of discordant observations. *Monthly Notices of the Royal Astronomical Society* **34**, 9-15. (JFM 5, 122).

**Summary:** The author examines in detail the objections raised by Glaisher (1873) to the author's criterion [Stone (1868)] for the rejection of discordant observations, and maintains that the criterion is valid. He suggests that Glaisher's alternative is not really an alternative at all, since his own criterion is to be used to reject mistakes and Glaisher's alternative supposes that obvious mistakes have already been rejected. With regard to Glaisher's objection that the value of n, such that the observer may be expected to make one mistake in n observations, is in practice impossible to determine, he points out that his criterion is relatively insensitive [robust, as modern statisticians would say] to moderately large variations in n. Turning to Glaisher's development of his alternative, the author states that he regards it as mathematically unsound. Even if Glaisher's assumptions are granted, Stone insists that, since the measure of precision h is assumed to vary from observation to observation, the expression to be maximized is not $(h/\sqrt{\pi})^n e^{-h^2[(x_1-a)^2+\ldots+(x_n-a)^2]}(da)^n$, which assumes $h_1 = h_2 = \ldots = h_n = h$, but $u = \pi^{-n/2} h_1 h_2 \ldots h_n e^{-[h_1^2(x_1-a)^2+\ldots+h_n^2(x_n-a)^2]}(da)^n$, in which the h's are regarded as variables. He shows that this leads to the conclusion that the most probable true value a may be determined by solving the equation $1/(x_1-a) + 1/(x_2-a) + \ldots + 1/(x_n-a) = 0$, which, according to Stone, will always have one real root which will make u an absolute maximum.

**References:** Chauvenet (1863), Stone (1868), Glaisher (1873).

**Citations:** Glaisher (1874), Stone (1874), Merriman (1877), Wright (1884), Edgeworth (1887a), Pizzetti (1892), Czuber (1899), Wellisch (1909), Archibald (1925), Coolidge (1925), Rider (1933).

* Todhunter, Isaac (1873). *A History of the Mathematical Theories of Attraction and the Figure of the Earth, from the Time of Newton to that of Laplace*, Volumes I and II. Macmillan and Company, London. [Reprinted in two volumes bound as one by Dover Publications, Inc., New York, 1962]. (JFM 5, 52—listing only). [Review, *Bulletin des Sciences Mathématiques et Astronomiques* **6** (1874), 276-285. JFM 6, 41-42)].

**Summary:** This history includes accounts of several investigations in which methods based on order statistics are used to find the equation $y = a + bx$ of the best-fitting straight line involved in determination of the ellipticity of the earth from measurements of degrees of meridian and lengths of a seconds pendulum at widely separated points on the earth's surface. Foremost among these are the works of Boscovich (1755, 1757, 1760), the first in collaboration with Maire and the last appended to Boscovich's commentary on Stay's Latin poetry, and of Laplace (1786, 1793), repeated by Laplace (1799), Part I, Book III, ch. v, Secs. 39-40. Todhunter dismisses the paper by Boscovich (1757) as simply an abstract of part five of the book by Maire and Boscovich (1755) and the method based on the two criteria proposed in the paper, but not in the book, as "a curious method of treating discordant observations, so as to obtain the best result from them." (Vol. 1, p. 321). He is more complimentary on pp. 331-332, where he discusses the corresponding portion of the supplementary notes of Boscovich (1760): "Boscovich here explains a method of his own invention for combining discordant observations so as to evolve an advantageous result. . . . Boscovich's exposition of his method takes a geometrical form: it is simple, clear, and instructive." He continues (p. 332): "Boscovich exemplifies his method by applying it to the five arcs he had adopted . . . . In the French translation [1770] of Boscovich's former treatise [Maire & Boscovich (1755)], besides this example another is given, which involves *nine* measured arcs. . . . The map, notes, and index render the translation more useful than the original." The method of Laplace (1786) and Laplace (1799), Sec. 39 is based on minimizing the maximum error, while that of Laplace (1793) and Laplace (1799), Sec. 40 is essentially Boscovich's, with the geometric method of minimizing the sum of absolute deviations, subject to the restriction that the sums of the positive and negative deviations be equal, replaced by an analytic one. Todhunter writes (Vol. 2, p. 62): "The method of treating discordant observations which is explained in Sec. 39 [of Laplace (1799)] does not appear to have found much favour;" and (p. 206) "I presume that neither of the two methods which Laplace discusses would now be practically used in such calculations, but the method of least squares." Also included is an account of the work of Adrain (1818), comparing the results obtained by the method of least squares, which he apparently discovered independently of Legendre and Gauss [see Adrain (1808)], with those obtained by the method used first by Boscovich and later by Laplace.

**References:** Maire & Boscovich (1755), Boscovich (1757, 1760), Laplace (1786, 1793, 1799), Puissant (1805), Svanberg (1805), Delambre (1806-10), Adrain (1818), Todhunter (1865).

**Citation:** Gore (1889).

* Fechner, Gustav Theodor (1874). Ueber den Ausgangswerth der kleinsten Abweichungssumme, dessen Bestimmung, Verwendung und Verallgemeinerung. *Abhandlungen der Königliche Sächsische Gesellschaft der Wissenschaften zu Leipzig* **18** [*Math. Phys. Classe* **11**], 1-76. (JFM 9, 780—listing only); extract, *Annalen der Physik und Chemie* **Jubelband**, 66-81. (JFM 6, 729).

**Summary:** In Section 1, Fechner shows that, while the sum of squares of deviations is a minimum when taken from the "arithmetische Mittelwerth" (arithmetic mean) A, the sum of the absolute deviations is a minimum when taken from the "Centralwerth" (median) C. In Section 2 he explores the interest that attaches to knowledge of C as well as A; in Section 3, methods of estimating C and various relations

between C and A. In Section 4 he derives formulae for variation of the sum of deviations depending on a shift of the starting point and the resulting evidence for the potential quality of the median. In Section 5, he deals in general with "Potenzmittelwerthe" (power means), which he defines as values such that the sums of powers of deviations are minimal when taken from them. In Section 6 he makes some observations on the validity of the principle of the arithmetic mean. In Section 7 he considers the probability law of the deviations relative to various power means under the assumption of validity of their principles. In Section 8 he gives confirmation of the Gaussian law of error as applied to monthly variation of temperature and an empirical comparison of the precision of the three lowest power means. In Section 9 he explores the use of other types of means, which he calls combination means. Let a, b, c, and d be four observations. He considers the following combination means: $M_1 = \sqrt[1]{(a+b+c+d)/4}$ = arithmetic mean; $M_2 = \sqrt[2]{(ab+ac+ad+bc+bd+cd)/6}$; $M_3 = \sqrt[3]{(abc+abd+acd+bcd)/4}$; and $M_4 = \sqrt[4]{abcd}$ = geometric mean. He says he knows of no application for this type of mean, other than the arithmetic and geometric means.

**Comments:** Coolidge (1925), p. 113, cites this paper as the source of the theorem that the sum of the absolute deviations is a minimum when taken from the median. Fechner was not, however, the first to prove this fact [see Glaisher (1872)], but Fechner apparently did not know of Glaisher's proof, since he states (p. 4): "It is not known to me whether this question has already been proposed, since undoubtedly until now little interest has attached to it" [Compiler's translation]. The basic idea underlying the proof goes back to Laplace (1812—second supplement, 1818); see also von Andrae (1860).

**References:** Laplace (1812), Gauss (1823), Encke (1832-34).

**Citations:** Venn (1866) [1888], Helmert (1876a), Merriman (1877), Edgeworth (1886a, 1887a,d), Gore (1889), Czuber (1891a), Pizzetti (1892), Czuber (1899), Keynes (1911), Julin (1921), Goldhizer (1923), Rietz (1924), Coolidge (1925), Bruen (1938).

* Glaisher, J. W. L. (1874). Note on a paper by Mr. Stone, "On the rejection of discordant observations." *Monthly Notices of the Royal Astronomical Society* **34**, 251. (JFM 6, 144-145).

**Summary:** In this brief note, the author refers to a paper containing his opinions, which he characterizes as "necessarily somewhat personal", concerning the remarks of Stone (1873b), which he has decided to withdraw and "let the matter rest . . . without further comment", having stated what his belief is in his earlier paper [Glaisher (1873)]. He asks the reader "not to assume that I hold, because I offer no reply, all the opinions attributed to me by Mr. Stone . . ." and states "I never held that the h of every observation was *a priori* equally likely to have any value from 0 to $\infty$, as I took the usual result [all values of h equal] as a first approximation." He concludes by asking anyone wishing to form an opinion on the subject in question to read also another paper by Stone (1873a).

**References:** Glaisher (1873), Stone (1873a,b).

**Citations:** Stone (1874), Merriman (1877), Archibald (1925), Coolidge (1925), Rider (1933).

* Jevons, W. Stanley (1874). *The Principles of Science*. Macmillan and Co., London-New York. Second edition, 1877; other editions, 1887, 1924, 1958.

**Summary:** Chapter XVI (pp. 357-373 of the second edition) deals with the method of means and Chapter XVII (pp. 374-398) with the law of error. In the former, the author notes that Boethius stated ten kinds of means or averages and Jordanus added an eleventh, but he discusses only three of them (arithmetic, geometric, and harmonic means). He notes that Quetelet (1846) distinguished between means of observations of a quantity which is itself constant, but subject to variable measurement error, and of a quantity which is itself variable, and that Herschel (1850) pointed out the importance of this distinction.

In Chapter XVII the author gives demonstrations of the normal law of error by Gauss, by Herschel, and by Laplace and Quetelet. He states that there is no justification for believing that the errors committed in all classes of observations should follow the same law, but does not suggest any other. He also discusses in some detail the method of least squares and the rejection of outlying observations. He takes a position against arbitrary exclusion of discrepant observations, in the following words (p. 393): "The mere fact of divergence ought not to be taken as conclusive against a result, and the exertion of arbitrary choice would open the way to the fatal influence of bias. . . . The apparently divergent number may prove in time to be the true one. . . . To neglect a divergent result is to neglect the possible clue to a great discovery."

**References:** Cotes (1722), Legendre (1805), Gauss (1809), Laplace (1812), Encke (1832-34), Quetelet (1846) [1849], DeMorgan (1847), Herschel (1850) [1857], Gould (1855), Airy (1861), Chauvenet (1863), DeMorgan (1864), Todhunter (1865), Venn (1866) {[1876]}, Whitworth (1867), Galton (1869), Glaisher (1872).

**Citations:** Pizzetti (1892), Czuber (1899), Secrist (1917), Julin (1921), Keynes (1921), Rietz (1924), Shewhart (1931).

* Stone, E. J. (1874). Note on a discussion relating to the rejection of discordant observations. *Monthly Notices of the Royal Astronomical Society* **35**, 107-108. (JFM 6, 145).

**Summary:** The author asks anyone interested in the discussion to read, in order of publication, his original paper [Stone (1868)], the "animadversions" upon it in the paper by Glaisher (1873), and his reply [Stone (1873b)]. In reply to the statement by Glaisher (1874) that "I never held that the h of every observation was *a priori* equally likely to have any value from 0 to $\infty$, as I took the usual result as a first approximation", the author quotes three extracts from the earlier paper by Glaisher (1873) to show that he has not misrepresented Glaisher's position. He closes with the following statements (p. 108): "It is true that Mr. Glaisher attempted to solve the question by successive approximations, but if the process had been properly conducted, the results obtained must have finally tended, without limit, to those obtained by any correct direct process. I cannot, therefore, see the force of the remark, that the arithmetical mean was taken as a first approximation. It might shorten or lengthen the work according as the arithmetical mean was, or was not, a close approximation to the final result. More than this it could not do. At least such is my opinion, and will, I venture to believe, be the opinion of mathematicians generally."

**Comments:** This is the last paper in an exchange between Stone and Glaisher which, at least in its later stages, seems to have generated more heat than light.

**References:** Stone (1868), Glaisher (1873), Stone (1873b), Glaisher (1874).

**Citations:** Merriman (1877), Pizzetti (1892), Czuber (1899), Archibald (1925), Coolidge (1925), Rider (1933).

* Galton, Francis (1875). Statistics by intercomparison, with remarks on the law of frequency of error. *Philosophical Magazine* (4) **49**, 33-46.

**Summary:** The author proposes the use of the median as a measure of central tendency and of the difference between the median and one of the quartiles, or the average distance between the median and the two quartiles, as a measure of dispersion (probable error). He proposes that the median and the quartiles be determined by intercomparison of objects or physical attributes without actually measuring each. For example, if one wishes to determine the median and the quartiles of the heights of a group of men, it suffices to line them up in order of height and count 1/4, 1/2, and 3/4 of the total frequency down the line, then measure the heights of only the three individuals occupying the median and quartile posi-

tions. The remainder of the paper deals with the composition of errors from a number of sources with effects of varying magnitude, and is not relevant to a study of order statistics.

**Reference:** Galton (1869).

**Citations:** Venn (1866) [1888], Merriman (1877), Edgeworth (1885, 1886a, 1887a,d), Pizzetti (1892), Edgeworth (1911), Gini (1912), Julin (1921), Rietz (1924), Gini & Galvani (1929), Gini (1939), Gumbel (1943c).

* Helmert, F. R. (1875a). Ueber die Formeln für den Durchschnittsfehler. *Astronomische Nachrichten* **85**, 353-366. (JFM 7, 113).

**Summary:** This paper is concerned primarily with the calculation of the probable error, the mean (absolute) deviation, and the standard deviation in the conventional manner (from deviations from the mean), and only incidentally from differences among the observations themselves. The last section, however, deals with an apparent paradox concerning the mean deviation and the standard deviation, calculated in the latter way from a sample of size n, which are equal for $n=2$ but not otherwise, the standard deviation being more than 25% larger than the mean deviation for sufficiently large samples [from a Gaussian distribution].

**Citations:** Helmert (1876a), Merriman (1877), Gore (1889), Pizzetti (1892), Czuber (1899).

* Helmert, F. R. (1875b). Ueber die Berechnung des wahrscheinlichen Fehlers aus einer endlichen Anzahl wahrer Beobachtungsfehler. *Zeitschrift für Mathematik und Physik* **20**, 300-303. (JFM 7, 113).

**Summary:** The author rejects the objections of Mees (1875) to the validity of the proof [Gauss (1816, Helmert (1872)], based on the criterion of minimizing the mean square deviation, that the probable error of n observations can be determined more accurately from the mean of the squares of the errors than from the mean of the absolute values of the $m^{th}$ powers of the errors, where m is an integer not equal to 2. He points out an error in Mees' study based on the criterion of minimizing the mean cube of the absolute deviation.

**References:** Gauss (1816), Helmert (1872), Mees (1875).

**Citations:** Helmert (1876b), Mees (1876), Merriman (1877), Gore (1889), Pizzetti (1892), Kendall (1946).

* Laurent, H. (1875). Sur la méthode des moindres carrés. *Journal de Mathématiques Pures et Appliquées* (3) **1**, 75-80. (JFM 7, 112).

**Summary:** The author compares two approaches to justification of the method of least squares. He states that Laplace and his students justified its use, for large numbers of observations, whatever the law of error, provided only that the probability of an error of magnitude $\epsilon$ tends to zero as $\epsilon \to \infty$, whereas Gauss and the Germans assumed *a priori* that the law of error is $(h/\sqrt{\pi})e^{-h^2\epsilon^2}$ and justified this by assuming that the arithmetic mean of several inexact measurements is the most probable value of the quantity measured. Laurent says that one can never justify such a hypothesis *a priori*; on the contrary, one ought to reject it, because it assigns positive probabilities to impossibly large errors. "Who is the astronomer", he inquires, "who commits an error of 361 degrees in measuring an angle?" (p. 76). The author examines 444 measurements of an angle of approximately 16°, and concludes that the observations cast doubt on the exactness of the Gaussian law, and that therefore one ought to reject the method of least squares when one has only a small number of observations.

**Comments:** This paper, while making no direct mention of methods based on order statistics, is indirectly relevant because rejection of the method of least squares implies adoption of one of its rivals, several of which are based on order statistics.

**References:** [Gauss (1809), Laplace (1812)].

**Citations:** Merriman (1877), Gore (1889), Pizzetti (1892), Czuber (1899).

* Lexis, W. (1875). *Einleitung in die Theorie der Bevölkerungsstatistik*. Karl J. Trübner, Strassburg.

**Summary:** The fifth and last chapter (pp. 93-124) is the only one containing anything relevant to our study. Consider L persons of a given age alive at a particular time. Suppose m of them die within one year, so that L−m are still alive after one year. Let p = m/L and q = 1−p = (L−m)/L. The author shows that for L sufficiently large so that one can use normal tables, the ratio m/L differs from the true probability R that a person of the given age will die within one year by less than $3\sqrt{2pq/L}$ with probability 0.999978. He states (p. 100): "One can designate the positive and negative deviations admissible with this probability as the 'extreme' deviations." He also gives the corresponding expression $3\sqrt{2pq/L+2p'q''/L'}$ for the extreme deviation of the difference of two such observed ratios from the true difference, and gives numerical examples of the use of both expressions. He replaces the factor 3 in both expressions by 0.476963 to obtain the probable deviations.

**References:** Cournot (1843) [1849], Quetelet (1846).

**Citations:** Czuber (1899), Keynes (1921), von Bortkiewicz (1922b).

* Mees, R. A. (1875). Ueber die Berechnung des wahrscheinlichen Fehlers einer endlichen Zahl von Beobachtungen. *Zeitschrift für Mathematik und Physik* **20**, 145-152. (JFM 7, 112-113).

**Summary:** The author questions the validity of the proof advanced by Gauss (1816) and several later authors, including Helmert (1872), of the proposition that the probable error of a number of observations can be determined more precisely from the mean of the squares of the errors than from the mean of the absolute values of the $m^{th}$ powers of the errors, where m is any integer different from 2. In particular, he asserts that it is begging the question to use the mean square deviation as a criterion, and claims to show that if one uses instead the mean cube of the absolute deviation, one reaches the conclusion that the result is more precise for m=1 (mean of the absolute first powers of the errors) than for m=2.

**Comments:** This paper, along with the rejoinder by Helmert (1875b) and the reply by Mees (1876), while not directly relevant to a study of order statistics, is included because of its bearing on the use of the mean deviation, which is a minimum [see Glaisher (1872) and Fechner (1874)] when taken from the median rather than from the arithmetic mean.

**References:** Gauss (1816), Helmert (1872).

**Citations:** Helmert (1875b, 1876b), Mees (1876), Merriman (1877), Gore (1889), Pizzetti (1892), Czuber (1899).

* Bréger, P. (1876). Memoire sur la probabilité d'atteindre un but de forme quelconque. *Revue Maritime et Coloniale* (fourth quarter, volume and page numbers unknown); partial English translation by C. A. Stone, U. S. Government Printing Office, Washington, 1883.

**Summary:** Delauney (1888) states that the author proves that if successive errors in shooting are grouped two by two, in the order in which they occur, the mean of the larger errors in each pair will be in the

ratio $\sqrt{2}$:1 to the general mean. This theorem is also stated by Bertrand (1887a) without any mention of the source.

**Note:** The theorem cited by Delauney is not found in Stone's translation, which is the only form of this memoir the compiler has seen.

**Citation:** Delauney (1888).

* Helmert, F. R. (1876a). Die Genauigkeit der Formel von Peters zur Berechnung des wahrscheinlichen Fehlers directer Beobachtungen gleicher Genauigkeit. *Astronomische Nachrichten* **88**, 113-132. (JFM 8, 114-115).

**Summary:** The author discusses several methods of calculating the probable error of a number of equally precise direct observations. He simplifies the formula given in his earlier paper [Helmert (1875a)] and compares it with one given by Fechner (1874). He then compares these two formulas based on the mean absolute deviation of the observations from their arithmetic mean with one developed by Jordan (1869, 1872) and by von Andrae (1869, 1872) based on the mean absolute difference of the observations from each other, calculation of which is facilitated [see von Andrae (1872)] by ordering the observations.

**References:** Jordan (1869), von Andrae (1872), Fechner (1874), Helmert (1875a).

**Citations:** Merriman (1877), Gore (1889), Czuber (1891a), Pizzetti (1892), Czuber (1899, 1903), Wellisch (1909), Gini (1912), de Finetti (1931a), Davies & Pearson (1934), Gini (1939), Pearson (1939), Simon (1941), Kendall (1943, 1946), Nair (1947).

* Helmert, F. R. (1876b). Ueber die Wahrscheinlichkeit der Potenzsummen der Beobachtungsfehler und über einige damit im Zusammenhange stehende Fragen. *Zeitschrift für Mathematik und Physik* **21**, 192-218. (JFM 8, 113).

**Summary:** For n errors of observation from (a) a uniform distribution or (b) a normal distribution, the author determines the probability that the sum of the absolute values of their $m^{th}$ powers lies in an interval $(x, x+\triangle x)$, and goes on to apply the result to the determination of the precision of the probable error derived from the sum of the (absolute) $m^{th}$ powers. He computes numerical results for small values of m and n and gives mathematical expressions for large values of the sample size n. At the end he adds a remark on the answer by Mees (1876) to his rejoinder [Helmert (1875b)] to an earlier paper by Mees (1875).

**References:** Gauss (1816), Helmert (1872, 1875b), Mees (1875, 1876).

**Citations:** Merriman (1877), Gore (1889), Pizzetti (1892), Czuber (1899, 1903), Gini (1912), Craig (1932a), Münzner (1934), Cramér (1946), Kendall (1946).

* Mees, R. A. (1876). Ueber die Berechnung des wahrscheinlichen Fehlers aus einer endlichen Zahl von Beobachtungen. *Zeitschrift für Mathematik und Physik* **21**, 126-128. (JFM 8, 113).

**Summary:** The author admits the error in his earlier paper [Mees (1875)] pointed out by Helmert (1875b), but remains unconvinced that the proof of Gauss (1816) and Helmert (1872) is valid.

**References:** Gauss (1816), Helmert (1872, 1875b), Mees (1875).

**Citations:** Helmert (1876b), Gore (1889), Pizzetti (1892).

* Helmert, F. R. (1877a). Die Bestimmung des Fehlergesetzes aus Beobachtungen auf graphischem Wege. *Zeitschrift für Vermessungswesen* **6**, 22-26.

**Summary:** This paper deals with graphical estimation of the error law from observations. The author starts with 51 triangulation errors and plots the observed frequency (both cumulative and non-cumulative). He estimates the standard deviation $\sigma$ of the population as the root-mean-square error of the observations, and plots a normal probability density function and a normal cumulative distribution with this value of $\sigma$ on the same axes as the observed p.d.f. and c.d.f., respectively, for purposes of comparison. The abscissas of the points where the observed c.d.f. has jumps, each of magnitude 1/51, are, of course, the order statistics of the sample of 51 observations.

**Citations:** Gore (1889), Czuber (1899).

* Helmert, F. R. (1877b). Ueber den Maximalfehler einer Beobachtung. *Zeitschrift für Vermessungswesen* **6**, 131-147.

**Summary:** The author disagrees strongly with the conclusions reached by Jordan (1877a). He points out that the new law proposed by Jordan is based on the first three terms of a series which approximates the Gaussian law more and more closely as the number of terms n increases: for $n=1$, $M=1.7321$ m [uniform, $\phi(M) \neq 0$]; for $n=2$, $M=2.2361$ m [parabolic, $\phi(M)=0$, $\phi'(M)\neq 0$]; for $n=3$, $M=2.6457$m [$\phi(M)=\phi'(M)=0$, $\phi''(M)\neq 0$]; for $n=4$, $M=3.000$m [$\phi(M)=\phi'(M)=\phi''(M)=0$, $\phi'''(M)\neq 0$]. He suggests an alternate system based on the composition of n independent elementary errors each uniformly distributed in the interval $(-a, +a)$, an idea which he attributes to Zachariae (1871), for which the p.d.f., for $n \geq 2$, consists of n curves, each of degree $n-1$, having contact of order $n-2$ with each other and with the axis of abscissas. In this system: for $n=1$, $M=1.7321$m; for $n=2$, $M=2.2495$m; for $n=3$, $M=3.0000$m; for $n=4$, $M=3.4641$m; as $n \to \infty$, $M \to \infty$ m, and the curve approaches the Gaussian curve, the ratio M/m increasing as the square root of n. Last but not least, the author points out that the maximum error M to be expected in a series of N observations is an increasing function of N; for $N=100$, Jordan's assertion that $M<3m$ may be quite reasonable, but for $N=100{,}000$ it is ridiculous. The author submits that, over a wide range of values of N, $2m<M<5m$, and as a corollary that the maximum difference of a pair of independent observations, given N such pairs, lies between 3m and 7m for a wide range of values of N.

**References:** Zachariae (1871), Helmert (1872), Jordan (1877a).

**Citations:** Helmert (1872) [1907], Jordan et al. (1879), Gore (1889), Czuber (1891a), Pizzetti (1892), Czuber (1899), Vogeler (1907), Wellisch (1909), von Bortkiewicz (1922b), von Eberhard (1938).

* Jordan, W. (1877a). Ueber den Maximalfehler einer Beobachtung. *Zeitschrift für Vermessungswesen* **6**, 35-40.

**Summary:** The author points out that the Gaussian law of error sets no upper limit on the magnitude of the error, although very large errors have very small probabilities of occurrence. He contends that an error of 1° in the measurement of an angle with a good theodolite has, without any question, probability equal to zero, and not a value which differs from zero. He proposes another law, $\phi(\Delta) = A+B\Delta^2+C\Delta^4$, excluding terms of odd degree because of symmetry. By imposing the conditions that the error curve be tangent to the axis of abscissas at $\Delta=M$ and that the area under it be 1, i.e. $\Delta(M) = 0, \Delta'(M) = 0$, $2\int_0^M \phi(\Delta)d\Delta = 1$, he finds $A=15/16M$, $B=-15/8M^3$, and $C=15/16M^5$. For this error law, he finds the mean (absolute) error to be $t=0.3125M$, the root-mean-square error to be $m=0.37797M$ [hence $M=2.6457m$], and the probable error to be $r=0.28108M$. He compares the fit of this new law and the Gaussian law (with $M=\infty m$) to 470 astronomical observations, and concludes that the Gaussian law fits better. Nevertheless, he insists that 3m should be taken as the upper limit of error, and suggests that observed errors greater than this may actually be gross blunders which should be rejected.

**Reference:** Encke (1832-34).

**Citations:** Helmert (1877b), Jordan (1877b), Czuber (1891a, 1899), Wellisch (1909), von Bortkiewicz (1922b).

* Jordan, W. (1877b). *Handbuch der Vermessungskunde*, Vol. I. Metzler, Stuttgart. (JFM 9, 771-773); third edition, 1888. (JFM 20, 1227-1228); fourth edition, 1895. (JFM 25, 1832; 26, 1075-1076); fifth edition, 1904. (JFM 35, 948-949); sixth edition, 1910. (JFM 41, 1008).

**Summary:** This first volume of the author's handbook of surveying deals primarily with the adjustment computation by the method of least squares. The author repeats his conclusion [Jordan (1877a)] that for good measurements the ratio M/m of the maximum error to the root-mean-square error hardly ever reaches the value 3. In the fourth and subsequent editions, the author, in Section 146 (Theory of the Maximum Error) replaces the normal law and the algebraic law which he used earlier [Jordan (1877a)] by the algebraic law

$$y_n = \phi(\epsilon)_n = \frac{1}{2} \quad \frac{3\cdot5\cdot7\ldots(2n+1)(2n+3)}{2\cdot4\cdot6\ldots2n(2n+2)} \cdot \frac{1}{M} \cdot (1-\epsilon^2/M^2)^{n+1},$$

where M is the maximum error. This approximation to the error law is supposed to improve as n increases. In Section 147 (Computation of the Maximum Error), he estimates the maximum error M from the data $m^2/M^2 = (3m^4-\nu^4)/2\nu^4$, where $m^2$ and $\nu^4$ are the sample second and fourth moments, given by $m^2 = \Sigma x^2/(N-1)$ and $\nu^4 = N\,\Sigma x^4/(N-1)^2$, where the x's are deviations from the sample mean. If $3m^4 - \nu^4$ is negative, which would make M imaginary, he takes this as evidence of a gross error. The conclusion is essentially that reached by the author in 1877 and stated in the first edition and his paper of that year [Jordan (1877a)]: "Our few experiments of this sort point clearly to the conclusion that the ratio M/m for good measurements hardly ever reaches the value 3, and usually lies between 2 and 3." [p. 574, fourth edition (1895), compiler's translation].

**References:** Encke (1832-34), Jordan (1877a).

**Citations:** Helmert (1872) [1907], Gore (1889), Jordan (1890), Czuber (1891a), Pizzetti (1892), Blümcke (1897, 1898), Czuber (1899), Blümcke (1901), Mitscherlich (1903), Kozák (1907), Vogeler (1907), Kozák (1908-10), Brunt (1917), von Bortkiewicz (1922b).

* Merriman, Mansfield (1877). A list of writings relating to the method of least squares, with historical and critical notes. *Transactions of the Connecticut Academy of Arts and Sciences* **4** (1), 151-232. (JFM 9, 154).

**Summary:** This is a chronological bibliography containing 408 titles (and covering the period 1722-1876) on the method of least squares and the theory of errors.

**Comments:** Since the author includes papers on rival methods, several of which are based on order statistics, the compiler has found this to be a most useful source of information.

**References:** Cotes (1722), Euler (1749), Mayer (1750), [Maire &] Boscovich (1755), Simpson (1756, 1757), Boscovich (1760), Lambert (1760, 1765b), Lagrange (1774), Laplace (1774), D. Bernoulli (1778), Euler (1778), Laplace (1781), J. Bernoulli (1785), Laplace (1793, 1799), Prony (1804), Trembley (1804), Legendre (1805), Puissant (1805), von Zach (1805), von Lindenau (1806), Adrain (1808), Gauss (1809), Laplace (1810, 1811a,b, 1812), Legendre (1814), van Beeck Calkoen (1816), Gauss (1816), Adrain (1818), Anonymous (1821), Gauss (1823), Cauchy (1824), Fourier (1824a), Ivory (1825), Muncke (1825), Hauber (1830, 1830-32), von Riese (1830), Cauchy (1831), Encke (1832-34), Cauchy (1837), Hagen (1837), Bessel

& Baeyer (1838), Gerling (1843), Ellis (1844), Quetelet (1846), DeMorgan (1847), Herschel (1850), Peirce (1852), Gould (1855), Lloyd (1855), Airy (1856), Winlock (1856), Petzval (1857), Airy (1861), Chauvenet (1863), DeMorgan (1864), Todhunter (1865), Stone (1868), von Andrae (1869), Jordan (1869), Zachariae (1871), von Andrae (1872), Glaisher (1872), Helmert (1872), Jordan (1872), Glaisher (1873), Stone (1873a,b), Fechner (1874), Glaisher (1874), Stone (1874), Galton (1875), Helmert (1875a,b), Laurent (1875), Mees (1875), Helmert (1876a,b).

**Citations:** Venn (1866) [1888], Edgeworth (1887b), Gore (1889), Pizzetti (1892), Czuber (1899), Edgeworth (1911), Keynes (1921).

* Wilson, J. M. (1877). Note on a special case of "the most probable result" of a number of observations. *Monthly Notices of the Royal Astronomical Society* **38**, 81-82. (JFM 9, 156).

**Summary:** Suppose that the weight of an observation varies inversely as its deviation from the final result. Arrange the observations in order of magnitude. If the number of them is odd the middle observation must be taken as the most probable result; if the number is even, any number between the two middle observations may be taken. In other words, the most probable result is the median of the observations.

**Reference:** Glaisher (1872).

**Citations:** Edgeworth (1887a), Pizzetti (1892).

* Herschel, Clemens (1878). The gauging of streams. *Transactions of the American Society of Civil Engineers* **7**, 236-242.

**Summary:** In connection with a study of the discharge of the Merrimack and Connecticut Rivers, among others, for the various months of the years 1871-1874, the author proposes the use of what hydrologists call duration curves and statisticians call cumulative distribution curves or ogives. On page 238, he writes: " . . . the proper way to arrange the months of the year for the purposes of hydraulic engineering is to arrange them, not by their names, not by their division into seasons, but by this very quality that we are investigating, their wetness and dryness. Let us call the driest month number one; the next driest, be its name what it may, number two, and the wettest will be number twelve. In that way we may get a diagram which will convey a great deal of meaning to the engineer. . . . The diagram indicates in one direction 'cubic feet per second per square mile', and in the other direction the relative discharge of the different months in each year. Each year occupies twelve of the horizontal divisions, 'one' being the driest, 'two' the next driest, 'three' the next, until 'twelve' is the wettest month of the year. The names of the months are given at each plotted point, so that it may be seen how the wetness and dryness is distributed by months for many years." He states that this method was first suggested to him by Mr. Joseph P. Davis, City Engineer of Boston.

**Citation:** Foster (1934).

* Peirce, Benjamin (1878). On Peirce's criterion. *Proceedings of the American Academy of Arts and Sciences* **13** (N. S. **5**), 348-349.

**Summary:** In this paper, in the form of a letter to C. P. Patterson, Superintendent of the U. S. Coast Survey, Peirce gives a fuller explanation of the criterion which he proposed over a quarter of a century earlier [Peirce (1852)]: "I perceive that the theory of my criterion has been frequently misunderstood. I presume that to be due in a great degree to the conciseness of the argument with which it was published; and I propose to remedy this defect. The problem which I undertook to solve was the following. There being given certain observations, of which the greater portion is to be regarded as normal and subject

to the ordinary law of error adopted in the method of least squares, while a smaller unknown portion is abnormal and subject to some obscure source of error, to ascertain the most probable hypothesis as to the partition of the observations into normal and abnormal. The principle adopted in my solution of the problem is the universally recognized doctrine that the measure of the probability of an hypothesis compared with other hypotheses equally probable in other respects is the probability that the event will occur under the hypothesis, and that the most probable hypothesis is that under which the event is the most probable. This is the literal expression of the mathematical analysis published in Gould's Astronomical Journal for 1852. . . . The evidence, by which certain observations are placed in the doubtful list and subjected to scrutiny, whether they should be rejected, must be exclusively the magnitude of the errors which they involve, when these errors are computed as if they were normal observations." [p. 348]. "That some of the observations which are not rejected may be abnormal, notwithstanding the smallness of their errors, must be admitted. This possibility was fully recognized in the geometrical development which was given in the Astronomical Journal; and I am not aware that there has been any criticism adverse to the mathematics of that article." [p. 349].

**References:** Peirce (1852), Gould (1855), Schott (1878).

**Citations:** Archibald (1925), Rider (1933).

* Schott, Charles A. (1878). On Peirce's criterion. *Proceedings of the American Academy of Arts and Sciences* **13** (N. S. **5**), 350-351.

**Summary:** In this paper, in the form of a letter to Carlile P. Patterson, Superindendent of the U. S. Coast Survey, the author makes the following comments concerning the criterion introduced by Peirce (1852): "Large errors may arise either from an *accumulation* of a number of smaller ones, having their origin in different sources, as is recognized in the theory of combination of errors, and as such they may be regarded as *normal* (even if quite large), provided they belong to a series of an indefinitely large number of observations. Practically, we have but a few observations (most frequently less than one hundred), and while certain sources of error may combine to the production of large ones, their *actual* appearance in a *short* series of observations must injuriously affect the most probable result (say the mean) deducible from the series. Here we need the means of separation, and the criterion logically performs this. Or large errors may arise from bad observations (due to inattention of observer, without being aware of it), from the presence in this particular case of an unsuspected constant error, or even from an accidental slip (necessarily not altogether outside the possibility of its being due to other causes admissible): all such large errors having no recognized place in the adopted law of the occurrence of error must be subject to rejection, for which we need the criterion. . . . Having used the criterion for the last twenty years in various investigations, I found it uniformly gave excellent discrimination, and do not remember a single case where it came in conflict with proper judgment based upon experience." [p. 350].

**Reference:** Peirce (1852).

**Citations:** Peirce (1878), Doolittle (1884), Rider (1933).

Jordan, W. et al. (1879). Bericht der Commission für geometrische Genauigkeitsbestimmungen. *Zeitschrift für Vermessungswesen* **8**, 352-374.

**Summary:** A commission headed by Jordan reports on its investigation of two questions: (1) the relation of the maximum error to the standard error, the average error, and the probable error; and (2) the error of measurement in relation to the magnitude being measured. The second question is not relevant to our subject. With regard to the first, the commission reports that (for a normal distribution) the standard error (mittlerer Fehler) m is equal to 1.253 times the average error (durchschnittlicher Fehler) or 1.483 times the probable error (wahrscheinlicher Fehler), but that no such exact relation

can be given between the maximum error (Maximalfehler) or limiting error (Grenzfehler) M and the others. They point out that M exceeds 2.0m 46 times in 1000, 2.5m 12 times in 1000, 3.0m 3 times in 1000, and 3.5m 1 time in 1000. They take as a rule of thumb: Limiting error (Grenzfehler) = 3 times standard error (mittlerer Fehler). For the standard error of a difference ($\sqrt{2}$ times the standard error of a single observation), they take: Admissible error of difference (Zulässige Messungsdifferenz) = 4 times standard error (mittlerer Fehler).

**References:** Helmert (1877b).

**Citation:** Vogeler (1907).

* Lalanne, L. (1879). De l'emploi de la Géométrie pour resoudre certains questions de moyennes et de probabilité. *Journal de Mathématiques Pures et Appliquées* (3) **5**, 107-130. (JFM 11, 164-165); abstract, *Comptes Rendus de l'Académie des Sciences de Paris* **87** (1878), 355-358. (JFM 10, 172).

**Summary:** The author uses geometry to solve the following problems: (1) Among the infinite number of possible triangles of which the sides are subject only to the condition of being included between the limits a and b, what are the mean values of the three sides, previously arranged in order of magnitude? Solution: Let x, y, and z be the lengths of the three sides in increasing order of magnitude. They must satisfy the following inequalities: $x \geq a, z \leq b, x \leq y, y \leq z, z \leq x+y$. The first four of these relations, if one retains only the equality signs, are the equations of four planes which form a tetrahedron. The inequalities restrict the point (x,y,z) to the interior of this tetrahedron. If $a > b/2$, the plane $z = x+y$ does not intersect this tetrahedron, and hence the solution is given by the coordinates of the latter's center of gravity: $\bar{x}_1 = (1/4)(3a+b)$, $\bar{y}_1 = (1/2)(a+b)$, $\bar{z}_1 = (1/4)(a+3b)$. If $a < b/2$, the point (x,y,z) is restricted to the interior of the pentahedron bounded by the planes $x = a$, $z = b$, $x = y$, $y = z$, and $z = x+y$, and the solution is given by the coordinates of the center of gravity of this pentahedron, $\bar{x}_2 = [4(3a+b)(b-a)^3 - (6a+b)(b-2a)^3]/D$, $\bar{y}_2 = [8(a+b)(b-a)^3 - 2(2a+3b)(b-2a)^3]/D$, $\bar{z}_2 = [4(a+3b)(b-a)^3 - 2(2a+3b)(b-2a)^3]/D$, where $D = 16(b-a)^3 - 8(b-2a)^3$. If $a = b/2$, both solutions reduce to $\bar{x} = 5b/8$, $\bar{y} = 3b/4$, $\bar{z} = 7b/8$. (2) A bar (of length $\ell$) is broken into three pieces. What is the probability that, from the three pieces, a triangle can be formed? Solution: Let x, y, and z be the lengths of the three pieces. The locus of points which satisfies the fundamental relation $x+y+z = \ell$ is a triangle [in the first octant] of which the vertices are situated at the distance $\ell$ from the origin on the three coordinate axes. But in order for the triangle to be possible, it is necessary that one have simultaneously $x \leq y+z, y \leq z+x, z \leq x+y$. The three planes determined by the equations of this group are respectively perpendicular to each of the three coordinate axes at a distance from the origin equal to $\ell/2$. Their intersections with the triangle $x+y+z = \ell$ determine a new triangle whose vertices are the midpoints of the sides of the first triangle. Since the area of the second triangle is 1/4 that of the first, the required probability is 1/4.

**Comments:** The first of these two problems is obviously a problem in order statistics, and it has been pointed out in the comments on the paper by Lemoine (1873), who stated the second problem and solved it analytically, that the latter can be restated in terms of order statistics.

**Reference:** Lemoine (1873).

**Citations:** Czuber (1899), Wölffing (1899).

* Chaplin, W. S. (1880). The relation between the tensile strengths of long and short bars. *Van Nostrand's Engineering Magazine* **23**, 441-444.

**Summary:** The author begins by presenting theoretical arguments and experimental data to show that it is reasonable to believe that variations in the strength of a material will follow the normal (Gaussian) law. He then states that, according to the laws of probability, if the probability is p that the tensile

strength of a bar of given length and cross section exceeds a certain value, the probability that the strength of all of n bars of the same dimensions exceeds that value is $p^n$. Equivalently, the probability that the strength of the weakest of the n bars (or the strength of a single bar of the same cross section but n times as long) exceeds the specified value is $p^n$. Hence one should expect a decrease in the median (or mean) strength of a bar with an increase in length. The author uses a table of areas under the normal probability curve [Chauvenet (1863)] to tabulate the expected decrease (in terms of the probable error) as a function of n, the factor by which the length is multiplied, for n = 1(1)28. He also presents the same results graphically.

**Reference:** Chauvenet (1863).

**Citations:** Chaplin (1882), Slocum & Hancock (1906).

* Bréger, P. (1881). Sur les différences successives des observations. *Comptes Rendus de l'Académie des Sciences* (Paris) **93**, 1119-1121.

**Summary:** Let $x_1, x_2, x_3, \ldots, x_n$ be the successive values of n observations of the same phenemenon and let $\epsilon_1, \epsilon_2, \epsilon_3, \ldots, \epsilon_n$ be their respective errors. The mean square error $\Gamma^2$ and the mean error $\gamma$ are defined by $\Gamma^2 = \Sigma(\epsilon^2)/n$ and $\gamma = \Sigma|\epsilon|/n$. The author defines the mean square successive difference $D^2$ and the mean successive difference d by $D^2 = \Sigma(x_i - x_{i+1})^2/n$ and $d = \Sigma|x_i - x_{i+1}|/n$ where, by definition, $x_{n+1} = x_i$. He shows that, as n increases, the ratios $D/\Gamma$ (for any probability law) and $d/\gamma$ (for the Gaussian law), both converge to $\sqrt{2}$, the former more rapidly than the latter. He claims that, if one assumes a Gaussian distribution, methods of computing the precision constant $h = 1/\Gamma\sqrt{2}$ based on the successive differences are more exact than those based on deviations from the mean, and should be used in preference to them. He gives an example of the application of these methods to ballistic data.

**Note:** Successive differences are differences of successive observations in the order in which they occur, not in order of magnitude, and hence are not directly relevant to a study of order statistics. This paper is included because later authors, including Czuber (1891a, 1899, 1903 [1914]) and von Bortkiewicz (1931), have confused the mean successive difference with the mean difference discussed by Jordan (1869, 1872), von Andrae (1869, 1872) and Helmert (1876a) and revived by Gini (1912), with whose name it is usually associated.

**Citations:** Czuber (1891a), Pizzetti (1892), Czuber (1899, 1903), von Bortkiewicz (1931).

* Galton, Francis et al. (1881). Report of the Anthropometric Committee. *Reports of the British Association for the Advancement of Science* **51**, 225-272.

**Summary:** "The chairman of the Committee, Mr. Francis Galton, contributes to the Appendix of this Report a paper [pp. 245-260] on the range in height, weight, and strength of the different classes at every age. He measures the range, not between the maximum and minimum values recorded, which afford no safe basis for comparison, but through an extension of the principle by which the so-called 'probable error' is ascertained. Thus, he first arranges the cases in the order of their magnitude, then he cuts off a certain fractional portion of them from either end of the series, and measures the difference between the maximum and minimum of the intermediate group. The ranges given are between the upper and lower tenths [deciles] and between the upper and lower fourths [quartiles], the value of the latter range being identical with twice the 'probable error'." (p. 230). Galton points out (pp. 245-246) that the range of a large sample, which depends to a great extent upon the accident of including or not including one or two extreme values, is much less trustworthy than the interquartile or interdecile range. He includes a section on the calculation of deciles, quartiles and medians from grouped data.

**Citations:** Venn (1866) [1888], Bowley (1901), Goldhizer (1923), Julin (1933).

* **Bruns, Heinrich (1882).** Ueber eine Minimumsaufgabe. *Mathematische Annalen* **20**, 455-460.

**Summary:** The author notes that Schreiber (1882), prompted by a concrete case, treats the following problem: In a trigonometric network of triangles a series of excess bearings are to be measured in addition to the necessary and sufficient ones, the total number of measurements being given, so that a certain piece, e.g. the distance between two points, will be determined from the measurements with the greatest possible weight. Schreiber shows that the solution of this problem follows from the determination of the values of n variables $x_i (i = 1,2, \ldots, n; n<p)$ so as to minimize the sum of the absolute values of p linear functions $L_\alpha = b_\alpha + a_{\alpha 1}x_1 + \ldots + a_{\alpha n}x_n (\alpha = 1, \ldots, p)$. The author points out that the problem itself is not new, having been previously considered by Boscovich (1757?), Laplace (1799?), and Gauss (1809). He proceeds to give a treatment of the problem from a purely mathematical viewpoint.

**References:** Boscovich (1757?), Laplace (1799?), Gauss (1809), Schreiber (1882).

**Citation:** Friedrich (1937).

* Chaplin, W. S. (1882). On the relative tensile strengths of long and short bars. *Proceedings of the Engineers' Club, Philadelphia* **3**, 15-28.

**Summary:** The author presents, in somewhat greater detail, much the same material as in his earlier paper [Chaplin (1880)]. The following quotation shows the dependence of his results on the "weakest link" concept: "A bar under tensile stress yields and breaks at one point; it is more or less extended at all points, but it breaks at only one, and it is the strength at this point which determines the strength of the bar to resist tensile stress. Suppose now that pieces a, $a_1$, $a_2$, etc., have been cut from a bar A, and tested in the usual way. The average strength of the pieces a, $a_1$, $a_2$, etc., will necessarily be greater than the strength of the weakest piece; or, as it is the weakest piece which determines the strength of A, the average tensile strength of the pieces a, $a_1$, $a_2$, etc. will be greater than the strength of A. Or *long bars are on the average weaker to resist tensile stress than short ones of the same material and cross-section.*" [p. 19]. A quantitative relation between the average strength of a bar of given length and one n times as long is worked out, assuming normality. If the former is S, the latter is S − a p, where $p = .6745\sigma \doteq .6745s$ and $a = \Phi^{-1}[\sqrt[n]{1/2}]/ .6745$, p being the probable error, $\sigma$ the population standard deviation, and s the usual estimate of $\sigma$ from the sample, with $\Phi(x) = \int_{-\infty}^{x} \phi(x)dx$ and $\phi(x) = e^{-x^2/2}/\sqrt{2\pi}$. Values of a are tabulated for n = 1(1)30(5)60(10)100(50)200(100)1000.

**References:** Chauvenet (1863), Chaplin (1880).

**Citation:** Slocum & Hancock (1906).

* van Pesch, A. J. (1882). Verschillende oplossingen (abstract). Onderwerpen behandeld op de wetenschappelijke Wintervergaderingen 1880. 1 December. *Nieuw Archief voor Wiskunde* **9** (2), 190.

**Summary:** The author states two problems: (1) What is the probability that, if a homogeneous bar is broken (at random) into three pieces, a triangle can be formed with the pieces as sides? (2) If the bar is broken into four pieces, what is the probability that a triangle can be formed with any three of them as sides? He gives the answers as 1/4 and 4/27, respectively.

**Comments:** The first problem had already been solved by Lemoine (1873). The author's answer to the second is incorrect; the correct answer is 1/15 [see van den Berg (1891)].

**Citation:** van den Berg (1891).

* Schreiber, O. (1882). Die Anordnung der Winkelbeobachtungen in Göttinger Basisnetz. *Zeitschrift für Vermessungswesen* **11**, 129-161. (JFM 14, 914-915).

**Summary:** The author observes (p. 130) that, since the introduction of the method of least squares, geodesists seem to have taken it as self evident that it is most advantageous if all available bearings (directions) in a base network are observed. He raises the question as to whether it would be more advantageous to observe fewer bearings with greater precision, and the related question as to whether angles between the chosen directions should all be measured with the same precision, or with different precision, and in the latter case with what relative precision. He proposes the following problem (p. 135): In a network of triangles, of which the angles are already known approximately, which angles must be observed and how often, so that, for a fixed total number of observations, the weight (precision) of the most plausible value of a certain function of the angles shall be as great as possible? He shows that this problem reduces to the following: $L_1, L_2, \ldots, L_n$ are all linear functions of p variables $x_1, \ldots, x_p (p<n)$ with given coefficients; find the set of x's for which the sum of the absolute values of the L's is a minimum. He gives several numerical examples, with data taken from the Göttingen base network of 1880, of the application of this theorem.

**References:** Gauss (1809, 1823).

**Citations:** Bruns (1882), Jordan (1888), Gore (1889), Runge (1890), Friedrich (1937).

* Edgeworth, F. Y. (1883a). The law of error. *Philosophical Magazine* (5) **16**, 300-309. (JFM 15, 162).

**Summary:** The author questions the universal and indiscriminate use of the normal (Gaussian) law of error in the following words: "The Law of Error is deducible from several hypotheses, of which the most important is that every measurable (physical observation, statistical number, &c.) may be regarded as a function of an indefinite number of elements, each element being subject to a determinate, although not in general the same, law of facility. Starting from this hypothesis, I attempt, first, to reach the usual conclusion by a path which, slightly diverging from the beaten road, may afford some interesting views; secondly, to show that the exceptional cases in which that conclusion is not reached are more important than is commonly supposed." [pp. 300-301]. "I submit, in the absence of evidence to the contrary, that non-exponential [non-Gaussian] laws . . . do occur in *rerum naturâ*, that the 'ancient solitary reign' of the exponential [Gaussian] law of error should come to an end." [pp. 305-306].

**Comments:** This paper is indirectly relevant to a study of order statistics because an end to the "solitary reign" of the Gaussian distribution signals also an end to that of the arithmetic mean and the method of least squares, opening the door for substitutes based on order statistics.

**References:** Quetelet (1846), De Morgan (1847), Glaisher (1872, 1873), Edgeworth (1883b).

**Citations:** Edgeworth (1883b), Pizzetti (1892), Czuber (1899), Edgeworth (1905), Keynes (1921), Bowley (1928).

* Edgeworth, F. Y. (1883b). The method of least squares. *Philosophical Magazine* (5) **16**, 360-375. (JFM 15, 162—listing only).

**Summary:** The author begins with a philosophical discussion of the difference between the approaches of Gauss and of Laplace, between most probable results and most advantageous results, and between minimizing mean square errors and mean (absolute) errors. He proceeds to the question of how to treat outlying observations, in regard to which he writes (p. 370): "Agreeably to the distinction clearly exhibited by Mr. Glaisher [(1872), pp. 102-103], the method of least squares belongs to [one or the other of two classes] according as it is, or is not, given that all the observations have the same weight. The latter

case presents the much vexed question, What shall be done with considerable outlying errors? Shall the exceptional observation be omitted from the average, as Peirce [(1852)] says? Or shall it count for one, as Airy [(1856)] says? Or shall it count indeed, but not count for one, as says De Morgan [(1847)]? Upon the hypothesis here entertained De Morgan's view is undoubtedly correct in theory, though in practice it may not differ from the practice of Peirce. The approximative method is justified in following the analogy of the exact method; which would, according to the inverse method here all along contemplated, include all the data among the premises, though it may be that the conclusion, which is a function of them all, is less affected by (the variation or omission of) some of them than others." The author proposes the following method of weighting the observations: Let $P = h_1 h_2 \ldots \exp[-h_1^2(x-x_1)^2 - h_2^2(x-x_2)^2 - \ldots]dh_1 dh_2 \ldots$ be the *a posteriori* probability of the given observations $x_1, x_2, \ldots$ having resulted from a particular system of weights $h_1^2, h_2^2, \ldots$ and a particular mean x; then determine that system so as to make P a maximum.

**Comments:** The system of weights proposed by Edgeworth in this paper is the same as that proposed by Stone (1873b). Edgeworth acknowledges Stone's priority, of which he was unaware at the time he wrote this paper, in a later one [Edgeworth (1887a), p. 373 (footnote)].

**References:** Laplace (1812), Gauss (1823?), De Morgan (1847), Peirce (1852), Airy (1856), Glaisher (1872), Edgeworth (1883a).

**Citations:** Edgeworth (1883a, 1886a, 1887a), Gore (1889), Pizzetti (1892), Keynes (1921), Rietz (1924), Bowley (1928), Rider (1933).

* Doolittle, M. H. (1884). The rejection of doubtful observations (abstract). *Bulletin of the Philosophical Society of Washington (Mathematical Section)* **6**, 153-156. (JFM 16, 184).

**Summary:** The author makes the following comments: "For the purposes of this discussion we may divide errors into two grand classes, *instructive errors* and *uninstructive errors*. The latter class includes blunders in recording, pointing on wrong objects, &c. The former consists of errors that indicate error in other observations. . . . It seems very clear to me that the larger an instructive error is the more instructive it is, and the more important is it that the observation containing it *should not* be rejected. . . . On the other hand, the larger an uninstructive error is, the more important is it that the observation *should* be rejected. Whenever an observation is intelligently rejected, there is a comparison of two antecedent probabilities, viz.: that of the occurrence of an instructive error of the magnitude involved and that of the occurrence of an uninstructive error of the same magnitude. When an error is evidently so large that it cannot possibly belong to the instructive class, the antecedent probability of such an instructive error is 0; the antecedent probability of an uninstructive error is always greater than 0; and the observation should certainly be rejected. But since the theory of least squares allows no limit whatever to the possible magnitude of instructive errors such rejection involves the admission that the method of least squares is not applicable to the case. When an observation involves a merely suspicious error, which is neither so large that instructiveness is impossible nor so small as to pass without question, it would seem reasonable that the observation should be weighted according to the relative magnitude of the two antecedent probabilities which I have mentioned; but this can never be determined with any approach to mathematical precision. . . . In practical problems the antecedent probability of blunders and other uninstructive errors is never known, and it is only matter of exceedingly vague conjecture. . . . It is well known that the method of least squares gives very untrustworthy information in regard to the antecedent probability of large instructive errors. In regard to the other antecedent probability required for an intelligent solution of the problem, it gives no information whatever. So far as I can understand Prof. Peirce's method of arriving at a criterion, he takes two probabilities, both functions of probabilities of instructive error, and balances them against each other. This procedure reminds me of what sometimes happens in war, when two detachments of the same army meet in the dark and fire into each other, each supposing the other to belong to the common enemy. Prof. Peirce also

seems to me to violate the fundamental principle of the science of probabilities, that probabilities must be independent in order that their product shall equal concurrent probability. If a computer resorts to the criterion when he feels that his own judgment is worthless, and only then, the criterion is harmless; since it is of no importance whether a decision is made by a worthless judgment or a worthless criterion." [pp. 152-155]. Several persons participated in the discussion that followed presentation of this paper, including Mr. Asaph Hall, who summarized briefly the literature on the subject.

**References:** DeMorgan (1847), Peirce (1852), Gould (1855), Airy (1856), Winlock (1856), Chauvenet (1863), Stone (1868), Glaisher (1873), Schott (1878).

**Citations:** Gore (1889), Pizzetti (1892).

* Merriman, Mansfield (1884). *A Text-Book on the Method of Least Squares*. John Wiley & Sons, New York. [Review, *Nature* **30**, 334. (JFM 16, 182)]. Sixth edition, 1893; seventh edition, 1897. (JFM 28, 212—listing only); Spanish translation by Babin, Buenos Aires, 1889. (JFM 21, 218).

**Summary:** Articles 130-132 (pp. 166-169 of the sixth edition) deal with the rejection of doubtful observations, and Art. 130 advocates the use of Chauvenet's criterion: "Let n be the number of direct observations, and also the number of errors. Let r denote the probable error of a single observation as found from the n residuals . . . . Let x be the limiting error, and let x/r be called t. Let P be the value of the integral in Table II [Values of the Probability Integral for Argument x/r] corresponding to t. Then $P=(2n-1)/2n$ and $x=tr$ is the criterion for rejection of the largest residual. To prove this, consider that the quantities in Table II need only be multiplied by the total number of errors to show the actual distribution; so that nP indicates the number of errors less than x, and $n-nP$ indicates the number greater than x, and any error greater than this x would be larger than allowed by the theoretical distribution. Hence the value of x corresponding to this value of P is the limiting value, which indicates whether the greatest residual in a series may be rejected or not." The necessary table [Table VII—For Applying Chauvenet's Criterion] is given on p. 198. An example of the use of this criterion is given in Art. 131 (pp. 167-168). In Art. 132 (pp. 168-169) the author discusses two other criteria— Peirce's and a new one based on Hagen's deduction of the law of probability of error: "In Art. 26 the maximum error is expressed by $m\triangle x$, and the quantity $m\triangle x^2$ is replaced by $1/2h^2$. It is hence easy . . . to find $m\triangle x=4.4\, r^2/dx$, where dx is the constant interval between successive values of the errors." [p. 168]. Merriman states (p. 169): "In general, it should be borne in mind that the rejection of measurements for the single reason of discordance with others is not usually justifiable unless that discordance is considerably more than indicated by the criterions. A mistake is to be rejected, and an observation giving a residual greater than 4r or 5r is to be regarded with suspicion, and be certainly rejected if the notebook shows anything unfavorable in the circumstances under which it was taken."

**References:** Gauss (1809), Hagen (1837), Peirce (1852), Chauvenet (1863).

**Citations:** Venn (1866) [1888], Edgeworth (1886a, 1887a,b,e,) Gore (1889), Wölffing (1899), Bowley (1901), Tolley (1916), Niceforo (1919), Campbell (1920), Hall (1921), Keynes (1921), Edgeworth (1923), Foster (1924), Rietz (1924).

* van Pesch, A. J. (1884). (Abstract—no title). Onderwerpen behandeld op de Wetenschappelijke Wintervergaderingen 20 Februari 1884. *Nieuw Archief voor Wiskunde* **11** (1), 48.

**Summary:** The author solves algebraically, analytically, and by integration the problem stated and solved in his earlier abstract [van Pesch (1882)]: A bar is broken into three pieces; determine the probability that with the pieces as sides a triangle can be formed. More generally, what is the probability that with n pieces, into which a bar is broken, a polygon can be formed? The author states that a number of complicated integrations would be required for the solution. He claims to have bypassed the difficulties by a trick, but the result is not stated in the abstract.

**Comments:** The problem of n pieces had already been solved by Halphen (1873); for a further generalization, see van den Berg (1891).

**Citation:** van den Berg (1891).

* Wright, Thomas Wallace (1884). *A Treatise on the Adjustment of Observations, with Applications to Geodetic Work and other Measures of Precision*. D. van Nostrand, New York; second edition (with the cooperation of John Fillmore Hayford), 1906. (JFM 37, 272, 960).

**Summary:** The author discusses the problem raised by the presence of outliers in astronomical and other observations. After reviewing the criteria of Peirce (proved by Gould), Chauvenet, and Stone, the author states (p. 135 of first edition): "The least objectionable criterion based on mathematical principles may, I think, be developed from the principle laid down in Art. 30, where the maximum error was estimated at about five times the p.e. or three times the m.s.e. If, therefore, an observation differs from the general run of the series by more than this amount it should at least be bracketed and attention be called to it." For computers who are also competent observers, however, Wright recommends a subjective approach based on knowledge of the sources of variation encountered or likely to be encountered in observations of the type under consideration, rather than strictly adhering to a criterion based on mathematical principles, which is the best that a computer who is not an observer can do. He also considers weighted arithmetic means, including those based on the suggestion of De Morgan, and medians, as we see from the following passage (pp. 136-137): "We may . . . consider whether, when discrepant observations occur we may not get a more satisfactory result by ignoring the arithmetic mean altogether. Suppose, for example, that we had three observed values, 100, 60, 61, and that we had no means of getting any further observations. These values would seem to show that the true value was likely to be nearer 60 than 100. Just how much nearer is the question. To reject the observation 100 would be without reason, and to take the arithmetic mean of all three would ignore the evidence afforded by the observations themselves. . . . A plausible result has been given (Arts. 11, 15) as that observed value which has as many observed values greater than as it has less than it. Thus in the preceding example a good value to choose would be 61." In the second edition, Wright and Hayford restate Wright's rejection rule in slightly modified form (p. 90): "Reject each observation for which the residual exceeds five times the probable error of a single observation. Examine each observation for which the residual exceeds 3-1/2 times the probable error of a single observation, and reject it if any of the conditions under which the observation was made were such as to produce any lack of confidence."

**References:** Anonymous (1821), De Morgan (1847), Peirce (1852), Gould (1855), Airy (1856), Winlock (1856), Chauvenet (1863), Stone (1868), Glaisher (1872, 1873), Stone (1873b).

**Citations:** Gore (1889), Pizzetti (1892), Brunt (1917), Secrist (1917), Julin (1921), Reilly, Rae & Wheeler (1925), Rider (1933).

* Edgeworth, F. Y. (1885). Methods of statistics. *Journal of the Royal Statistical Society* **Jubilee**, 181-217.

**Summary:** Among other subjects, the author discusses the choice of measures of central tendency and of variability. He insists that, while the arithmetic mean and the root-mean-square deviation from it are most accurate for samples from a normal population, other measures (median and mean absolute deviation or quartile deviation) are more convenient and are little less accurate, while for other populations they may be more accurate as well as more convenient. With regard to measures of variability he writes (pp. 188-189): ". . . When the observations really conform to a [normal] probability-curve, there are several formulae for the modulus [$c = \sigma\sqrt{2}$, where $\sigma$ is the standard deviation] which are little inferior to the above [root-mean-square error] in respect of accuracy, and two of them which are superior in respect of convenience. If we call the preferential method the method of mean square of errors, one of the rival methods might be called the method of mean first power; the other the method of

mean zero powers. . . . [The last] method is that described by Mr. Galton [(1875)] . . .; the same in principle as that which was employed by Quetelet [(1835 [1869], 1846)]. The essence of this method is to note the points between which are comprised quarters (eighths or other fractions) of the total number of observations, and then to equate the distances thus given by observation to the corresponding multiple of the modulus as assigned by theory. For example, if we take two points so that between them there occur half the total number of given observations, and outside each of them a quarter of the total number, the distance between these two points ought theoretically to be equal—is equatable—to twice the modulus $\times$ 0.476." On the use of the median he writes (pp. 190-191): "Still under the heading ($\beta$) [the case where the observations do not themselves fulfill the normal law of error] we have next to consider other ways of dealing with our m observations besides taking their arithmetical mean. The most important case is where we take the *median*, as it has been called, or *centralwerth*, of our m observations: that is the point on either side of which will lie m/2 observations. If we form this kind of Mean for several sets consisting each of m observations, the means so formed will range under a probability-curve whose fluctuation [$c^2 = 2\sigma^2$] is a function of a datum which may be described as the central ordinate of the curve from which the m observations are taken. The sought fluctuation is to be equated to the reciprocal of $2my^2$, where y is the central ordinate of the said curve divided by its area."

**References:** Laplace (1812), Quetelet (1835 [1869], 1846), Todhunter (1865), Glaisher (1872), Galton (1875), Edgeworth (1886a).

**Citations:** Venn (1866) [1888], Edgeworth (1886a,b, 1887e, 1887-90, 1898, 1905), Yule (1907), Edgeworth (1911, 1913), Julin (1921), Keynes (1921), Rietz (1924), Bowley (1928).

* Edgeworth, F. Y. (1886a). Observations and statistics: An essay on the theory of errors of observation and the first principles of statistics. *Transactions of the Cambridge Philosophical Society* **14** (2), 138-169; abstract, *Proceedings of the Cambridge Philosophical Society* **5** (4), 310-312.

**Summary:** The author begins with the statement (p. 139): "The object of this paper is to distinguish and examine the different cases which are presented by the problem: What is the best Mean?" He makes passing reference to the use of the quartile deviation by Quetelet (1846) and of quartiles and deciles by Galton (1875) in estimating the probable error, and to the method of situation and the most advantageous method [see Laplace (1799, 1812)]. The main attention, however, is given to the question stated above, and the relative advantages of arithmetic mean, median, and mode are explored in considerable detail, with less attention given to other possible means. On the grounds of precision, the arithmetic mean is declared to be superior to the others for the normal law and others near it, but the median is better "when the apex of the curve is very high and its extremities very much extended." (p. 167). "In respect of convenience, [the Mode] has a considerable advantage over the Arithmetical Mean and a less marked advantage over the Median." (p. 168).

**References:** Laplace (1799, 1812), Gauss (1823), Ellis (1844), Quetelet (1846), De Morgan (1847), Herschel (1850), Peirce (1852), Airy (1856), Todhunter (1865), Glaisher (1872, 1873), Fechner (1874), Galton (1875), Edgeworth (1883b), Merriman (1884), Edgeworth (1885).

**Citations:** Venn (1866) [1888], Edgeworth (1885, 1886b, 1887a,b,d,f, 1887-90), Venn (1891), Pizzetti (1892), Edgeworth (1898), Bowley (1901), Yule (1907), Edgeworth (1911), Keynes (1921), Bowley (1928).

* Edgeworth, F. Y. (1886b). Problems in probabilities. *Philosophical Magazine* (5) **22**, 371-384.

**Summary:** In connection with the study of problems in probabilities encountered in banking, the author determines the laws of facility of error for the median and the quartiles of a sample of size n from a normal population with modulus c, where $c = \sigma\sqrt{2}$, $\sigma$ being the standard deviation. He states that his

method was suggested by the determination of Laplace (1812—second supplement, 1818) of the error incurred by his "method of situation." He finds that both the median and the upper (or lower) quartile are approximately normally distributed, with moduli $k = \sqrt{\pi/2n}\, c$ and $\kappa = \sqrt{\pi/1.7n}\, c$, respectively. The probable error is the difference between the median and one of the quartiles, but Edgeworth points out that one cannot calculate its modulus-squared by adding the squares of the moduli of median and quartile, since the median and the quartiles (also the octiles and the deciles) are not independent, but positively correlated. He takes the sum of the two moduli-squared and the smaller of them as superior and inferior limits, respectively, on the modulus-squared of the probable error. He notes that the error in determining the probable error can be somewhat reduced if one uses [half] the distance between the observed quartiles instead of the distance between one of them and the observed median.

**References:** Laplace (1812) [1818 suppl.], Edgeworth (1885, 1886a).

**Citations:** Edgeworth (1887e, 1887-90), Keynes (1921), Bowley (1928).

* Newcomb, Simon (1886). A generalized theory of the combination of observations so as to obtain the best result. *American Journal of Mathematics* **8**, 343-366. (JFM 18, 183-185).

**Summary:** The author considers the problem of combining a number of observations of the same quantity so as to obtain the best result. He raises two objections to the criterion for rejection of doubtful observations proposed by Peirce (1852): (1) It disregards any *a priori* knowledge of the probable error of the observations and seeks to determine it from the observations themselves; and (2) It does not take account of the fact that the *a priori* probability of an abnormal observation varies from one observer to another. He points out (page 344, footnote) two previously unnoticed results of Peirce's criterion when applied to three or four observations: "Of a set of three observations none are ever rejected by it, no matter how much one may deviate from the other two. In a set of four observations, if three agree exactly, the fourth will always be rejected if it differs ever so little from the others. More generally, if no one of the three results which agree best among themselves differs from the mean of the three by more than $\epsilon$, then a fourth, which differs from that mean by more than $4\epsilon$, will be rejected." Newcomb proposes a new criterion which has been well summarized by Rider (1933), pp. 18-19, as follows: "Newcomb . . . proposed as a generalized law of error $\pi^{-1/2}(p_1 h_1 e^{-h_1^2 x^2} + p_2 h_2 e^{-h_2^2 x^2} + \ldots + p_m h_m e^{-h_m^2 x^2})$, in which the h's are the different values of the measures of precision and $p_i$ is the probability that an observation chosen at random has the precision $h_i$. For the best result from n observations Newcomb gives $X = (w_1\eta_1 + w_2\eta_2 + \ldots + w_\ell\eta_\ell)/(w_1 + w_2 + \ldots + w_\ell)$, $(\ell = m^n)$. The $m^n$ $\eta$'s are weighted means of observed quantities $X_1, X_2, \ldots, X_n$, each mean being obtained by making a hypothesis concerning the distribution of the measures of precision, $h_1, h_2, \ldots, h_m$, among the n separate observations. Since each observation may, independently of all the others, have any one of the m measures of precision, there will be $m^n$ such hypotheses, each leading to a different mean, $\eta$. The final value . . . of X is the mean by weights of the different hypotheses, the weight of each result being proportional to the probability of the hypothesis upon which it depends, which is represented by w. Each w is the product of two factors, one of which is proportional to the probability of the combination, while the other is the probability of the combination of errors to which the hypothesis leads. Obviously, with even a few observations the method is exceedingly cumbersome. Newcomb gives a modified method to be used in case the observations are numerous, but even so, the method is both complicated and tedious."

**References:** Gauss (1823), Peirce (1852).

**Citations:** Helmert (1872) [1907], Edgeworth (1887a,b,d), Gore (1889), Pizzetti (1892), Wellisch (1909), Edgeworth (1911), Newcomb (1912), Ogrodnikoff (1928), Rider (1933), Hulme & Symms (1939), Jeffreys (1939).

* Bertrand, J. (1887a). Théorème relatif aux erreurs d'observation. *Comptes Rendus de l'Académie des Sciences de Paris* **105,** 1043-1044; comments by H. Faye, 1102. (JFM 19, 210-211).

**Summary:** The author states the following theorem: "If one repeats an even number of times the measurement of a quantity, associates the results two by two at random, and distinguishes in each group of two the larger and the smaller of the accidental errors, the ratio of the sum of the former to the sum of the latter, when the number of trials increases, converges to $\sqrt{2} + 1 = 2.41$." (p. 1043). He does not state under what hypothesis (presumably normality) concerning the underlying error distribution this result holds, and offers no proof other than an unconvincing demonstration, which he attributes to Broch, that the result is approximately correct for five series of observations ranging from 54 to 100 in number with the ratio in question ranging from 2.28 to 2.49 and averaging 2.39 for the combined set of 379 observations. In a letter to the author, published in the same volume (p. 1102), Faye makes the following comment: "If one considers all the combinations of errors two by two, the ratios of the sums corresponding to the largest and smallest of these errors are included between the limits 1 and 3.915, which themselves have infinitesimally small probability, but whose mean 2.457 differs little from the number 2.414 of your chance drawing."

**Citations:** Delauney (1888), Czuber (1891a).

* Bertrand, J. (1887b). Sur la loi des erreurs d'observation. *Comptes Rendus de l'Académie des Sciences de Paris* **105**, 1147-1148. (JFM 19, 212).

**Summary:** The author states the two following theorems without proof: "I. If the same quantity is measured a large number of times and if one groups the measurements two by two in random order and chooses in each group the larger of the two errors committed, the ratio of the mean of the squares of these maximum errors to the mean of the squares of all the errors converges to the value $1+2/\pi$ when the number of trials increases indefinitely. II. If one groups the measurements three by three, in random order, the mean of the squares of the largest errors of each group divided by the mean of the squares of all the errors has as its probable value $1+2\sqrt{3}/\pi$ and approaches arbitrarily closely to this value when the number of trials increases."

**Citations:** Gore (1889), Czuber (1891a).

* Edgeworth, F. Y. (1887a). On discordant observations. *Philosophical Magazine* (5) **23**, 364-375. (JFM 19, 214-215).

**Summary:** In discussing the diversity of methods for the treatment of discordant observations, the author (p. 365) makes the following statement: "Different methods are adapted to different hypotheses about the cause of a discordant observation; and different hypotheses are true, or appropriate, according as the subject-matter, or the degree of accuracy required, is different." He specifies three hypotheses (pp. 365-366), leaving it to the reader to estimate their *a priori* probability: (1) "There are only two species of erroneous observations—errors of observations proper, and mistakes. The frequency of the former is approximately represented by the curve $y=(h/\sqrt{\pi})e^{-h^2x^2}$; where the constant h is the same for all the observations. But the mathematical law only holds for a certain range of error. . . . The smallest mistake is greater than the largest error of observation proper." (2) "The type of error is still the [normal] probability curve with unvarying constant. But the range of its applicability is not so accurately known beforehand. We cannot at sight distinguish errors proper from mistakes." (3) "All errors are of the type $y=(h/\sqrt{\pi})e^{-h^2x^2}$. But the h is not the same for different observations. *Mistakes* may be regarded as emanating from a source of errors whose h is very small." Next he divides the different methods of treating discordant observations into four groups (p. 367): "I. The first sort of method [Airy (1856)] is based upon the principle that the calculus of probabilities supplies no criterion for the correction of discordance. All that we can do is to reject certain huge errors by common sense or simple induction. . . .

II. Or, secondly [Peirce (1852), Chauvenet (1863), Stone (1868),], we may reject observations upon the ground that they are proved by the Calculus of Probability to belong to a much worse category than the observations retained. III. Or, thirdly [De Morgan (1847), Glaisher (1872), Stone (1873b), Edgeworth (1883b), Newcomb (1886)], we may retain all the observations, affecting them respectively with *weights* which are determined by *a posteriori* probability. IV. In a separate category may be placed a method [Fechner (1874), Galton (1875), Wilson (1877)] which ... reduces the effect ... of discordant observations—the method which consists in taking the Median or 'Centralwerth' of the observations." He states that the first method is well adapted to the first hypothesis, but not so well to the others; the second class of methods [especially the method of Stone (1868)] are specially adapted to the second hypothesis. The author suggests, however, that Stone's criterion be modified so that the probability is $1-1/m$ that in m observations not a single one will deviate from the mean by more than $\kappa\sigma$ (the rejection limit), where m is Stone's modulus of carelessness. He traces the idea behind this distinction back to Cournot (1843). He raises an objection to the methods of Peirce and Chauvenet. Methods of the third class, he says, except possibly that of Newcomb, are not well adapted to the first two hypotheses, but are quite good for the third hypothesis, for which they are avowedly designed. He defers discussion of the fourth method to a later paper.

**References:** Cournot (1843), De Morgan (1847?), Peirce (1852), Airy (1856), Chauvenet (1863), Todhunter (1865), Stone (1868), Glaisher (1872), Stone (1873b), Fechner (1874), Galton (1875), Wilson (1877), Edgeworth (1883b), Merriman (1884), Edgeworth (1886a), Newcomb (1886), Edgeworth (1887f).

**Citations:** Pizzetti (1892), Edgeworth (1893), Czuber (1899), Edgeworth (1911), Keynes (1921), Bowley (1928), Rider (1933).

* Edgeworth, F. Y. (1887b). On observations relating to several quantities. *Hermathena* **6** (13), 279-285.

**Summary:** Eisenhart (1961), page 208, has given the following brief summary of this paper: "Edgeworth ... attributed Boscovich's algorithm to Laplace. Dropping Boscovich's Condition (I) [that the sums of positive and negative deviations be equal in magnitude], he devised a 'double median' method for finding values of a and b that correspond to the minimum minimorum of the sum $\Sigma_1^n |y_i - a - b x_i|$ ...." Further insight into Edgeworth's procedure is given by his own words (pp. 279-282): "It is proposed here to treat those difficulties in the reduction of observations which are peculiar to the case of plural quaesita. I have elsewhere [Edgeworth (1886a, 1887f)] considered the generic problem and the simpler species.... Consider, first, the simple case in which there are only two quaesita. Let the given equations be of the form $a_1x+b_1y-w_1=0, a_2x+b_2y-w_2=0$, &c., where $w_1$, $w_2$, &c., are observations subject to equal error. According to the usual procedure we obtain for one locus (of the sought point xy) the 'normal equation' $a_1[a_1x+b_1y-w_1] - a_2[a_2x + b_2y - w_2] + \&c. = 0$; which may be thus interpreted. Substitute any assigned value for y in the original equations. Of the n values for x thus presented, the (weighted) Arithmetical Mean is given by substituting the assigned value for y in the 'normal' equation. The analogous procedure is to find a locus such that if we substitute any assigned value of y in the original equations, the *Median* of the corresponding n values of x may be given by the locus. The series of points, which in the case of the Arithmetical Mean is obtained by a single stroke of analysis, must, in the case of the Median, be traced one by one. That is, we must substitute in the given equations successive values of y (e.g. $0, \delta, 2\delta, \&c.$), find the Median value for x corresponding to each assigned y, and plot the series of points. A second Median Curve is afforded by the Medians of the y components; and the intersection of these Median Curves gives the *Median Point*. The method is perfectly general. As an illustration we may take the case of two quaesita, x and y, the equations for which involve only one of the variables. The *Mean loci* are in this case lines parallel to the axes. And it follows, from considerations which I have elsewhere [Edgeworth (1887d)] put together, that the Median, as compared with the Arithmetical Mean, affords a solution nearly as good when the typical [normal] probability-curve prevails, and better when the observations are 'discordant'." The author mentions the methods proposed by Laplace (1799), Secs. 39 and 40, the latter of which he calls (page 280) a "remarkable hybrid between

the *Method of Least Squares* and the *Method of Situation*," because Boscovich's Condition (I) requires that deviations be taken from the arithmetic mean as in the method of least squares, instead of from the median, as in Edgeworth's version of the method of situation, where that condition has been dropped.

**References:** Laplace (1799, 1812), Airy (1861), Chauvenet (1863), Glaisher (1872), Merriman (1877, 1884), Edgeworth (1886a), Newcomb (1886), Edgeworth (1887d,f).

**Citations:** Edgeworth (1887c, 1887-90, 1888), Bowley (1928), Bruen (1938).

* Edgeworth, F. Y. (1887c). A new method of reducing observations relating to several quantities. *Philosophical Magazine* (5) **24**, 222-223.

**Summary:** In this letter to the editors, the author calls attention to his method [Edgeworth (1887b)] of reducing observations relating to several quantities, which he characterizes as a generalization of the method of situation which Laplace (1812), second supplement, has applied to observations relating to a single quantity. He writes as follows: "The method may be thus described in the case of two variables, x and y. Find an approximate solution by some rough process (such as simply adding together several of the equations so as to form two independent simultaneous equations). Take the point thus determined as a new origin, and substitute in the n (transformed) equations for one of the variables x a series of values $\pm\delta$, $\pm 2\delta$, &c. Corresponding to each of these substitutions we have n equations for y. For each of these systems determine the *Median* according to Laplace's *Method of Situation*. This series of Medians forms one locus for the sought point. A second locus is found by transposing x and y in the directions just given. The intersection of these loci is the required point. The method may be extended to any number of variables. For example, in the case of the four variables, x,y,z,w, we should construct one table containing the Median values of w corresponding to each triplet of values assigned to x,y,z; and three similar tables for the Median values of x,y,z, respectively. The system of values for x,y,z,w, which is identical in all the tables, forms the required solution. The advantages claimed for the new method are that, while in the typical case of the laws of facility being all [normal] Probability Curves, the generalized *Method of Situation* is only slightly less accurate, and considerably less laborious, than the *Method of Least Squares;* in the abnormal case of Discordant Observations the proposed method is not only more convenient, but better. It is much to be wished that some practical astronomer would give this method a trial by employing it in some laborious and important calculation." (pp. 222-223).

**References:** Laplace (1812), Edgeworth (1887b).

**Citations:** Turner (1887), Edgeworth (1888), Pizzetti (1892), Edgeworth (1893, 1923), Whittaker & Robinson (1924), Bowley (1928), Bruen (1938).

* Edgeworth, F. Y. (1887d). The choice of means. *Philosophical Magazine* (5) **24**, 268-271. (JFM 19, 215-216).

**Summary:** The author, who has attempted elsewhere [Edgeworth (1886a, 1887f)] to answer the general question, "What is the best Mean?," considers here the special case of discordant observations. He writes as follows (pp. 268-270): "One large class of Discordant Observations belongs to the category of errors whose law of facility is a compound of different [normal] Probability-Curves. . . . The compound source of error may be treated as belonging to the category of curves other than Probability-Curves. This category also comprises the species of Discordant Observations other than that above defined. What, then, are the methods proper to this category? They are two—the *Method of Least Squares* and the *Method of Situation*. . . . We can apply the rules primarily appropriate to observations obeying the typical *Law of Error*, to find both the *most probable* and the *most advantageous* Mean. . . . The *Method of Least Squares* is seen to be our best course when we have thrown overboard a certain portion of our data—a sort of sacrifice which has often to be made by those who sail upon the stormy seas of Probability.

But . . . it might be better, instead of directing our exclusive attention to the Arithmetic Mean and the correlated characteristic of the inverse mean-square-of-error, to prescind in the same sense some other Mean, and in particular the Median, with its correlate the Greatest Ordinate. The Median of any cluster fluctuates according to a Probability-Curve whose modulus is $\sqrt{s}/\sqrt{2}\,P$, P being the Greatest Ordinate of the common facility-curve. The criterion whether the Median or Arithmetic Mean is the better reduction is presumably the character of the correlated Probability-Curve. The reduction which corresponds to the smaller Modulus is presumably the better; since thus we obtain a smaller 'probable' error,and what is often more important, a smaller *improbable*, or, as Mr. Merriman proposes to cali it, 'huge' error. Which of the reductions will have the smaller Modulus will depend upon the character of our facility-curve. For [normal] Probability-Curves, and presumably functions in their neighborhood, it is shown by Laplace [(1812, Supplement 2] that the Arithmetic Mean has the advantage. But for curves whose head reaches high, while their extremities stretch out far, the Median has the advantage. Now the grouping of Discordant Observations is apt to assume this form. Accordingly the Median is proposed as the Mean proper to this class of observations. If we have been deceived by the appearance of Discordance . . . and the facility-curve was really a normal Probability-Curve, yet we shall have lost little by taking the Median instead of the Arithmetic Mean. For the error of the former is of the same order as (only 1.3 [times] greater than) the error of the latter. And, if the observations are really discordant, the derangement due to the larger deviations will not be serious, as it is for the Arithmetic Mean." Three numerical examples are given.

**References:** Laplace (1812), Airy (1861), Chauvenet (1863), Fechner (1874), Galton (1875), Edgeworth (1886a), Newcomb (1886), Edgeworth (1887a,f).

**Citations:** Edgeworth (1887b, 1887-90), Pizzetti (1892), Edgeworth (1893), Yule (1907), Edgeworth (1911), Bowley (1928), Rider (1933).

* Edgeworth, F. Y. (1887e). The empirical proof of the law of error. *Philosophical Magazine* (5) **24**, 330-342. (JFM 19, 214).

**Summary:** Most of this paper is not directly relevant to a study of order statistics, but the following portion (pp. 332-333) dealing with a test of symmetry based on the difference between the arithmetic mean and the median is relevant: "It is a very good plan . . . to compare the mean-error above and that below the apparent, the arithmetic, mean. The significance of this result may be tested by the formula which Laplace [(1812)] has given for the error of the sum of errors, *abstraction faite du signe*. . . . It appears that the mean error above and that below the Arithmetic Mean can only be different when the Arithmetic Mean and the Median are different. Accordingly an equally good test of symmetry may be afforded by comparing the Arithmetic Mean and the Median and estimating the probability of the observed difference occurring if the curve were really symmetrical. . . . For the Modulus-squared, which tests the difference to be expected between these Means, we have (by adding the squares of the Modulus applicable to each Mean separately) $2Se^2/n^2 + n/2P^2$; where the e's are errors measured from the Arithmetic Mean, P is the greatest ordinate (the number of observations per unit of abscissa at the densest part of the given group)." The author gives two numerical examples of the use of this test of symmetry.

**Comments:** The test of symmetry based on the difference between the arithmetic mean and the median which the author proposes is erroneous. Since the arithmetic mean and the median are not independent, but positively correlated, the variance of their difference is not the sum of their variances, but that sum decreased by twice their covariance.

**References:** Laplace (1812), Chauvenet (1863), Todhunter (1865), Glaisher (1872), Merriman (1884), Edgeworth (1885, 1886b).

**Citations:** Pizzetti (1892), Edgeworth (1893), Pearson (1895), Edgeworth (1898), Czuber (1899), Keynes (1921), Bowley (1928).

* Edgeworth, F. Y. (1887f). *Metretike: or the Method of Measuring Probability and Utility*. The Temple Co., London.

**Summary:** Edgeworth (1887d) mentions (footnote, p. 268) corrections to an earlier paper [Edgeworth (1886a)] given in the Appendix to this book. In another footnote on the same page, he refers the reader to this book for a fuller statement of the reasoning behind his statement that the Method of Least Squares is a good method. Edgeworth (1888) writes (pp. 184-185, footnote): "The Method of Least Sum is . . . exactly on a par with the Method of Least Squares. . . . No doubt the use of either method divorced from the law of facility appropriate to it [double exponential for the former, normal for the latter] is open to logical objections. But the difficulties are not greater for one method than for the other. The present writer's explanation of the philosophical difficulty common to both methods is stated in the Appendix to a little treatise on the Art of Measurement, entitled 'Metretike'. . .". Bowley (1928) writes (pp. 118-119): "The measurement of belief is as subtle as the measurement of utility. In each case the calculation is inexact and the criteria used are arbitrary. In probability it is often necessary at some stage to assume equal frequency of chances, in the treatment of utility it is assumed that for some purposes all men are equal; many analogies of this kind are treated in *Metretike*, and it is shown that the methods of solution are analogous in the two sciences. Further, both depend on unverified assumptions, the one that experience is the standard of credibility, the other that utility is the standard of good conduct. Both tend to result in non-numerical mathematical statements, that one quantity is equal to, or greater or less than, another, but not how much greater or less. Again, utility governs probability in such questions as, What is the best mean?, where the detriment of error is considered. There is close analogy in both the fundamental ideas and in the mathematical expression between the maximising of probability to discover frequency constants and the maximising of utility to determine equilibrium in economics."

**Note:** The compiler has been unable to obtain a copy of this book, and hence has had to rely solely on the sources cited for information as to its contents.

**Reference:** Edgeworth (1886a).

**Citations:** Edgeworth (1887a,b,d, 1888), Keynes (1921), Bowley (1928).

* Edgeworth, F. Y. (secretary) (1887-90). Reports of the Committee appointed for the purpose of investigating the best methods of ascertaining and measuring variation in the value of the monetary standard. *Reports of the British Association for the Advancement of Science* **57**, 247-301; **58**, 181-219; **59**, 133-164; **60**, 485-488. Italian translation by Paolo Conte, *Biblioteca dell' Economista* (5) **20** (1922), 153-340.

**Summary:** One of the issues studied by the Committee, and the one of principal interest here, is the type of average to be used in constructing a price index for all items of national consumption and auxiliary price indexes for subgroups thereof. Almost every type of average, weighted or unweighted, that has ever been proposed is considered — arithmetic mean, median, mode, geometric mean, etc. The secretary (Edgeworth) argues strongly for the use of the median. He writes (p. 291 of the first report): "In view of the evidence adduced in the last section that price-variations are apt to be grouped asymmetrically, the 'arithmetic' species of mean becomes precarious when our *quaesitum* is a quasi-objective type. The additional complexities which have been introduced in this section make against the geometric mean which was above recommended a certain hypothesis. There exists another species of mean more adapted to the rough character of our calculation, the Median; that is, in the simpler cases, that quantity which has as many of the given observations above it as below it, but a certain analogue of this operation, when the observations have different weights. *The required formula is the Weighted Median*, the

operation designated by Laplace [1812, Suppl. 2 (1818)], as the 'Method of Situation'. . . . ." He repeats some of the arguments in favor of the median which he presented in earlier papers [Edgeworth (1886b, 1887d)] and illustrates by an example the determination of the weighted median. Further arguments in favor of the median are given in the second report (pp. 206-209) and in the third report (pp. 156-161). The use of the quartiles and the quartile deviation (semi-interquartile range) is mentioned in a footnote on p. 203 of the second report. The quartiles of a sample of 39 prices are taken (correctly) as the tenth and thirtieth order statistics of the sample and the quartile deviation as half their difference.

**References:** Laplace (1812) [1818], Venn (1866) [1888], Edgeworth (1885, 1886a,b, 1887b,d).

**Citations:** Venn (1866) [1888], Rietz (1924), Bowley (1928).

* Lehmann-Filhes, R. (1887). Ueber Abnorme Fehlervertheilung und Verwerfung Zweifelhafter Beobachtungen. *Astronomische Nachrichten* **117** (2792), 121-132. (JFM 19, 1200–1201).

**Summary:** The author begins with the statement that often more large errors occur than one would expect according to the Gaussian law of error, and astronomers discard observations with abnormally large errors. In this connection he mentions the criteria of Peirce (1852) and Chauvenet (1863). The author is led by several examples to an attempt to explain this phenomenon on the basis of a measure of precision h varying from observation to observation. He assumes that all the values of h are distributed about a mean $h_0$ in such a manner that the probability that h lies between h and h+dh is given by $\psi(h)dh = (k/\sqrt{\pi})e^{-k^2(h-h_0)^2}\,dh$, which reduces to the Gaussian law of error only when $k=\infty$ [not when $k=0$, as stated incorrectly by Dziobek in the review cited above]. With the aid of these considerations he is able to obtain better agreement between theoretical and observed values, as he shows in an example.

**References:** Laplace (1799), Gauss (1809, 1816, 1823), Peirce (1852), Chauvenet (1863).

**Citation:** Wellisch (1909).

* Turner, H. H. (1887). On Mr. Edgeworth's method of reducing observations relating to several quantities. *Philosophical Magazine* (5) **24**, 466-470. (JFM 19, 216-217).

**Summary:** The author states: "Edgeworth [(1887c)] invites attention to a method of reducing observations relating to several quantities, which he has suggested as a substitute for the ordinary process of the 'Method of Least Squares'. I have applied this method to an example for a particular case of two variables, and venture to offer the following remarks and suggestions for consideration. [Here follows a quotation from Edgeworth's paper cited above.] Some of the labour of this process, and sometimes the preliminary search for an approximate solution, may be avoided by the use of a graphical method [which the author describes]. . . . In the method of least squares the normal equations have a unique solution; but the intersection of two broken lines may be a series of points, and the two median loci may also have a common portion. The solution then becomes to some extent indeterminate. . . . It is possible that the number and distribution of these points of intersection afford real information as to the value and accordance of the observations. But, in practice, a single solution, although its singularity may be somewhat fictitious, is preferable to a variety; and unless some additional criterion for extracting a single solution from the median loci can be obtained, it is to be feared that we have here a somewhat serious objection to this method on the score of convenience. . . . Mr. Edgeworth claims as advantages for the new method that (1) It is considerably less laborious than the Method of Least Squares. (2) In the case of Discordant Observations it is theoretically better. So far as my slight experience entitles me to express an opinion on these points, I should say that (1) is very doubtful. In trying a new method much time is liable to be wasted; but there would, I imagine, never be quite the same straightforwardness about the new method which makes the method of least squares so easy, although somewhat long. (2) is somewhat counterbalanced by the failure to give a *unique* solution." (pp. 466-470).

**Reference:** Edgeworth (1887c).

**Citations:** Edgeworth (1888), Pizzetti (1892), Edgeworth (1923), Bruen (1938).

* Bertrand, J. (1888a). Sur la loi de probabilité des erreurs d'observation. *Comptes Rendus de l'Académie des Sciences de Paris* **106**, 153-156. (JFM 20, 219-220).

**Summary:** The Gaussian law of probability is the only one for which, among several observations made under the same conditions, the mean value is the most probable. Given any other law for the probability of errors, it is possible to specify the combination of a series of measurements which will give the most probable value. The converse is not true; given a combination of observations, in most cases there is no probability law for which that combination gives the most probable value. For example, there is no probability law for which the geometric mean or the harmonic mean of a number of observations is the most probable value. After pointing out that the foregoing statements are true only *a priori*, the author continues (pp. 155-156): "Everything changes in the presence of known results. When, for example, all the observations are concordant except one which deviates notably from the others, nobody would think of advising the adoption of the arithmetic mean or of replacing it by any other general rule. Theory demonstrates that, for a large number of observations, such anomalies should be attributed, with probability equivalent to certainty, to influences which change the conditions of the problem." (Compiler's translation).

**Reference:** Gauss (1809?).

**Citations:** Gore (1889), Czuber (1891a), Pizzetti (1892), Wölffing (1899).

* Bertrand, J. (1888b). Sur la combinaison des mesures d'une même grandeur. *Comptes Rendus de l'Académie des Sciences de Paris* **106**, 701-704. (JFM 20, 222-223).

**Summary:** The author begins (p.701): "When one has made many measurements of the same quantity, . . . all . . . meriting equal confidence, the universal rule is to take the mean. One may remark, however, that after the measurements have been made and the mean and the probable error have been calculated, . . . [the measurements] have ceased to merit the same confidence; those which deviate most from the mean are regarded, with reason, almost with certainty, as less good than the others: it seems natural to discard them. The question is very delicate and observers are permitted such a suppression only after having recognized a probable intrinsic cause of the inferiority of the suspected measure . . . . The consequence of this scruple is to permit very rarely the later suppression of measurements at first accepted: how can one discover after the event causes of error which have escaped attention when they were before the eyes?" (Compiler's translation). The author proposes computing the mean and the constant k [$= 1/\sigma\sqrt{2}$], calculating the error $\lambda$ such that the probability of an error smaller than $\lambda$ is p (arbitrarily chosen), then discarding all observations differing from the mean by more than $\lambda$, and starting over. This results in multiplying the probable value of the square of the error by a factor which is a function of $t = k\lambda$. This factor, which is zero for $t=0$ and one for $t=\infty$, is tabulated by the author for $t = 0.1(0.1)0.8$. He warns that in applying this method, the number of observations not discarded must be large.

**Citations:** Gore (1889), Czuber (1891a), Pizzetti (1892), Wölffing (1899), Wellisch (1909).

* Bertrand, J. (1888c). Sur la valeur probable des erreurs les plus petites dans une série d'observations. *Comptes Rendus de l'Académie des Sciences de Paris* **106**, 786-788. (JFM 20, 223).

**Summary:** If k [$= 1/\sigma\sqrt{2}$] is the characteristic constant [of a normal distribution], the probability of an error between z and z + dz is $(k/\sqrt{\pi})\exp(-k^2z^2)dz$, and the probable value of the smallest error in n

observations is given by $(2k/\sqrt{\pi})\int_0^\infty ze^{-k^2z^2}dz[1-\Theta(kz)]^{n-1}$, where $\Theta(t)=(2/\sqrt{\pi})\int_0^t e^{-t^2}dt$. When n is large, the probable value of the smallest error approaches $1/(n+1)k$, that of the second smallest $2/(n+1)k$, that of the third smallest $3/(n+1)k$ . . . .

**Citations:** Czuber (1891a), Pizzetti (1892), Wölffing (1899).

* Delauney, C. (1888). Sur un théorème relatif aux écarts du tir. *Comptes Rendus de l'Académie des Sciences* (Paris) **106**, 515.

**Summary:** The author calls attention to a theorem concerning shooting errors stated by Bréger (1876). If one groups two by two, in the order in which they occur, the successive errors, the mean of the larger errors in each pair will be in the ratio $\sqrt{2}$: 1 to the general mean. The author obtained for 33 series the numerical value 1.41 for this ratio. Bréger showed that the accepted law of probability assigns, for a large number of trials, a ratio equal to $\sqrt{2}$. This theorem is one of those stated by Bertrand (1887a).

**References:** Bréger (1876), Bertrand (1887a).

**Citation:** Czuber (1891a).

* Edgeworth, F. Y. (1888). On a new method of reducing observations relating to several quantities. *Philosophical Magazine* (5) **25**, 184-191. (JFM 20, 219).

**Summary:** The author begins with a restatement of the method he proposed a year earlier [Edgeworth (1887b,c)]: "A substitute for the *Method of Least Squares* has been proposed by me, based upon the following principle. The data being of the form $a_1x + b_1y \ldots - v_1 = 0$; $a_2x + b_2y \ldots - v_2 = 0$, &c., (where $v_1, v_2$, &c. are observations of equal worth), a solution is obtainable by taking x,y . . . such that the sum of the residuals (the left-hand members of the above written equations), *each residual taken positively*, should be a minimum." (p. 184). In a footnote he writes (pp. 184-185): "This rule is derivable from the hypothesis that the law of error, the facility-curve under which the observations range, is of the form $y=(h/2)e^{-hx}$, x taken positively in both directions [Laplace's first law of error; see Laplace (1774)]. But the use of the rule does not commit us to the assumption of the hypothesis. The Method of Least Sum is in this respect exactly on a par with the Method of Least Squares. The rule of the latter Method is—Determine x and y so that the *sum of the squares* of the residuals may be the least possible. This rule is derivable from, and specially correlated with, the hypothesis that the law of facility is the [normal] Probability-curve. But it is thought legitimate by Laplace and other eminent authorities to employ the rule even where the hypothesis is not assumed. No doubt the use of either method divorced from the law of facility appropriate to it is open to logical objections. But the difficulties are not greater for one method than for the other." The author continues (pp. 185-186): "The point thus designated must be on each of two, or more, loci analogous to the Normal equations of the ordinary method. Accordingly the intersection of the 'Median loci' was at first proposed by me as the solution. But Mr. Turner [(1887)] has shown that these loci are apt to have in common, not only several points, but even lines and spaces. . . . In this event common sense teaches that we should adopt the *middle* of the indeterminate tract as the best point; and this presumption is confirmed by a formal calculation of utility such as Laplace [(1812)], in the simplest case of a single unknown quantity, has employed to discover the 'most advantageous point.' " Having thus disposed of one of Turner's criticisms, Edgeworth endeavors (somewhat less successfully, it seems to the compiler) to answer the other, namely that the new method is *not* less laborious than the method of least squares, as Edgeworth (1887b,c) had asserted.

**Comments:** Eisenhart (1961) has pointed out (p. 208) that the method proposed by Edgeworth in this paper results from dropping Boscovich's Condition I [see Boscovich (1757, 1760)].

**References:** Laplace (1812), Edgeworth (1887b,c,f), Turner (1887).

**Citations:** Pizzetti (1892), Czuber (1899), Edgeworth (1911, 1923), Whittaker & Robinson (1924), Bowley (1928), Rhodes (1930), Bruen (1938).

* Faye, H. E. (1888). Sur certains points de la théorie des erreurs accidentelles. *Comptes Rendus de l'Académie des Sciences de Paris* **106**, 783-786. (JFM 20, 223).

**Summary:** The author points out that the greatest discrepancy from the mean of a set of observations is not likely to be counterbalanced by a discrepancy of nearly the same magnitude but of opposite sign, and hence unless the number of observations in the set is very large it is likely to have an undue influence on the arithmetic mean, so that the arithmetic mean of all the observations is not the most probable value. Given the forty observations 2.28, 2.48, 2.64, 2.66, 2.75, 2.81, 2.95, 2.98, 3.11, 3.22, 3.26, 3.27, 3.28, 3.43, 3.68, 3.76, 3.78, 3.78, 3.91, 3.95, 3.98, 4.08, 4.10, 4.15, 4.18, 4.21, 4.43, 4.43, 4.45, 4.49, 4.51, 4.59, 4.65, 4.76, 4.84, 5.08, 5.21, 5.23, 5.48, 6.35, he computes the arithmetic means of the largest and smallest observations, the second largest and second smallest, and so on (the midrange and the quasi-midranges). The former has the value 4.315 and the latter range from 3.815 to 3.995. He attributes this discrepancy to the fact that the largest deviation from the arithmetic mean, 6.35 − 3.93 = 2.42, is not matched by a comparable deviation in the opposite direction. He therefore recommends rejecting the largest observation, which changes the arithmetic mean from 3.93 to 3.87, a value which he considers to be more probable. In general, however, he holds that observations should be rejected only if they are considered doubtful at the time they are made or at least before any computations have been made. Otherwise the calculator can too easily make the results agree with his preconceived and sometimes erroneous opinion.

**Reference:** Gauss (1809).

**Citations:** Czuber (1891a, 1899), Wellisch (1909), Rider (1933).

* Galton, Francis (1888). Co-relations and their measurement, chiefly from anthropometric data. *Proceedings of the Royal Society of London* **45**, 135-145.

**Summary:** This paper, as the title indicates, deals primarily with the measurement of correlation. Incidentally, however, it deals with the median as a measure of central tendency and the quartile deviation as a measure of variability. The author writes (p. 137): "After marshalling the measures of each limb in the order of their magnitudes, I noted the measures in each series that occupied respectively the positions of the first, second, and third quarterly divisions. Calling these measures in any one series, $Q_1$, M, and $Q_3$, I take M, which is the median or middlemost value, as that whence the deviations are to be measured, and $\frac{1}{2}\{Q_3 - Q_1\} = Q$, as the probable error of any single measure in the series. This is practically the same as saying that one-half of the deviations fall within the distance of $\pm$ Q from the mean value, because the series run with fair symmetry."

**Citations:** Edgeworth (1893), Keynes (1921), Rietz (1924), Snedecor (1937).

* Jordan, W. (1888). Ueber günstigste Gewichtsvertheilung. *Zeitschrift für Vermessungswesen* **17**, 641-649. (JFM 20, 1232).

**Summary:** The author makes an unsuccessful attempt to prove, in a different way, the theorem proved by Schreiber (1882).

**Reference:** Schreiber (1882).

**Citations:** Runge (1890), Friedrich (1937).

* Bertrand, J. (1889). *Calcul des Probabilités*. Gauthier-Villars et Fils, Paris. (JFM 21, 198-199).

**Summary:** In this book the author repeats and elaborates on several relevant results contained in some of his earlier papers. In Articles 156 and 157, he shows that the probable magnitude of the larger of two errors of observation from a normal population is $\sqrt{2}$ times the probable error of a single observation, and hence that the probable magnitude of the smaller of the two errors of observation is $(2-\sqrt{2})$ times that of a single observation. Thus the ratio of the probable larger error to the probable smaller error is $1 + \sqrt{2}$, as stated without proof by the author in an earlier paper [Bertrand (1887a)]. In Articles 158 and 159, he proves the following result, which he had given without proof in another paper [Bertrand (1887b)]: If observations from a normal population are divided at random into groups of two or three, the ratio of the probable value of the square of the largest error to the probable value of the square of the error of a single observation is $1 + 2/\pi$ or $1 + 2\sqrt{3}/\pi$, respectively. In Article 166, the author inquires whether it is permissible to discard measurements rendered suspect by their differences from the mean. On page 212 he writes: "It is incontestable that, if one is certain to have operated in the same manner and with the same care, all the observations have the same right to exercise their influence upon the mean adopted. But if one operates always in the same way, one obtains always the same result." (Compiler's translation). Having said this, he proceeds to examine the consequences of discarding measurements presumed to be the less good ones. If the probability of an error z is regarded with confidence as equal to $(k/\sqrt{\pi}) \exp[-k^2z^2]dz$, he calculates the error $\lambda$ such that the probability that an error shall be less than $\lambda$ will have an arbitrarily chosen value p, for which it suffices to solve the equation $p = (2/\sqrt{\pi}) \int_0^{k\lambda} e^{-t^2}dt = \Theta(k\lambda)$, where $\Theta$ is the error function. Having thus determined $\lambda$, he suppresses among the n observations those whose difference from the mean is greater than $\lambda$, as proposed in an earlier paper by the author [Bertrand (1888b)]. In Articles 167 and 168 he repeats a demonstration given in an earlier paper [Bertrand (1888c)] that for normally distributed errors, the probable magnitudes approached by the smallest, second smallest, third smallest, . . . of n errors (n large) are $1/[(n+1)k]$, $2/[(n+1)k]$, $3/[(n+1)k]$, . . ., where $k = 1/\sigma\sqrt{2}$, $\sigma$ being the population standard deviation. In Article 169, he points out that the probable magnitude of the smallest error is less than the probable error of the mean, which is of the order $1/\sqrt{n}$.

**Citations:** Estienne (1890), Czuber (1891a), Pizzetti (1892), Czuber (1899), Kozák (1908-10), Goedseels (1909), Edgeworth (1911), Dodd (1913), Contarino (1914), Dodd (1914), Keynes (1921), von Bortkiewicz (1922b), Coolidge (1925), Ogrodnikoff (1928), Lidstone (1930), de Finetti (1932), Münzner (1934), Gumbel (1935a), Calichiopulo (1937), Fréchet (1941), Boldrini (1942), Gumbel (1943c), Kendall (1946).

* Galton, Francis (1889). *Natural Inheritance*. Macmillan and Co., London-New York.

**Summary:** Chapters IV and V contain material relevant to a study of order statistics. Chapter IV deals with Schemes of Distribution and of Frequency and Chapter V with Normal Variability. The former contains a discussion of properties of the median and the quartile deviation for distributions in general, and the latter gives a similar discussion of their properties for samples from a normal distribution. Much is also said about (centesimal) Grades or percentiles. The following excerpts are of special interest: "The median, M, has three properties. The first follows immediately from its construction, namely, that the chance is an equal one, of any previously unknown measure in the group exceeding or falling short of M. The second is, that the most probable value of any previously unknown measure in the group is M. . . . The third property is that whenever the curve of the Scheme is symmetrically disposed on either side of M . . ., then M is identical with the ordinary Arithmetic Mean or Average." (p. 41). "As the M [median] measures the Average Height of the curved boundary of a Scheme, so the Q [quartile deviation] measures its general slope. . . . Our Q has the further merit of being practically the same as the value which mathematicians call the 'Probable Error' . . . ." (p. 53). "If we know the value of M as well as that of Q [for a Normal Scheme of Measures] we know the entire Scheme. M expresses the mean value of all the objects contained in the group and Q defines their variability. . . . If the Measures at any *two* specified Grades are given, the whole Scheme of Measures is thereby determined. Let A, B be the two given

Measures of which A is the larger, and let a, b be the values of the tabular Deviations [in units of the Probable Error] for the same Grades . . . not omitting their signs . . . . Then the Q of the Scheme = $(A-B)/(a-b)$. . . . $M=A-aQ$; or $M=B-bQ$." (pp. 61-62).

**Reference:** Galton (1869).

**Citations:** Pearson (1895), Galton (1896, 1899a), Sheppard (1899a), Bowley (1901), Galton (1902), Kapteyn (1903), Gini (1909), Charlier (1910), Edgeworth (1911), Weinberg (1916), Gini (1917), Niceforo (1919), Czuber (1921), Julin (1921), Rietz (1924), Whittaker & Robinson (1924), Julin (1933), Snedecor (1937).

* Gore, James Howard (1889). A bibliography of geodesy. *Report of the U. S. Coast and Geodetic Survey for 1887*, Appendix No. 16 (pp. 313-512). Second edition (1903), *Report of the U. S. Coast and Geodetic Survey for 1902*, Appendix No. 8 (pp. 427-787). U. S. Government Printing Office, Washington.

**Summary:** The author gives an extensive (200-page) bibliography of geodesy, which includes numerous publications on the theory of errors and the adjustment of observations, especially those in which the theory is applied to the determination of the ellipticity of the earth. The second edition is even more extensive (361 pages).

**References:** Maire & Boscovich (1755) [1770], Boscovich (1760), Lalande (1771), Laplace (1786, 1793), Laplace (1799), Puissant (1805) [1819], Svanberg (1805), Delambre (1806-10), von Lindenau (1806), Laplace (1811a,b, 1812), Delambre (1813), Mathieu (1813-14), Legendre (1814), Gauss (1816), Adrain (1818), Gauss (1823), Ivory (1825), Hauber (1830, 1830-32), Hagen (1837) [1882], Bessel & Baeyer (1838), Stampfer (1839), Cournot (1843) [1849], Gerling (1843), Ellis (1844), Gould (1855), Airy (1861), Chauvenet (1863), De Morgan (1864), Jordan (1869), Zachariae (1871), von Andrae (1872), Glaisher (1872), Helmert (1872), Jordan (1872), Glaisher (1873), Peirce (1873), Stone (1873a), Todhunter (1873), Fechner (1874), Helmert (1875a,b), Laurent (1875), Mees (1875), Helmert (1876a,b), Mees (1876), Helmert (1877a,b), Jordan (1877b), Merriman (1877), Schreiber (1882), Edgeworth (1883b), Doolittle (1884), Merriman (1884), Wright (1884), Newcomb (1886), Bertrand (1887b, 1888a,b), {Pizzetti (1889), Czuber (1890), Estienne (1890), Jordan (1890), Czuber (1891a,b), Goedseels & Mansion (1893), Poincaré (1896), Blümcke (1897, 1898), Czuber (1899)}.

**Citation:** Pizzetti (1892).

* Pizzetti, P. (1889). Alcune ricerche sulla probabilità a priori degli errori d'osservazione. *Giornale di Matematiche di Battaglini* (Naples) **27**, 77-89. (JFM 21, 220-222).

**Summary:** The author extends certain results given by Laplace (1812). Let $h=1/\sigma\sqrt{2}$ (where $\sigma$ is the population standard deviation) be a measure of the precision of a single observation. Then the corresponding measures of precision for the arithmetic mean of a sample of $(2m+1)$ observations, each with precision h, from a normal population and for the $(m+1)$st ordered observation (the median) are respectively $H = h(1+2m)^{1/2}$ and $H' = h(1+4m/\pi)^{1/2}$. As $m\to\infty$, $H/H' \to 2/(2\pi)^{1/2} \doteq 4/5$. The author shows further that $H''$, the precision of the best quasi-midrange (the arithmetic mean of values in a large sample of size n which are symmetrically placed at approximately the 27th and 73rd percentiles of the sample) is given by $H'' = h(.80964n)^{1/2} \doteq .9h/\sqrt{n}$ as compared with $H = h/\sqrt{n}$ for the arithmetic mean, so that $\lim_{n\to\infty} H/H'' \doteq .9$.

**Comments:** The author incorrectly gives .27n and .73(n+1) as the ranks of the 27th and 73rd percentiles in the sample of size n. The former should be given as .27(n+1). The error is, of course, negligible for sufficiently large n.

**Reference:** Laplace (1812).

**Citations:** Gore (1889) [1903], Pizzetti (1892), Czuber (1899).

* Czuber, E. (1890). Bemerkung über die wahrscheinlichsten Werte beobachteter Grössen. *Archiv der Mathematik und Physik* (2) **9**, 97-101. (JFM 22, 236-237).

**Summary:** Let $\phi(x)$ be the law of error and let $x_1,x_2,\ldots,x_n$ be the errors of n observations of a quantity which is a function of m unknown elements p,q,r,.... Then the most probable system of values (maximum likelihood estimates) of p,q,r,... will be obtained from the condition that $\Omega=\phi(x_1)\phi(x_2)\ldots\phi(x_n)$ be a maximum. The author points out, however, that there are three conditions under which the usual method of finding the maximum likelihood estimates, based on solving likelihood equations formed by equating to zero derivatives of the likelihood function $\Omega$, fails: (1) The properties of $\phi$ may be such that in general $\Omega$ assumes no extremum (i.e., $\Omega$ may be a constant), in which case the estimates found by solving the likelihood equations are illusory; (2) $\Omega$ may have only a minimum (no maximum), in which case the solutions of the likelihood equations correspond to minimum rather than maximum likelihood; and (3) $\Omega$ may reach a largest value which is not a true maximum (i.e., it may be either a monotone increasing or a monotone decreasing function of one or more of the elements p,q,r, ...), in which case the largest or smallest observation may be the value sought.

**Comments:** The author attributes the method of maximum likelihood to Gauss (1809), but we have already noted its use by Lambert (1760) and by Daniel Bernoulli (1778).

**Note:** The above summary and comments were written before the compiler (at the suggestion of Dr. Joseph Berkson) read the paper by Edwards (1974), in which he discusses the distinction between the method of maximum likelihood and the method of maximum probability. See the compiler's note on the paper by Lagrange (1774). In retrospect, it is clear that Czuber's method is actually the method of maximum probability, since he states in a footnote that he assumed that all values of the unknown parameter are *a priori* equally probable. Except for the need to make that assumption in the case of one method but not in the other, the two methods are essentially the same and yield identical results.

**Reference:** Gauss (1809).

**Citations:** Gore (1889) [1903)], Czuber (1891a).

* Estienne, J. E. (1890). Étude sur les erreurs d'observation. *Revue d'Artillerie* **36**, 235-259. (JFM 22, 1199-1200).

**Summary:** The author states at the outset (p. 235): "The present memoir has as object to prove that the best value to adopt, as a measure of a quantity of which experiment has furnished values tainted by accidental errors, is, *in every case*, the *median* value, furnished by the following rule: One arranges in order of magnitude the values obtained; when their number is odd, the value of the middle one is the median value; when their number is even, one has as median value the two middle ones and every intermediate value". (Compiler's translation). In Chapter I, entitled "Demonstration of the rule", the author proves the following theorem (p. 241): "The most probable value, determined by the rule of median value, is that for which the arithmetic sum of the deviations is a minimum." Chapter II is devoted to the proposition that the rule of the median value is independent of the law of errors and should be applied to the exclusion of every other rule, whatever be this law. Chapter III deals with consequences of the law of the median value, of which the author gives several, including the use of the method of least first powers in solving n inconsistent equations in m unknowns (n>m). An example dealing with artillery fire is given in the appendix.

**References:** Laplace (1799), Gauss (1809?), Bertrand (1889).

**Citations:** Gore (1889) [1903], Czuber (1891a,b), Mendeleev (1895), Estienne (1926-27).

* Jordan, W. (1890). Bestimmung eines Maximalfehlers. *Zeitschrift für Vermessungswesen* **19**, 559-569. (JFM 22, 1201).

**Summary:** The author points out that it is impossible to estimate the maximum error from actual measurements because one cannot be sure that very large deviations have not resulted from gross blunders rather than from errors in the usual sense. To obtain data not subject to this objection, he records the number of zeros in the sixth decimal place of 50 seven-place logarithms in each of the 1800 columns in the Vega-Bremiker table of log 10000, log 10001, . . ., log 99999 and studies the distribution of the number of times 0,1,2,3, . . ., 24 zeros occur. He compares this distribution (obviously asymmetric!) with a Gaussian distribution with mean 5 (the theoretical value under the hypothesis that each of the ten digits is equally likely) and with a theoretical algebraic function [see Jordan (1877a,b)] having finite limits. He computes two different values of the mean error m (standard deviation $\sigma$), $m(+) = 2.310$ and $m(-) = 1.819$, for values above and below the mean. Far from paying any attention to the criticism by Helmert (1877b) of the conclusion stated in his earlier publications that the maximum error is $3\sigma$, he now concludes that it is actually $2.75\sigma$, on the dubious ground that the smallest value (0) is only 2.75 $m(-)$ below the mean (5), even though by his own figures the largest value (24) is 8.23 $m(+)$ above the mean.

**Reference:** Jordan (1877b) [1888].

**Citations:** Gore (1889) [1903], Czuber (1899), Vogeler (1907), von Bortkiewicz (1922b).

* Runge, C. (1890). Die Schreiber'sche Satz. *Zeitschrift für Vermessungswesen* **19**, 20-24. (JFM 22, 1203-1204).

**Summary:** The author, using the notation of Jordan (1888) and the approach tried by Jordan, succeeds in supplying an elegant proof of the theorem first proved by Schreiber (1882). This proof, however, requires three pages as compared with only half a page for Schreiber's proof.

**References:** Schreiber (1882), Jordan (1888).

**Citation:** Friedrich (1937).

* van den Berg, F. J. (1891). Over de kans dat, bij willekeurige verdeeling van eene gegeven rechte lijn, uit de segmenten gesloten veelhoeken kunnen worden gevormd. *Nieuw Archief voor Wiskunde* **18**, 63-118. (JFM 23, 234).

**Summary:** The author sets out to determine the probability that for a random division of a straight line into m segments, a closed polygon with $m-n+2$ sides can be formed from every group of $m-n+2$ segments. In order for this to be possible it is necessary that the sum of $m-n+1$ randomly chosen segments (and hence the sum of the $m-n+1$ shortest segments) be always greater than each of the other $n-1$ segments. This is a generalization of problems considered by Lemoine (1873) and Halphen (1873). Actually the author solves a still more general problem; he determines the probability that the sum of the $m-n+1$ shortest segments is greater than each of p of the remaining segments and less than each of the $n-p-1$ others, which reduces to his original problem in the special case in which $p=n-1$ (and hence $n-p-1=0$). He derives a formula for this probability $K_{m,n,p}$, which is given by

$$K_{m,n,p} = [m!/n!)\binom{n-1}{p}\sum_{k=n-p}^{n}(-1)^{-n+p+k}\binom{p}{n-k}/\{(n+k)(n+2k)\ldots[n+(m-n)k]\}.$$

An equivalent formula which is more convenient in some cases is given by

$$K_{m,n,p} = [m!/(m-n-1)!(n-p-1)!n^{m-n}]\sum_{i=1}^{m-n}(-1)^{m-n-i}\binom{m-n-1}{i-1}i^{m-n+p-1}/$$
$$\{[n+(n-p)i][n+(n-p+1)i]\ldots(n+ni)\}.$$

The author corrects an error in a paper by van Pesch (1882), who gave 4/27 instead of 1/15 (the correct value) for the probability in the case $m=4$, $n=3$, $p=2$.

**References:** Halphen (1873), Lemoine (1873), van Pesch (1882, 1884).

**Citations:** Czuber (1899), Mantel (1915).

* Czuber, Emanuel (1891a). *Theorie der Beobachtungsfehler*. B. G. Teubner, Leipzig. (JFM 23, 235-237).

**Summary:** This book is divided into three parts, dealing with the theory of linear observational errors, the method of least squares, and the theory of errors in the plane and in space. Only the first part is relevant to a study of order statistics. The first six sections (pp. 1-113) deal with laws of error [with particular emphasis on the normal (Gaussian) law, though Laplace's first law and others are mentioned]; the early work of Simpson (1756, 1757) and Lagrange (1774) on the advantages of taking averages, of Laplace (1774, 1781) on his "most advantageous method", and of Daniel Bernoulli (1778) on the method of maximum likelihood; the work of Legendre (1805), Adrain (1808), Gauss (1809), Laplace (1812), and later writers on the method of least squares; and the problem of the choice of means and its relation to the choice between the method of least squares and rival methods. Section 6 closes with a historical notice in which the work of Glaisher (1872) and Estienne (1890) on the median is discussed. Section 7 deals with the estimation of the precision of a series of observations from the true errors; of particular interest is Article 59 (pp. 141-145), which deals with the work of Gauss (1816) and Encke (1832-34) on the probable error of the median absolute error. Section 8 deals with estimation of precision from apparent errors; Arts. 73-75 (pp. 174-181) discuss formulas given by Jordan (1869), von Andrae (1869, 1872), and Helmert (1876a) based on differences of pairs of ordered observations and Art. 76 (pp. 181-182) compares these with other formulas. Section 9 deals with comparison of the error law with experience; Article 84 (pp. 199-202) summarizes the work of Bertrand (1887a,b) and Delauney (1888) on pairs of errors chosen at random and then ordered. Section 10 (pp. 202-211) deals with the smallest and largest errors in a set of observations. The author states that the probable value of the smallest error in a large number of observations is easily found [see Bertrand (1888c)], but that that of the largest error is more difficult. He gives a table of the ratio of the largest error M to the probable error $\mu$ in a sample of size n from a Gaussian population, based on the assumption that it corresponds to the value $(n-1)/n$ for the c.d.f., for n=20(20)100, 500, 1000, 5000, . . ., 5000000, 10000000. He refers to earlier work of Fourier (1824a), Helmert (1877b), and Jordan (1877a,b). Section 11 (pp. 211-231) deals with the treatment of outlying observations, reviewing the contributions of Svanberg [Anonymous (1821)], Bessel & Baeyer (1838), De Morgan (1847), Peirce (1852), Gould (1855), Airy (1856), Winlock (1856), Chauvenet (1863), Stone (1868), Glaisher (1873), Bertrand (1888b, 1889), and Faye (1888).

**References:** Simpson (1756, 1757), Lagrange (1774), Laplace (1774), D. Bernoulli (1778), Euler (1778), Laplace (1781), J. Bernoulli (1785), Trembley (1804), Legendre (1805), Adrain (1808), Gauss (1809), Laplace (1812), Gauss (1816), Anonymous (1821), Fourier (1824a), Encke (1832-34), Ellis (1844), Quetelet (1846), De Morgan (1847), Herschel (1850), Peirce (1852), Gould (1855), Airy (1856), Winlock (1856), Chauvenet (1863), Stone (1868), von Andrae (1869), Jordan (1869), von Andrae (1872), Glaisher (1872, 1873), Stone (1873a), Fechner (1874), Helmert (1876a, 1877b), Jordan (1877a,b), Bréger (1881), Bertrand (1887a,b, 1888a,b,c), Delauney (1888), Faye (1888), Bertrand (1889), Czuber (1890), Estienne (1890).

**Citations:** Helmert (1872) [1907], Gore (1889) [1903], Cranz (1896), Czuber (1899, 1903), Edgeworth (1905), Kozák (1907, 1908-10), Wellisch (1909), Edgeworth (1911), Dodd (1913), Weinberg (1916), Keynes (1921), von Bortkiewicz (1922b), Dodd (1922), Ogrodnikoff (1928), von Bortkiewicz (1931), Craig (1932a), Rider (1933), von Eberhard (1938), Arley & Buch (1940), Fréchet (1941), Gaede (1942), Deming (1943).

* Czuber, Emanuel (1891b). Über ein Ausgleichungsprincip. *Technische Blätter* **23**, 1-9. (JFM 23, 1207).

**Summary:** The author states that his purpose is to examine the rule proposed by Estienne (1890) and its consequences, especially the adjustment principle following from it, in relation to the theory of errors.

He restates Estienne's rule as follows (p. 1): "One orders the observations according to their magnitude; if their number is odd, then one chooses the middle one; if their number is even, then the two middle ones and every value between them have an equal right to be chosen. In the sequel we will call the value so determined the median [Medianwert]". He points out that choice of the median leads to an adjustment principle (minimizing the sum of the absolute values of the errors) which was advanced by Laplace before the establishment of the method of least squares. He notes, however, that Laplace [(1799), Livre III, No. 40] minimized the sum of the absolute values of the errors subject to the condition that their algebraic sum be zero; in other words, the "errors" were taken as deviations from the arithmetic mean, not from the median. But Gauss (1809), in his first "proof" of the method of least squares, showed that if one assumes that the arithmetic mean is the probable value, it follows that the law of errors is Gaussian and that one should minimize the sum of the squares of the errors, not the sum of their absolute values. The author goes on to show that the median and the method of least absolute first powers, far from being valid whatever the law of errors, as asserted by Estienne, are, as pointed out by Glaisher (1872), tied just as firmly to the first law of error of Laplace (1774) as are the arithmetic mean and the method of least squares to Laplace's second (Gauss') law. In conclusion, he states (p. 9): "According to the foregoing, one can regard Estienne's investigation as an interesting study, since it has many points of contact with the older works in the area of theory of errors. His rule of the median and the resulting adjustment principle of the least arithmetic sum cannot, however, alter the position of the rule of the arithmetic mean and the method of least squares."

**References:** Laplace (1774, 1799), Legendre (1805), Gauss (1809), Glaisher (1872), Estienne (1890).

**Citation:** Gore (1889) [1903].

* Venn, John (1891). On the nature and uses of averages. *Journal of the Royal Statistical Society* **54**, 429-448; discussion, 448-456.

**Summary:** The author discusses various averages, including arithmetic mean, median, mode, mid-range, and geometric mean, as well as criteria for choosing among them. He summarizes his conclusions as follows (pp. 447-448): "Every sort of average—and there are many such sorts—is a single fictitious substitute of our own for the plurality of actual values existent in the results which are naturally or artificially set before us. It is impossible, therefore, for the former, in any case, effectually to take the place of the latter. But the extent to which it may succeed or fail in doing so will depend upon the nature of the facts presented to us, and still more upon the precise object we have in view. Three cases may be suggested:—(1) If we are only concerned with comparative results, and these presumably form the bulk of statistical inquiry, then almost any kind of average will answer the purpose. (2) If we want accurate quantitative results, then the selection of the kind of average is no longer at our disposal, but must depend upon the precise object we have in view. The common arithmetic mean *may* be the best, and probably in most cases is so, but circumstances can easily be suggested in which some other kind of mean becomes not merely more convenient but distinctly more accurate. (3) There is the well known class of cases, of which the Science of Mensuration offers the most familiar examples, in which the 'mean' is resorted to in order to discover what is regarded, accurately or metaphorically, as the true value. Without in the least wishing to criticise the customary methods and conclusions, I have ventured to suggest that it may depend upon the object we have in view, not merely what kind of average should be selected, but whether there is a necessary gain in employing any kind of average at all." Contributors to the discussion include Galton and Edgeworth, the latter presenting the case for use of the median.

**References:** Venn (1866), Edgeworth (1886a).

**Citations:** Julin (1921), Keynes (1921), Rietz (1924).

* Holmes, George K. (1892). Measures of distribution. *Publications [Journal] of the American Statistical Association* **3** (18-19), 141-157.

**Summary:** The author proposes a triple measure of distribution of wealth which includes the percentage of the population owning wealth, the average holding of those owning wealth, and the difference between the median wealth according to the number of owners (the usual median) and the median wealth according to the amount of wealth (i.e., the amount of wealth such that the total wealth held by owners holding less than this amount and the total wealth held by those holding more than this amount are equal).

**Citations:** Holmes (1905), Lorenz (1905), von Bortkiewicz (1931).

* Pizzetti, P. (1892). I fondamenti matematici per la critica dei risultati sperimentali. *Atti della Regia Università di Genova* **11**, 113-333. (JFM 24, 204-205); review by V. Reina, *Nuovo Cimento* (3) **33** (1893), 43-48.

**Summary:** The author gives few if any new results, but this paper is a very useful summary of work to date on the theory of errors, with a bibliography of 503 items, many of which are relevant to a study of order statistics. Reina lists 11 additional works on the subject in his review.

**References:** Cotes (1722), Mayer (1750), [Maire &] Boscovich (1755), Simpson (1757), Boscovich (1760), Lambert (1760, 1765b), Lagrange (1774), Laplace (1774), D. Bernoulli (1778), Laplace (1781), J. Bernoulli (1785), Laplace (1793), Trembley (1804), Legendre (1805), Puissant (1805), von Lindenau (1806), Adrain (1808), Gauss (1809), Laplace (1810, 1811a,b, 1812), Legendre (1814), van Beeck Calkoen (1816), Gauss (1816), Adrain (1818), Gauss (1823), Fourier (1824a), Ivory (1825), Fourier (1826a), Hauber (1830, 1830-32), Cauchy (1831), Encke (1832-34), Cauchy (1837), Hagen (1837), Bessel & Baeyer (1838), Cournot (1843), Gerling (1843), Ellis (1844), Quetelet (1846), De Morgan (1847), Herschel (1850), Peirce (1852), Gould (1855), Lloyd (1855), Airy (1856), Winlock (1856), Petzval (1857), von Andrae (1860), Airy (1861), Chauvenet (1863), De Morgan (1864), Todhunter (1865), Venn (1866), Stone (1868), von Andrae (1869), Jordan (1869), Zachariae (1871), von Andrae (1872), Glaisher (1872), Helmert (1872), Jordan (1872), Glaisher (1873), Peirce (1873), Stone (1873a,b), Fechner (1874), Jevons (1874), Stone (1874), Galton (1875), Helmert (1875a,b), Laurent (1875), Mees (1875), Helmert (1876a,b), Mees (1876), Helmert (1877b), Jordan (1877b), Merriman (1877), Wilson (1877), Bréger (1881), Edgeworth (1883a,b), Doolittle (1884), Wright (1884), Edgeworth (1886a), Newcomb (1886), Edgeworth (1887 a,c,d,e), Turner (1887), Bertrand (1888a,b,c), Edgeworth (1888), Bertrand (1889), Gore (1889), Pizzetti (1889).

**Citations:** Czuber (1899, 1903), Keynes (1921), Whittaker & Robinson (1924), Deming (1943).

* Edgeworth, F. Y. (1893). Exercises in the calculation of errors. *Philosophical Magazine* (5) **36**, 98-111. (JFM 25, 356).

**Summary:** The author (page 111) gives the following summary: "We have estimated the error incident to each coefficient employed in determining the correlation between organs. We have shown how this error becomes greater with the number of the organs; the instability of the construction increases rapidly with its height. In reaching these conclusions respecting correlated averages, we have come upon two principles of wider application. (1) When observations are combined according to a system of weights different from that which is known to be best, it is in general advantageous to *reject* a certain class of the given observations. (2) When, as usual, the observations range under a [normal] probability curve, the median m corrected by the quartiles $q_1$ and $q_2$ affords a formula for the Mean, viz. $(1.2m + q_1 + q_2) \div 3.2$, which is more accurate than that method of combining such observations which has hitherto been supposed to be the most accurate, viz. the Arithmetic Mean. The principle may be applied with great ease and advantage to Discordant Observations."

**Comments:** The second principle stated by the author is incorrect. The source of his error lies in the assumption that the quartiles are independent of each other and of the median, whereas they are actually considerably correlated, as Pearson (1920) has pointed out.

**References:** Edgeworth (1887a,c,d,e), Galton (1888).

**Citations:** Sheppard (1899a), Bowley (1901), Pearson (1920), Bowley (1928).

* Goedseels, E.; Mansion, P. (1893). Discussion sur la théorie des erreurs. *Annales de la Société Scientifique de Bruxelles* **17** (1), 52-53. (JFM 25, 351).

**Summary:** Goedseels observes that in reality no one has ever established the method of least squares in an absolutely conclusive manner, and that one should avoid assigning too great objective value to the results to which it leads. Mansion is of the same opinion. In reality, the only definition which one can give of accidental errors is this: They are the errors which are eliminated by the method of least squares. But it is just to recall that Gauss took care to express himself with precision on what is arbitrary in the theory to which he gave such a perfect form. The great advantage of the method of least squares is that it allows the combination, in a simple and reasonable manner, of the results of observations of unequal value in a condensed form.

**Comments:** This discussion is of interest because the budding dissatisfaction with the method of least squares which is expressed here later led to substantial contributions by the authors [see, for example, Mansion (1906), Goedseels (1909, 1911), and Mansion (1913)] to the theory of rival methods, some of which are based on order statistics.

**Citations:** Gore (1889) [1903], Wölffing (1899).

* Vallier, E. (1894). *Balistique Experimentale*. Publisher unknown, Paris.

**Summary:** Kozák (1908-10) writes (p. 580, compiler's translation): "Vallier [see Cranz (1896), p. 312] altered the rule of Chauvenet by determining the argument t from $\Phi(t) = (n^2 - 1)/n^2$ [Stone's first criterion for $m = n^2$]; the deviation at which the rejection begins is thus a =. . . = rt/0.4769 [where r = the probable error]."

**Note:** The compiler has been unable to obtain a copy of this book, and so has had to rely solely on information about it given by Cranz and by Kozák.

**References:** Chauvenet (1863), Stone (1868).

**Citations:** Cranz (1896), Heydenreich (1903), Kozák (1908-10).

* Mendeleev, D(mitri) I(vanovich) (1895). Course of work on the renewal of prototypes or standard measures of lengths and weights. (Russian). *Vremennik Glavnoi Palaty Mer i Vesov* **2**, 157-185. Reprinted 1950 in *Collected Writings (Sochineniya)*, Vol. 22, pp. 175-213. Izdatel'stvo Akademii Nauk SSSR, Leningrad-Moscow.

**Summary:** The author (p. 209 of reprint) states the end result that the length of platinum-iridium prototype meter No. 28 (assigned by lot to Russia) is $1\ M + 0.50\mu$. This result is based on 11 determinations (tabulated on p. 208) of the form $1\ M + n\mu$, where n takes the values 0.61, 0.44, 0.64, 0.65, 0.49, 0.29, 0.29, 0.45, 0.56, 0.42, 0.68. The author comments (footnote, p. 209): "According to the 'median rule' given by Mr. Estienne (in 1890), the most probable conclusion drawn from a set of data corresponds not to the overall arithmetic mean, but rather to an average real quantity based on a number of observations,

which is found, by arranging all the results in order of their numerical magnitudes. Thus, in the above example, having discarded 4 of the lowest values (from 0.29 to 0.45) and 4 of the highest values (from 0.61 to 0.68) we are left with three medium values, namely, these values in order of magnitude: 0.45, 0.49, 0.56, which give the probable value + 0.49$\mu$. Here this value is in complete agreement with the arithmetic mean, and this indicates that the errors follow a definite law, used by the Gaussian probability theory, i.e., that the observations do not contain gross accidental errors, but are rather determined by the unavoidable errors of observation. I would not consider it out of place to point out that, on my part, I use a different method to evaluate the harmony of a series of observations that must give identical numbers, namely I divide all the numbers into three, if possible equal, groups (if the number of observations is not divisible by three, the greatest number is left in the middle group): those of greatest magnitude, those of medium magnitude, and those of smallest magnitude; the mean of the middle group is considered the most probable (here it is equal to 0.51$\mu$), and if the mean of the remaining groups is close to it (here it is equal to 0.53$\mu$), the observations are considered harmonious. But in the calculations of the Main Bureau we observe the ordinary Gaussian calculations." (translation by George Marinenko and Walter Sadowski contained in a 1971 memorandum by Churchill Eisenhart, who notes that the largest of the four lowest values is 44, not 45 and that the mean of the six observations not in the middle group is 0.495$\mu$, not 0.53$\mu$). For the eleven observations under consideration, there is excellent agreement among the arithmetic mean (Gauss), 1 M + 0.50$\mu$; the median (Estienne), 1 M + 0.49$\mu$; and the mean of the middle third of the observations (Mendeleev), 1 M + 0.51$\mu$.

**Reference:** Estienne (1890).

* Pearson, Karl (1895). Contributions to the mathematical theory of evolution. II. Skew variation in homogeneous material. *Philosophical Transactions of the Royal Society of London A* **186**, 343-414; abstract, *Proceedings of the Royal Society of London* **57**, 257-260. (JFM 26, 243). Reprinted in *Karl Pearson's Early Statistical Papers*, pp. 41-112, Cambridge University Press, Cambridge, England, 1948.

**Summary:** This paper contains the first exposition of Pearson's system of frequency curves. The only material relevant to a study of order statistics concerns the relative position of mean, median, and mode of samples from a Pearson Type III distribution, which the author discusses at some length on pages 375-376 (pp. 73-74 of the reprint). Let the equation of the Type III curve be $y=y_0(1+x/\alpha)^{\gamma\alpha}e^{-\gamma x}$, or if we take $z=\gamma(\alpha+x)$ and $\gamma\alpha=p$, $y=y_0(e/p)^p z^p e^{-z}$. Then for p positive, he finds that the median lies about one third of the way from the mean to the mode. In a footnote on p. 345 (p. 43 of the reprint) he writes: "The 'mean', the 'mode', and the 'median' have all distinct characters important to the statistician."

**Comments:** Pearson makes routine use of the median in several later papers, but only those which make some new contribution to the theory will be included in this bibliography.

**References:** Edgeworth (1887e), Galton (1889).

**Citations:** Yule (1896a), Edgeworth (1898), Czuber (1899), Sheppard (1899a,b), Kapteyn (1903), Edgeworth (1905), Charlier (1906, 1910), Edgeworth (1911, 1913), Tolley (1916), Czuber (1921), Julin (1921), Keynes (1921), Dodd (1922, 1923), Rietz (1924), Goodrich (1927), Sophister (1928), Pearson & Pearson (1931, 1932), Rosin & Rammler (1934), Roller (1941), Boldrini (1942), Kendall (1943), Cramér (1946), Kendall (1946).

* Cranz, Carl (1896). *Kompendium der Theoretischen Äusseren Ballistik*. B. G. Teubner, Leipzig. Second edition (with K. Becker), *Lehrbuch der Ballistik*, Band I: *Äussere Ballistik*, Julius Springer, Berlin, 1910; fifth edition (with the assistance of O. von Eberhard and K. Becker), 1925. Ergänzungsband, 1936. Reprinted 1943 by Edwards Brothers, Inc., Ann Arbor, Mich. English translation of second German edition, *Handbook of Ballistics*, Volume 1: *Exterior Ballistics*, H. M. Stationery Office, London, 1921.

**Summary:** In Section 68 (pp. 419-425), the author discusses the treatment of outliers. He mentions the criteria of Peirce, Chauvenet, Stone, Vallier, Heydenreich, Mazzuoli, and Rohne, and the objections of Airy, Bessel, and Faye to the rejection of any observation simply because it deviates too far from the others. He points out that several other criteria are special cases of Stone's, which rejects one observation out of m if the observations have come from a homogenous normal population. These criteria are those of Chauvenet ($m=2n$), Vallier ($m=n^2$), Mazzuoli ($m=n$), and Heydenreich ($m=2n-2$). He applies these criteria and those of Peirce and Rohne to the same example. The supplementary volume (Section 85, pp. 204-214, written by O. von Eberhard) contains further discussion of criteria for rejection of outliers and another example of their use.

**Note:** All references in the above summary to specific sections and pages are to those of the 1943 reprint of the fifth edition (1925) and the supplementary volume (1936). The relevant material is found in Section 65 (pp. 392-396) of the English translation of the second German edition.

**References:** Czuber (1891a), Vallier (1894), {Czuber (1899, 1903), Heydenreich (1903), Kozák (1907), Rohne (1907), Kozák (1908-10), Mazzuoli (1908)}.

**Citations:** Heydenreich (1903), Kozák (1908-10), von Eberhard (1938), Arley (1940), Arley & Buch (1940), Simon (1941), Smirnoff (1941).

* Galton, Francis (1896). Application of the method of percentiles to Mr. Yule's data on the distribution of pauperism. *Journal of the Royal Statistical Society* **59**, 392-396.

**Summary:** The author proposes the application of the method [see Galton (1889)] of percentiles (actually deciles in this case) to data on the distribution of pauperism given by Yule (1896a). Instead of computing the moments of the observed distribution and fitting a member of the Pearson system of frequency curves, he computes the deciles by linear interpolation in the observed (grouped) distribution, assuming that the observations in each class are uniformly distributed throughout the class. He then obtains, from the second, fifth, and eighth deciles, measures of central tendency (the median or fifth decile), of variability (the spread between the second and eighth deciles), and of skewness ($|v/w|$, where v and w are respectively the deviations of the second and eighth deciles from the median). A second measure of skewness which he proposes is the deviation of the mean from the median.

**Comments:** This paper and the two related ones by Yule (1896a,b), like many that we shall encounter, deal with grouped data, which are only partially ordered. We do not know the exact value of each ordered observation, only its order within a specified class. The quantiles found by linear interpolation are thus only approximations, albeit usually very good ones, to the true quantiles of the ungrouped, completely ordered, data.

**References:** Galton (1889), Yule (1896a).

**Citations:** Yule (1896a,b), Bowley (1901), Julin (1921), Keynes (1921).

* Poincaré, Henri (1896). *Calcul des Probabilités.* Gauthier-Villars, Paris. Second edition, 1912. (JFM 49, 383—listing only).

**Summary:** Chapters X-XIII (pp. 169-251), dealing respectively with the theory of errors and the *arithmetic mean, justification of the law of Gauss, errors in the position of a point, and the method of* least squares, are of interest. The author expresses doubt as to the universal validity of the Gaussian law of error (and with it the arithmetic mean and the method of least squares), but offers no specific alternatives. Twice (pp. 187 and 211-212) he raises the question as to whether one should reject outlying observations, but offers no definitive answer. He alludes to the work of various authors, especially Gauss and Bertrand, but gives no specific references.

**Note:** Numbers of chapters and pages in the above summary are those in the second (1912) edition.

**Citations:** Gore (1899) [1903], Czuber (1899), Bohlmann (1909), Edgeworth (1911), Dodd (1913), Contarino (1914), Daniell (1920), Keynes (1921), Dodd (1922), Whittaker & Robinson (1924), Coolidge (1925), Wilson & Hilferty (1929), Lidstone (1930), Madow (1937), Fréchet (1941), Simon (1941), Cramér (1946).

* Yule, G. Udny (1896a). Notes on the history of pauperism in England and Wales from 1850, treated by the method of frequency curves, with an introduction on the method. *Journal of the Royal Statistical Society* **59**, 318-349; discussion, 350-357.

**Summary:** The author gives an introduction to the system of frequency curves developed by Pearson (1895) and proceeds to fit suitably chosen members of the system to data on pauperism in England and Wales for the years 1850, 1860, 1870, 1881, and 1891. Along with other measures of central tendency and variability he considers the median, the quartile deviation, and the range. He mentions that the median has been used extensively by Galton because of its simplicity. He notes that the quartile deviation is the same as the probable error for symmetric distributions. He points out that large samples which have the same range may have very different standard deviations, so that the range of a large sample is not a good measure of variability. He closes with a supplementary note on the determination of the mode. One of the two methods he proposes is based on the rule of thumb that the distance between the mean and the median is approximately one-third of that between the mean and the mode, so that we have approximately Mode = Mean − 3 (Mean-Median). In determining the median of grouped data, he proposes linear interpolation in the frequency polygon (composed of trapezoids) rather than in the histogram (composed of rectangles). Contributors to the discussion include Francis Galton, W. F. Sheppard, and the author. Galton proposes use of the method of percentiles (or deciles), which he expounds more fully in a later memorandum [Galton (1896)], instead of the method of frequency curves. Sheppard suggests that the median is a better average than the mode (proposed by the author), since the latter is quite sensitive to small changes in the observed frequencies. The author replies to Sheppard's suggestion by pointing out that the mode of the fitted frequency curve, which is what he actually uses, does not exhibit the same sensitivity to small changes as does the observed mode. He reserves detailed comment on Galton's proposed method for a note [Yule (1896b)] on Galton's memorandum, saying only that the chief fault of Galton's method is that it does not lend itself to theoretical treatment.

**References:** Pearson (1895), Galton (1896), Yule (1896b).

**Citations:** Galton (1896), Yule (1896b), Pareto (1897), Bowley (1901), Julin (1921), Keynes (1921), Rietz (1924).

* Yule, G. U. (1896b). Remarks on Mr. Galton's note. *Journal of the Royal Statistical Society* **59**, 396-398.

**Summary:** The author points out that, whereas Galton (1896) has stated that the deciles of a distribution are a sufficient representation of the facts, his (Yule's) frequency curves [see Yule (1896a)] give values of the deciles which agree quite closely with Galton's observed deciles and are probably better estimates of the true deciles of the underlying population. He closes with the following statement (p. 398): "The method of percentiles may, in fact, often be a convenient method for representing observations, but it must be remembered that it is only a method of representation, it offers no theoretical law of distribution or legitimate means of smoothing observations. Its wide adoption to give the results of statistical researches is even to be deprecated, for it frequently causes the omission of the very data upon which a real statistical theory could be developed."

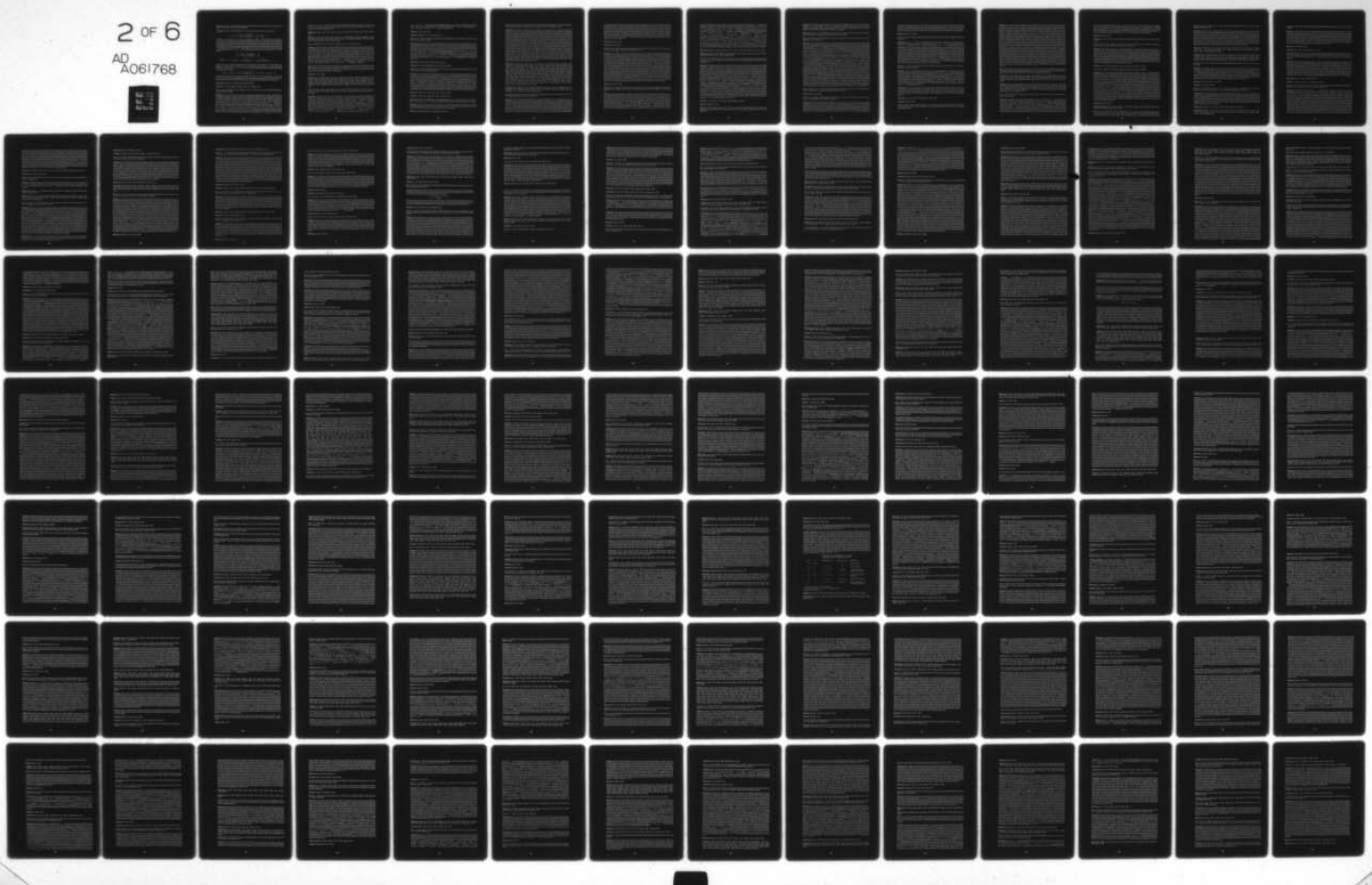
**References:** Galton (1896), Yule (1896a).

**Citations:** Yule (1896a), Bowley (1901), Keynes (1921), Rietz (1924).

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* Blümcke, Ad (1897). Zur Jordan'schen Theorie der Maximalfehlers. *Zeitschrift für Vermessungswesen* **26**, 51-54, 276-281, 561-562. (JFM 28, 205-206).

**Summary:** In the first installment (pp. 51-54), the author shows that the error function

$$y_n = \phi(\epsilon)_n = \frac{1}{2} \cdot \frac{3 \cdot 5 \cdot 7 \ldots (2n+1)(2n+3)}{2 \cdot 4 \cdot 6 \ldots 2n(2n+2)} \cdot \frac{1}{M_n} \left(1 - \frac{\epsilon^2}{M_n^2}\right)^{n+1},$$

where $M_n$ is the maximum error, which was proposed by Jordan (1877b) [1895], can be obtained in another way, by a variation of the derivation of the Gaussian law of error, based on the composition of elementary errors, which was given by Hagen (1837) [1867]. In the second part (pp. 276-281), the author shows that, just as the Gaussian law of error can be derived from the principle of the arithmetic mean, so can the Jordan law, if only one assigns to the individual observations weights which are functions of the errors of observation. In the third part (pp. 561-562), he shows that the joint probability density function of p independent errors $\epsilon_i$ ($i = 1,2, \ldots, p$), each following the Jordan law, is given by $D^p e^{-(n+1)J(n)\Sigma\epsilon_i^2/M_n^2}$, where

$$D = \frac{1}{2} \cdot \frac{3 \cdot 5 \cdot 7 \ldots (2n+1)(2n+3)}{2 \cdot 4 \cdot 6 \ldots 2n(2n+2)} \cdot \frac{1}{M_n}$$

$$\text{and } J(n) = 1 + \frac{1}{2} \cdot \frac{3}{2n+7} + \frac{1}{3} \cdot \frac{3 \cdot 5}{(2n+7)(2n+9)} + \frac{1}{4} \cdot \frac{3 \cdot 5 \cdot 7}{(2n+7)(2n+9)(2n+11)} + \cdots$$

[after correction of an obvious typographical error]. He points out that Jordan (1877b) [1895] has shown that, for $n = \infty$, $D = h/\sqrt{\pi}$ and $e^{-(n+1)J(n)\Sigma\epsilon_i^2/M_n^2} = e^{-h^2\Sigma\epsilon_i^2}$, so that the joint p.d.f. reduces to that of p independent Gaussian variables, namely $(h/\sqrt{\pi})^p e^{-h^2\Sigma\epsilon_i^2}$. He also gives two new approximations to the square of the maximum error, namely:

$$(1)\ M_n^2 = 2\pi \left[\frac{1}{2} \cdot \frac{3 \cdot 5 \cdot 7 \ldots (2n+1)(2n+3)}{2 \cdot 4 \cdot 6 \ldots 2n(2n+2)}\right]^2 \cdot m^2;\ (2)\ M_n^2 = 2(n+1)J(n) \cdot m^2$$

[again after correction of an obvious typographical error], where m is the standard deviation. He points out that values given by these approximations are somewhat smaller, especially for small values of n, than those given by the Jordan formula, but that they agree somewhat better with each other.

**References:** Hagen (1837) [1867], Jordan (1877b) [1895].

**Citations:** Gore (1889) [1903], Blümcke (1898, 1901), Vogeler (1907).

* Fechner, Gustav Theodor (1897). *Kollektivmasslehre* (ed. Gottl. Friedr. Lipps). Wilhelm Engelmann, Leipzig. (JFM 28, 208-209).

**Summary:** Various laws of error and various averages (arithmetic mean, median, and mode) are discussed in Chapter V. Chapter XX deals with the distribution of extreme values. The author discusses the largest and smallest values, E′ and E,, in a sample, their absolute or relative difference (the range), and their sum (twice the midrange), and cites examples of their use by Dove (1838) and Schmid in meteorological studies, as well as their mention by Encke (1832-34). He remarks (p. 321): "One must not forget that the magnitude of the extreme is dependent on the number of observations . . . ." He looks at the averages of the largest and of the smallest heights of 360 recruits divided into n samples of size m, for $m = 2$, $n = 180$; $m = 3$, $n = 120$, $m = 4$, $n = 90$; $m = 6$, $n = 60$; $m = 9$, $n = 40$; $m = 18$, $n = 20$; $m = 36$, $n = 10$; $m = 72$, $n = 5$; $m = 360$, $n = 1$; also at the average ranges of the arithmetic means, medians, and modes of these samples. He finds expressions for the p.d.f. and the mean, median, and mode of the largest absolute deviation of m values from their mean (assuming a normal distribution) as a function of $1/h = \sigma\sqrt{2}$ and

tabulates them for m=1,2,3,4,6,9,18,36,72,360,500,1000,5000, and 10000 in the case of median and mode; he also compares theoretical and observed values for the median. Additional examples and references are given in Chapter XXVII.

**References:** Gauss (1809, 1823), Encke (1832-34), Quetelet (1835) [1869], Dove (1838), Quetelet (1846).

**Citations:** Czuber (1921), Julin (1921), Keynes (1921), von Bortkiewicz (1922a,b), Goldhizer (1923), Rietz (1924), Grassberger (1932), Gumbel (1933e,g), Julin (1933), Rosin & Rammler (1934), Birkeland & Frogner (1935), Barricelli (1943), Kendall (1946).

* Pareto, Vilfredo (1897). *Cours d'Économie Politique*, Vol. II. F. Rouge, Lausanne.

**Summary:** Book III, Chapter 1 (pp. 299-345) deals with the curve of incomes which has come to be called by the author's name. Let N be the number of persons in a population who have incomes greater than x. On pp. 305-306, the author writes: "The real curve (of incomes) is interpolated by a line of which the equation is (1) $\log N = \log A - \alpha \log x$. The general equation of the curve is perhaps (2) $\log N = \log A - \alpha \log (a + x) - \beta x$; but it is only in a single case . . . that we have found an appreciable value for $\beta$. It is therefore quite probable that $\beta$ is, in general, negligible, and that one has simply (3) $\log N = \log A - \alpha \log (a + x)$. When it is a matter of total income, a is also, in general, quite small and most often of the order of errors of observation. Thus we are left with the equation (1). . . . When it is a matter of personal fortune, the constant a can no longer be neglected. It can take very considerable values. . . . The equation (1) gives (1′) $N = A/x^{\alpha}$. The equation (2) gives (2′) $N = [A/(x+a)]e^{-\beta x}$. . . . "(Compiler's translation).

**Note:** The book contains nothing directly relevant to a study of order statistics, but it is included in this bibliography because the parameter $\alpha$ of the Pareto distribution is often used as an index of concentration of income or wealth, and some knowledge of it is needed for purposes of comparison with other indices, some of which are based on order statistics.

**Reference:** Yule (1896a).

**Citations:** Lorenz (1905), Watkins (1908), Gini (1909), Persons (1909), Gini (1910), Edgeworth (1911), Furlan (1911), Porru (1912), Savorgnan (1915), Ricci (1916), Gini (1917), Niceforo (1919), Dalton (1920), Winkler (1924), Gini (1926), Saibante (1928), d'Addario (1930), Gibrat (1930), d'Addario (1931), von Bortkiewicz (1931), d'Addario (1932), Yntema (1933), d'Addario (1934a), Castellano (1935), d'Addario (1936), Gini (1936), Bresciani-Turroni (1937, 1939), Gini (1939), Davis (1941), Pizzetti (1941b), Boldrini (1942).

* Whitworth, William Allen (1897). *DCC Exercises Including Hints for the Solution of All the Questions in Choice and Chance*. Bell, London. (JFM 28, 212—listing only). Reprinted 1929 by G. E. Stechert & Co., New York.

**Summary:** The author gives hints for the solution of the exercises in his earlier book [Whitworth (1867)]. The following (pp. 195-197) are relevant to a study of order statistics [see comments on Whitworth (1867)]: "666. A line of length c is divided into n segments by n−1 random points. Find the chance that no segment is less than a given length a, where c>na. (Say c−na=ma). . . . [Solution:] Suppose the whole line divided into $\omega c$ equal elements, $\omega$ ultimately to be indefinitely great. Then the given length will contain $\omega a$ of the same elements. The points of partition can be selected in $C_{n-1}^{\omega c-1}$ ways. But if we first distribute $\omega a$ elements to each segment we have then an excess of $\omega ma$ elements to distribute as increments of the n segments. We can make this distribution . . . in $C_{\omega ma,n-1}$ ways. This is therefore the no. of favourable ways . . . Chance $= C_{\omega ma,n-1}/C_{\omega c-n,n-1} = [(\omega ma+1)(\omega ma+2) \ldots (\omega ma+n-1)]/[(\omega c-n+1)(\omega c-n+2) \ldots (\omega c-1)]$, and when $\omega$ is made infinite this becomes $(ma)^{n-1} \div c^{n-1}$

$= m^{n-1} \div (m+n)^{n-1}$. . . . 667. In the last question find the chance that r of the segments shall be less than a and n−r greater than a. [The solution is too long to reproduce here]. . . . The . . . chance required is . . . $[m^{n-1} - C_1^r (m+1)^{n-1} + C_2^r (m+2)^{n-1} - \ldots \pm (m+r)^{n-1}] / [\pm (m+n)^{n-1}]$".

**Reference:** Whitworth (1867).

**Citations:** Edgeworth (1911), Garwood (1940).

* Blümcke, Ad. (1898). Zur Jordan'schen Theorie der Maximalfehlers. *Zeitschrift für Vermessungswesen* **27**, 313-321. (JFM 29, 189-190).

**Summary:** This paper is a continuation of the author's earlier paper, in three installments, with the same title [Blümcke (1897)]. The author first points out that for Jordan's second algebraic law of error, discussed in the earlier paper, $m^2/M^2 = 1/(2n+5)$ when n is an integer, where m is the standard deviation and M is the maximum error. He then proceeds to prove, by the use of Gamma functions, that this relation holds also for non-integral values of n. He makes a study of the special cases in which $n = -1, -\frac{1}{2}, 0, +\frac{1}{2}, +1, +2, \infty$. In particular, $n = -1$ gives the rectangular distribution ($M^2 = 3m^2$), $n = 0$ gives the parabolic distribution ($M^2 = 5m^2$), and $n = \infty$ gives the normal distribution ($M = \infty$). The frequency curves are symmetric for all values of n; those for $n = +1$ and $n = +2$ have respectively first and second order contact with the horizontal axis at $\epsilon = \pm M$, while the one for $n = +\infty$ (normal distribution) is asymptotic to the axis.

**References:** Jordan (1877b) [1895], Blümcke (1897).

**Citations:** Gore (1889) [1903], Blümcke (1901), Vogeler (1907).

* Edgeworth, F. Y. (1898). On the representation of statistics by mathematical formulae. Part I. *Journal of the Royal Statistical Society* **61**, 670-700.

**Summary:** On pp. 681-685 the author discusses the use of the median and the quartiles (or other percentiles) instead of the mean, the standard deviation, and the standardized third moment as descriptive measures for asymmetric distributions, and remarks that they are much easier to calculate. In particular, he points out (p. 681) that if x is normally distributed with mean a and modulus $c = \sigma\sqrt{2}$, where $\sigma$ is the standard deviation, then $x^2$ has lower and upper quartiles $Q_1$ and $Q_2$ and median M given by $Q_1 = (a - 0.4769c)^2$, $M = a^2$, and $Q_2 = (a + 0.4769c)^2$, so that $Q_2 - Q_1 = 4(0.4769)ac$ and $Q_1 + Q_2 - 2M = 2(0.4769)^2c^2$, these last two quantities being measures of dispersion and of skewness, respectively. In calculating the median and the quartiles (or other percentiles) from grouped data, he advocates using, instead of linear interpolation, four-point Lagrangian interpolation, a procedure which he illustrates in the Appendix (Note 7, pp. 697-699). He also discusses the relative positions of mean, median, and mode.

**References:** Edgeworth (1885, 1886a, 1887e), Pearson (1895).

**Citations:** Bowley (1901), Edgeworth (1911), Bowley (1928).

* Rodewald, H. (1898). Zur Methodik der Keimprüfungen. *Die Landwirtschaftlichen Versuchs-Stationen* **49**, 257-282; Appendix by J. C. Kapteyn (translated from Dutch into German by H. Rodewald), 282-286.

**Summary:** The author states (p. 275): "Geometers regard three times the standard deviation as a limit [in deciding] whether a measurement must be repeated or not." There appears to be nothing else that is relevant to our subject either in the paper itself or in the appendix.

**Citations:** Mitscherlich (1903), von Bortkiewicz (1922b).

* Czuber, Emanuel (1899). Die Entwicklung der Wahrscheinlichkeitstheorie und ihrer Anwendungen. *Jahresbericht der Deutschen Mathematiker-Vereinigung* **7** (2), 1-279. (JFM 30, 43-45).

**Summary:** This report is divided into seven parts, of which the sixth (pp. 150-224) contains the material relevant to a study of order statistics. This part is devoted to the application of probability theory to the results of measurements. Article 55 (pp. 159-163) deals with the arithmetic mean and other averages, including the median and the power means [see Fechner (1874)]. Article 58 (pp. 170-177) discusses the estimation of the precision of a series of observations on the basis of the true errors. The median absolute deviation [Gauss (1816), Encke (1832-34)] is among the measures of precision considered. The method of least squares is introduced in Article 59 (pp. 177-180). The method of situation of Laplace (1793, 1799) based on the two conditions of Boscovich is mentioned as the first attempt at a systematic solution of the problem. Arts. 60-65 are devoted to demonstrations of the method of least squares based on the work of Gauss (1809), Laplace (1812), and Gauss (1823) and other material on that method, not directly relevant to our study. Article 66 (pp. 202-207) deals with measures of the precision of a series of direct observations on the basis of apparent errors and differences of observations, the latter based on the work of Jordan (1869, 1872), von Andrae (1869, 1872), and Helmert (1876a). Article 68 (pp. 211-217) deals with the comparison of experience with theory. The author discusses the largest and smallest errors in a series of n observations, and cites the work of Fourier (1824a), Helmert (1877b), Jordan (1877a, 1890), and Bertrand (1888c, 1889). He also deals with the treatment of outlying observations, citing the work of Peirce (1852), Gould (1855), Chauvenet (1863), Stone (1868, 1873, 1874), and Bertrand (1888b, 1889), as well as the words of caution of Bessel [& Baeyer] (1838), Airy (1856) [answered by Winlock (1856)], and Faye (1888).

**References:** N. Bernoulli (1709), Cotes (1722), Simpson (1757), Lambert (1765b), Lagrange (1774), Laplace (1774), D. Bernoulli (1778), Laplace (1781, 1793), Trembley (1804), Legendre (1805), Adrain (1808), Gauss (1809), Laplace (1810, 1811a, 1812), Legendre (1814), Gauss (1816), Anonymous (1821), Gauss (1823), Fourier (1824a), Ivory (1825), Hauber (1830, 1830-32), Encke (1832-34), Cauchy (1837), Hagen (1837), Bessel [& Baeyer] (1838), Cournot (1843), Gerling (1843), Ellis (1844), Quetelet (1846), De Morgan (1847), Herschel (1850), Peirce (1852), Gould (1855), Airy (1856), Winlock (1856), von Andrae (1860), Airy (1861), Chauvenet (1863), De Morgan (1864), Todhunter (1865), Venn (1866), Stone (1868), von Andrae (1869), Jordan (1869), Zachariae (1871), von Andrae (1872), Glaisher (1872), Helmert (1872), Jordan (1872), Glaisher (1873), Peirce (1873), Stone (1873a,b), Fechner (1874), Jevons (1874), Stone (1874), Helmert (1875a), Laurent (1875), Lexis (1875), Mees (1875), Helmert (1876a,b, 1877a,b), Jordan (1877a,b), Merriman (1877), Lalanne (1879), Bréger (1881), Edgeworth (1883a, 1887a,e, 1888), Faye (1888), Bertrand (1889), Pizzetti (1889), Jordan (1890), van den Berg (1891), Czuber (1891a), Pizzetti (1892), Pearson (1895), Poincaré (1896).

**Citations:** Gore (1889) [1903], Cranz (1896) [1910], Wölffing (1899), Czuber (1903), Edgeworth (1911), Contarino (1914), Weinberg (1916), Keynes (1921), von Bortkiewicz (1922b), Coolidge (1925), von Bortkiewicz (1931), Arley (1940), Arley & Buch (1940).

* Galton, Francis (1899a). The median estimate. *Reports of the British Association for the Advancement of Science* **69**, 638-640; summary, *Nature* **60**, 584.

**Summary:** "Dr. Galton proposes to substitute a scientific method for the very unsatisfactory ways in which the collective opinion of committees and assemblies of various kinds is ascertained, in respect of the most suitable amount of money to be granted for any particular purpose. How is that medium amount to be ascertained which is the fairest compromise between many different opinions? An average value—i.e., the arithmetic mean of the different estimates—may greatly mislead, because a single voter is apt to produce an effect far beyond his due share by writing down an unreasonably large or unreasonably small sum. Again, few persons know what they want with sufficient clearness to enable them to express it in numerical terms, from which alone an average may be derived; much deeper thought-

searching is needed to enable a man to make such a precise affirmation as that 'in my opinion the bonus given should be 80ℓ.,' than to enable him to say 'I do not think he deserves so much as 100 ℓ., certainly not more than 100ℓ.' Dr. Galton's plan for discovering the medium of the various sums desired by the several voters is to specify any two reasonable amounts A and B, and to find what percentage a of voters think the sum ought to be less than A, and what percentage b vote for less than B. It may now be assumed that the estimates will be distributed on either side of their (unknown) *median* m, with an (unknown) quartile q, in approximate accordance with the normal law of frequency of error; and thus (using the table of centiles given in the author's 'Natural Inheritance' [Galton (1889)]) the required median value can be found." (*Nature* **60**, 584).

**Reference:** Galton (1889).

**Citation:** Galton (1899b).

* Galton, Francis (1899b). A geometric determination of the median of a system of normal variants, from two of its centiles. *Nature* **61**, 102-104. (JFM 30, 211-212).

**Summary:** This paper is a sequel to an earlier paper of the author [Galton (1899a)], concerning which he writes (p. 102): "Its object was to solve a problem of the following kind: —40 per cent. of the members at a meeting vote that a proposed grant should be less than 100ℓ., 80 per cent. vote that it should [not] exceed 500ℓ. What is the Median Estimate, supposing the normal law of frequency to hold good? That is to say, What is the sum that one-half of the members would think too little, and the other half too much, and which therefore represents the best compromise between many discordant opinions? I showed that the calculation was exceedingly simple if certain tabular values are used . . . . But, on after refection, it seems to me that further simplification is both desirable and feasible." He proposes a graphical method using normal probability paper, on which data points which are normally distributed fall on a straight line. Any two points, such as the 40$^{th}$ and 80$^{th}$ percentiles given above, determine the straight line, and it is then a simple matter to read off the value of the 50$^{th}$ percentile (the median). The author adds the following interesting comment (p. 103): "Prof. Karl Pearson informs me that various curves representing technological formulae were similarly translated into straight lines by Lalanne, and discussed by him in a series of papers (1846-1878). He termed the process by which a proper choice of scales enables us to represent a given curve by a straight line, *anamorphic geometry*. Prof. Pearson also tells me that in Lalanne's hands and in those of his followers (Hermann, Vogler, Kapteyn, &c.) this geometry has been of great service in exhibiting engineering and other data in a form suitable for easy reckoning."

**Reference:** Galton (1899a).

* Sheppard, W. F. (1899a). On the application of the theory of error to cases of normal distribution and normal correlation. *Philosophical Transactions of the Royal Society of London A* **192**, 101-167. (JFM 30, 222).

**Summary:** The author extends the method of Galton (1889) for estimating the average and dispersion of a normal distribution from two quantiles to their estimation from three or more quantiles. On pp. 130-132 he writes: "If the values of a measure L are known to be distributed normally, the distribution is determined when the mean value $L_1$ and the semi-parameter (standard deviation) a are determined. When the values of L for n individuals obtained by random selection are given, the values of $L_1$ and of a can be found in either of two . . . ways: (1) We can find the average and the standard deviation (square root of average square of deviation from the average) of the n individuals. . . . (2) The other method is that which has been used mainly by Mr. Galton [(1889), p. 62]. Let $\alpha$ and $\beta$ be any two class-indices, and let X and Y be the corresponding values of L in the complete community. Then, if x and y are the

abscissae corresponding to class indices $\alpha$ and $\beta$ in the standard normal figure (i.e., if ordinates at distances x and y from the central ordinate divide the figure into areas whose ratios are $1+\alpha$: $1-\alpha$ and $1+\beta$: $1-\beta$ respectively), we have $X=L_1 + ax$, $Y=L_1 + ay$. When $L_1 = (xY-yX)/(x-y)$, $a=(X-Y)/(x-y)$. . . . (3) As an extension of this last result, let X,Y,U, . . . be values of L corresponding (in the complete community) to class-indices $\alpha$, $\beta$, $\gamma$, . . ., and let the corresponding abscissae in the standard figure be x,y,u, . . . . Then $x=L_1 + ax$, $Y=L_1 + ay$, $U=L_1 + au$, . . .; and therefore $L_1= (\ell X+mY+pU+ \ldots)/(\ell+m+p+\ldots)$, $a=(\ell'X+m'Y+p'U+ \ldots)/(\ell'x+m'y+p'u+ \ldots)$, where $\ell$, m, p, . . ., $\ell'$,m',p',. . . are any quantities which satisfy the conditions $\ell x+my+pu+\ldots=0$, $\ell'+m'+p'+\ldots=0$. . . ." The error in X is of the form $\omega+x\rho+\phi$, the error in Y is of the form $\omega+y\rho+\psi$, and the error in U is of the form $\omega+u\rho+\chi$, where $\phi,\psi,\chi$ *are uncorrelated with* $\omega$ *and also with* $\rho$. If $\Phi^2/n$ and $\Phi'^2/n$ *are the mean square errors of* $(\ell\phi+m\psi+p\chi+\ldots)/(\ell+m+p+\ldots)$ and $(\ell'\phi+m'\psi+p'\chi+\ldots)/(\ell'x+m'y+p'u+\ldots)$, the author shows that the mean squares of the errors in $L_1$ and in a, due to the class-index method, are $(a^2+\Phi^2)/n$ and $(\frac{1}{2}a^2+\Phi'^2)/n$, which are greater than $a^2/n$ and $\frac{1}{2}a^2/n$, respectively, the values for the average- and average-square method. He discusses the choice of quantiles so as to minimize the mean square errors of $L_1$ and a.

**References:** Quetelet (1846), De Morgan (1847), Galton (1889), Edgeworth (1893), Pearson (1895).

**Citations:** Sheppard (1899b), Bowley (1901), Edgeworth (1911), Pearson (1920), Keynes (1921), Fisher (1922), Rietz (1924), Cramér (1946), Kendall (1946).

* Sheppard, W. F. (1899b). On the statistical rejection of extreme variations, single or correlated. (Normal variation and normal correlation). *Proceedings of the London Mathematical Society* **31**, 70-99. (JFM 30, 222).

**Summary:** The author discusses the treatment of outlying observations in biological data, which differ from those in the physical sciences (e.g. astronomy) in that they may represent extreme variations which actually occur and not simply errors of observation. On the other hand, observations are usually plentiful, and there is no great harm in excluding a few good observations, provided this is done in such manner as not to bias the mean or the standard deviation. Excluding observations with equal probability from both tails of a symmetric distribution will not bias the mean but will bias the standard deviation downward; however, an adjustment can easily be made for this bias. For a sample of size n from a univariate normal population, the author considers Chauvenet's criterion, for which the probability of rejecting a random observation is 1/2n, and hence the chance of rejecting at least one of the n observations is $1-(1-1/2n)^n \doteq 1-1/\sqrt{e} = .39347\ldots$ for large n. He proposes two modifications of Chauvenet's criterion: (1) Exclude observations which deviate from the mean by a multiple of the standard deviation exceeding the standard normal deviate corresponding to a two-tail probability of 1/n (instead of 1/2n), which makes the chance of excluding at least one observation $1-(1-1/n)^n \doteq 1-1/e = .63212\ldots$ for large n; (2) A compromise such that the probability of excluding at least one observation is 1/2. He calls the interval between rejection limits the *theoretical range* for modification (1) and the *probable range* for modification (2). He discusses the computation of the rejection limits for the three criteria, tabulates them for $n=10^p$, $p=1(1)6$, and gives two numerical examples of their use. He extends the method based on the probable range to the case of two or three correlated variables, in which observations lying outside a circle or a sphere in the space of the standardized variables are excluded. He provides the necessary tables for 1, 2, or 3 variables and $\log_{10} n=1.0(0.1)6.0$.

**References:** Chauvenet (1863?), Pearson (1895), Sheppard (1899a).

**Citation:** Edgeworth (1911).

* Wölffing, E. (1899). Ergänzung des von E. Czuber in seinem Referat über Wahrscheinlichkeitsrechnung gegebenen Litteraturverzeichnisses. *Mathematisch Naturwissenschaftliche Mitteilungen* (2) **1**, 76-84. (JFM 30, 45).

**Summary:** The author comments briefly on the work of Czuber (1899). He does not comment on the part relevant to our study, except to say that a broad scope is rightfully given to theory of errors and adjustment calculations. Most of the paper is given to a list of references, supplementing the list given by Czuber; a few of the new references are relevant to our subject.

**References:** Cauchy (1831), Halphen (1873), Lemoine (1873), Lalanne (1879) [1878], Merriman (1884), Bertrand (1888a,b,c), Goedseels & Mansion (1893), Czuber (1899).

* Estienne, J. E. (1900a). Sur la théorie des erreurs. *Comptes Rendus de l'Académie des Sciences de Paris* **130**, 66-69. (JFM 31, 233-234).

**Summary:** The author recalls that Gauss (no specific reference given) showed that if one assumes only that the law of errors of observation is such that the sum of the positive errors is numerically equal to the sum of the negative errors, the probability $P'_m$ that the arithmetic mean of m observations is affected by an error less than $\epsilon$ is $P'_m = \Theta(\epsilon\sqrt{n/2}/q)$, where q is the standard deviation of the individual observations and $\Theta$ is the normal c.d.f. The author proposes to show that the median enjoys an analogous property whatever the law of errors, without any restriction. Let $p_1(\epsilon)$ be the probability of making an error less than $\epsilon$ in a single observation. One has $p_1(0) = 0$, $p_1(-\infty) = p_1(+\infty) = 1/2$ (considering separately the positive and negative errors, each with probability 1/2) and the function is monotone increasing from 0 to $\pm\infty$. Let $p_{2n+1}(\epsilon)$ be the probability that the median of $2n+1$ observations is affected by an error less than $\epsilon$. The author shows that $p_{2n+1}(\epsilon)$ is given by $p_{2n+1}(\epsilon) = \int_0^{2p_1(\epsilon)} (1-z^2)^n\,dz/2 \int_0^1 (1-z^2)^n\,dz \doteq \Theta(2p_1\sqrt{n})/2$. If $P_{2n+1}(\epsilon)$ is the probability that the median of $2n+1$ observations is in error by less than $\epsilon$ and $P_1(\epsilon)$ is the corresponding probability for a single observation, the above relation becomes $P_{2n+1}(\epsilon) = \Theta(P_1\sqrt{n})$. Since $P_1(\epsilon)$ is a function which increases monotonically from 0 to 1 as $\epsilon$ increases from 0 to $\infty$ and $\Theta(z)$ increases very rapidly with z, the differences in the value of $P_1$ have little influence on the value of $P_{2n+1}$ for large values of $P_1\sqrt{n}$, and one can replace $P_1(\epsilon)$ by $\epsilon/k$, where $1/k$ (not $\epsilon/k$, as the author erroneously states) is the slope of the tangent at the origin of the curve $y = P_1(\epsilon)$. (Thus $1/k$ is twice the central ordinate of the probability density function, the factor 2 occurring because when absolute values are considered, the portion of the p.d.f. for negative errors is folded over on that for positive errors.) So one obtains $P_{2n+1}(\epsilon) = \Theta[(\epsilon/k)\sqrt{n}]$. The corresponding expression for the arithmetic mean, found by setting $m = 2n+1$ in Gauss' formula, is $P'_{2n+1}(\epsilon) = \Theta[(\epsilon/q)\sqrt{n+1/2}]$. The difference between $\sqrt{n}$ and $\sqrt{n+1/2}$ is negligible for large values of n, and hence if $q<k$ the arithmetic mean is more precise than the median; if $q>k$ the median is preferable. If r denotes the probable error, k is always (see note below) less than (not greater than, as the author erroneously states) 2r. Hence $q>2r$ implies $q>k$, so that the median is more precise than the arithmetic mean if the standard deviation exceeds twice the probable error. (The converse is not true). If one does not know the law of error but only the quantities q and r or their ratio, the author proposes the use of a weighted mean of the median and the arithmetic mean, with weights q and 2r respectively.

**Note:** The inequality $k<2r$ does not hold for a uniform error law (for which $k=2r$) or for a U-shaped error law (for which $k>2r$), a restriction not mentioned by the author.

**Citation:** Estienne (1900b).

* Estienne, J. E. (1900b). Valeur plausible d'une grandeur variable. *Comptes Rendus de l'Académie des Sciences de Paris* **130**, 393-395. (JFM 31, 234-235).

**Summary:** The author defines the *plausible value* of a quantity as the value such that it is equally likely that the true value falls above or below it. He regards the plausible value and the median as practically equivalent, just as are the probable value and the arithmetic mean. He states that one can easily demonstrate that the sum of absolute values of deviations from the plausible value (median) is a minimum, just as the sum of the squares of deviations from the probable value (arithmetic mean) is a minimum. As an advantage of the median, he points out that if $y=f(x)$ is a monotone increasing or

monotone decreasing function of x, the median value of the function is that function of the median, whereas the arithmetic mean does not enjoy the same property unless f(x) is linear. He also calls attention to the curious fact that artillerymen, in their theoretical calculations, use the arithmetic mean of the range of a cannon, while in practice, on the shooting range and in combat, they use the median.

**Reference:** Estienne (1900a).

* Láska, W. (1900). Ueber das arithmetische Mittel. *Zeitschrift für Vermessungswesen* **29**, 593-597. (JFM 31, 225-227).

**Summary:** Let $x_1, x_2, \ldots, x_n$ be a series of observations of an unknown quantity x, arranged in order of magnitude, n being an even number. Consider the midrange $x'_1 = (x_1 + x_n)/2$ and the quasi-midranges $x'_k = (x_k + x_{n-k+1})/2$, where $k = 2,3,\ldots, n/2$, and call the series $x'_1, x'_2, \ldots, x'_{n/2}$ the first derivative of the fundamental series. Now consider this series as the fundamental series and find its first derivative, which is the second derivative of the original fundamental series, and so on. If $n = 2^p$, then obviously the $p^{th}$ derivative is equal to the arithmetic mean. The author proves the following theorem: If $n = 2^p$ observed values of an unknown are given, which are symmetrically grouped about the unknown, then the arithmetic mean represents an approximation to the unknown which deviates from it less the larger the number of observations and the more decided the symmetry of the grouping. If n is not equal to $2^p$, then the same theorem remains valid, provided only that n is a somewhat larger number. If there is pronounced asymmetry, then the first derivative (and only the first) is found by the use of geometric quasi-midranges $x'_k = \sqrt{x_k \cdot x_{n-k+1}}$ or quadratic quasi-midranges $x'_k = \sqrt{x^2_k + x^2_{n-k+1}}$ according as the mode lies nearer the smallest or nearer the largest of the observations.

* Blümcke, Ad. (1901). Zur Jordan'schen Theorie des Maximalfehlers. *Zeitschrift für Vermessungswesen* **30**, 229-241. (JFM 32, 233-234).

**Summary:** This is a continuation of the author's earlier papers [Blümcke (1897, 1898)] with the same title. Having established that Jordan's second algebraic error law for $n = \infty$ (the normal distribution) has maximum error $M = \infty$, the author rejects this hypothesis as not in agreement with practice. He modifies Jordan's second law, which is based on a solution of the differential equation $dy/dx = -y \cdot 2x/(A + Bx^2)$, by adding a third term, $Cx^4$, to the denominator on the right-hand side of the differential equation. He seeks a solution of this family for which M/m is the same for $n = -1$ and $n = \infty$ and changes as little as possible for intermediate values of n. He succeeds in finding one for which $M/m = \sqrt{3} \doteq 1.73$ for $n = -1$ and $n = \infty$ and has a maximum value of approximately 1.75. As an alternative, he suggests using Jordan's second law for $n < 1/2$ and taking $M \doteq 2m$.

**Comments:** The results of Blümcke (1897, 1898, 1901), like those of Jordan (1877a,b, 1890), are of little practical value because they do not differentiate clearly between the maximum error for the population and the maximum error in a sample, which depends quite strongly upon the sample size. Nevertheless, they are of historical interest as early attempts to solve a difficult and important problem.

**References:** Jordan (1877b) [1895], Blümcke (1897, 1898).

**Citation:** Vogeler (1907).

* Bowley, Arthur L. (1901). *Elements of Statistics*. P. S. King & Son, Ltd., London; Charles Scribner's Sons, New York. Second edition, 1902; third edition, 1907; fourth edition, 1920 (JFM 47, 501—listing only). Fifth edition, 1926.

**Summary:** Part I deals with general elementary methods. Our interest is restricted to portions of Chapters V, VI, VII, IX, and X. Chapter V is concerned with averages, including the median and the average of the quartiles. The advantages and disadvantages of the median are discussed on pp. 102-104. Reference is made on p. 104 to the work of Galton (1889, 1896) and Yule (1896a,b). A table is given on p. 105 of the heights and weights of 76 boys of ages 13 to 15 years, including various averages of the weights. A graphic method due to Galton [see Galton et al. (1881), p. 247] of finding the median, quartiles, and deciles from ungrouped data is given on pp. 106-107, with a diagram facing p. 106. Chapter VI deals with measurements of dispersion and skewness and applications of averages. The mean deviation from the median is mentioned on p. 112 and the quartile deviation is discussed on p. 113. Gini's mean difference and its relation to the mean deviation from the median are discussed on pp. 114-115. On p. 116 the author mentions the ratio of the quartile deviation to the mean of the quartiles and the ratio of the mean deviation to the median as alternatives to the ratio of the standard deviation to the arithmetic average (coefficient of variation) as a measure of relative dispersion. On p. 116 the author proposes $s=(q_2-q_1)/(q_2+q_1)$, where $q_2$ is the excess of the upper quartile over the median and $q_1$ is the excess of the median over the lower quartile, as a measure of skewness. On p. 117 he states: "It should be noticed that the three characteristics of a group can be measured simply from the quartiles and median; the median for the central position, the quartile deviation for the dispersion, and the measurement [s] just discussed for the skewness." Chapter VII deals with the graphic method. A graphic method of finding the median and the quartiles from grouped data is discussed on pp. 138-139 and illustrated in a diagram facing p. 138. Chapter IX deals with index numbers, and the advantages of the median of price ratios as an index number are discussed on p. 206. Chapter X deals with interpolation. The use of Horner's method and Lagrange's formula in determining the median is discussed on pp. 227-228 and 236-237 respectively; a variant of the latter formula is attributed to Edgeworth (1898). Part II contains little that is relevant to our study. The Appendix contains a mathematical note on the method of least squares.

**Note:** All references to specific pages in the above summary are to those in the fifth edition, the only one the compiler has seen. According to the preface (pp. v-viii), substantial changes were introduced with the fourth edition, while the others differ little from their immediate predecessors.

**References:** Quetelet (1835, 1846), Chauvenet (1863), Todhunter (1865), Whitworth (1867), Galton et al. (1881), Merriman (1884), Edgeworth (1886a), Galton (1889), Edgeworth (1893), Galton (1896), Yule (1896a,b), Edgeworth (1898), Sheppard (1899a), {Edgeworth (1905), Yule (1911), Gini (1912), Persons (1919)}.

**Citations:** Lorenz (1905), Edgeworth (1911), King (1912), Edgeworth (1913), Secrist (1917), Niceforo (1919), Persons (1919), Dalton (1920), Jones (1921), Julin (1921), Keynes (1921), Goldhizer (1923), Rietz (1924), Jordan (1927), Eells (1930), Gibrat (1930), d'Addario (1931), von Bortkiewicz (1931), Shewhart (1931), Gini (1932), Crowe (1933), Julin (1933), Yang (1933), Anderson (1935), Castellano (1935), Wold (1935), Pizzetti (1941a), Boldrini (1942).

* Galton, Francis (1902). The most suitable proportion between the values of first and second prizes. *Biometrika* **1**, 385-390.

**Summary:** The author states the following problem (p. 385): "A certain sum . . . is available for two prizes to be awarded at a forthcoming competition; the larger one for the first of the competitors, the smaller one for the second. . . . What ratio should a first prize bear to . . . a second one? Does it depend on the number of competitors, and if so, in what way? Similar questions may be asked . . . when the number of prizes exceeds two. . . ." After some analysis, he states (p. 386): "I concluded that when only two prizes $\alpha$ and $\beta$ are given, their values should not be proportional to the absolute merits of the two competitors, but to their respective *excesses* of merit above the third competitor, who receives no prize at all. Let [A], [B], and [C] be the first, second, and third competitors, and a,b,c the marks allotted to them, then I conceive

that the most suitable relation of $\alpha$ to $\beta$ is as (a-c) to (b-c), and not as a to b." The author goes on to show both by computation (for a normal distribution) and by observation that "the value of (a-c) is roughly three times as great as that of (b-c), almost independently of the number of individuals in the series and quite independently of its mean and of its 'modulus of variability'." (p. 390). His computed results are based on assigning cumulative probability $(i-\frac{1}{3})/n$ to the $i^{th}$ smallest of n observations.

**Reference:** Galton (1889).

**Citations:** Pearson (1902), Irwin (1925a), Pearson (1925), Pearson & Pearson (1931, 1932), Kendall (1946).

* Hayford, John F. (1902). What is the center of an area, or the center of a population? *Publications [Journal] of the American Statistical Association* **8** (58), 47-58.

**Summary:** The author considers two possible definitions—the center of gravity and the point of intersection of two lines which bisect the area (or the population). He advocates the former, which gives a unique center, in preference to the latter, for which the result (the median point) varies according to the pair of lines chosen as axes. He points out that the U.S. Census Bureau, in Bulletin No. 62 of the Twelfth (1900) Census, gives both definitions, obtaining a unique median point by taking it to be the intersection of north-south and east-west lines which divide the population equally. In 1900 the median point of U.S. population, according to this definition, lay 80 miles north-eastward of the center of population according to the gravity definition. The movement of the latter is an accurate representation of the average movement of the population, but that of the former is not.

**Citations:** Ross (1930), Gini, Boldrini, Galvani & Venere (1933), Scates (1933).

* Pearson, Karl (1902). Note on Francis Galton's problem. *Biometrika* **1**, 390-399.

**Summary:** The author restates the problem of Galton (1902) as follows (p. 390): "A random sample of n individuals is taken from a population of N members which when N is very large may be taken to obey any law of frequency expressed by the curve $y=N\phi(x)$, $y\delta x$ being the total frequency with characters . . . lying between x and $x + \delta x$. It is required to find an expression for the average difference in character between the $p^{th}$ and $(p+1)^{th}$ individuals when the sample is arranged in order of magnitude of the character." In a footnote (p. 390) he states: "Clearly a knowledge of the average difference in character of any two adjacent individuals involves also a knowledge of the average difference in character between any two individuals". Let $\alpha$ be the proportion of individuals in the population having a character value less than x, that is, let $\alpha = \int_{-\infty}^{x}\phi(x)\,dx$. Then the solution of Galton's problem given by the author is $\{n!/[(n-p)!\,p!]\}\int_{-\infty}^{\infty}\alpha^{n-p}(1-\alpha)^{p}\,dx$. On p. 392 he remarks: "An interesting theorem which results from this has been given me by Mr. W. F. Sheppard; namely, the average differences between successive individuals are the successive terms in $\int_{-\infty}^{\infty}\{\alpha+(1-\alpha)\}^{n}dx$ when the subject of integration is expanded by the binomial theorem." Pearson points out (p. 392) that these results assume only that the *population* has a known distribution (normal, for example) while Galton assumed a normal distribution for the *sample*. He gives several numerical examples of the application of the theory. In a footnote (p. 398) he states: "I propose on another occasion to consider the application of Galton's problem to a new theory for the rejection of outlying individuals."

**Reference:** Galton (1902).

**Citations:** Edgeworth (1911), Irwin (1925a,b), Pearson (1925), Tippett (1925), Pearson & Pearson (1931), Sastry (1935), Kendall (1946), Wilks (1948).

* Czuber, Emanuel (1903). *Wahrscheinlichkeitsrechnung und ihre Anwendung auf Fehlerausgleichung, Statistik und Lebensversicherung.* B. G. Teubner, Leipzig. (JFM 34, 252-257). Second edition: Vol. 1 (1908) (JFM 39, 290); Vol. 2 (1910) (JFM 41, 266). Third edition: Vol. 1 (1914) (JFM 45, 343); Vol. 2 (1921)

(JFM 48, 594). Fourth edition: Vol. 1 (1924) (reprinted 1932) (JFM 58, 1152); Vol. 2 (1938) (ed. H. Münzner) (JFM 64, 1188).

**Summary:** Part II of the first edition and of Volume 1 of subsequent editions is concerned with the calculus of adjustment (Ausgleichungsrechnung), with Portion I dealing with the theory of errors of observation. Section 1 discusses the law of error, and Section 2 (pp. 266-275 of the second edition) considers various measures of precision, including the mean absolute error (Durchschnittsfehler), the square root of the mean square error (mittlere Fehler), and the probable error (wahrscheinliche Fehler). The $m^{th}$ roots of the means of the $m^{th}$ powers of the absolute values of the errors (Mittelwerten der Fehlerpotenzen), for various values of m, are also discussed, as is the mean of the $n(n-1)/2$ differences of n observations taken in pairs. Portion II deals with the combination of observations by the method of least squares. The other parts of this book are not relevant to a study of order statistics.

**References:** D. Bernoulli (1778), Legendre (1805), Adrain (1808), Gauss (1809), Laplace (1812), Gauss (1816, 1823), von Andrae (1869), Helmert (1872, 1876a,b), Bréger (1881), Czuber (1891a), Pizzetti (1892), Czuber (1899), {Edgeworth (1905)}.

**Citations:** Cranz (1896) [1910], Kozák (1907, 1908-10), Edgeworth (1911), Dodd (1913, 1914), Ricci (1916), Weinberg (1916), Czuber (1918), Keynes (1921), von Bortkiewicz (1922b), Dodd (1922), Rietz (1924), Coolidge (1925), von Bortkiewicz (1931), Grassberger (1932, 1933), Gumbel (1933e,f,g, 1934g), Rosin & Rammler (1934), Gumbel (1935-36), Ruark & Devol (1936), Arley & Buch (1940), Kendall (1946).

* Heydenreich, W. (1903). Über Ausreisser bei Messungen und Treffenbildern. *Kriegstechnische Zeitschrift* **6**, 253-265.

**Summary:** The author suggests a criterion for the rejection of outliers which is a modification of Chauvenet's. He proposes that, among n observations or measurements of the same quantity, a value which deviates from the mean by an amount greater than that which would occur with probability 1/2 in $(n-1)$ observations should be rejected. For samples of size $n=3(1)25,50,100$, he tabulates multiples of the mean deviation and of the probable error which warrant rejection of an observation if it deviates from the mean by a greater amount. He tabulates similar tables for Chauvenet's and Vallier's criteria, and applies these and other criteria discussed by Cranz (1896) to data on artillery fire.

**References:** Vallier (1894), Cranz (1896).

**Citation:** Cranz (1896) [(1910)].

* Kapteyn, J. C. (1903). *Skew Frequency Curves in Biology and Statistics*. Noordhoff, Groningen. (JFM 34, 268-269). Second edition (with M. J. van Uven), 1916. Hoitsema Bros., Groningen.

**Summary:** This work contains only a small amount of material directly relevant to a study of order statistics, but it is of collateral interest because it develops, from a different point of view than that of Pearson (1895), the theory of skew frequency curves, which has found use in the study of flood flows and various other phenomena in which order statistics also play an important role. The directly relevant material includes a discussion (pp. 43-44) of the median and the method of calculating it for the author's skew frequency curves, and a comparison (p. 45) of the value of the median with those of the mode and the arithmetic mean for several examples.

**References:** Laplace (1812), Quetelet (1846), Galton (1889), Pearson (1895).

**Citations:** Edgeworth (1911), Keynes (1911, 1921), Rietz (1924), Goodrich (1927), Cramér (1928), Gibrat (1930), d'Addario (1931), Gibrat (1932a), Slade (1934), Foster (1936), Arley & Buch (1940), Cramér (1946), Kendall (1946).

* Kirchberger, Paul (1903). Über Tschebyscheffsche Annäherungsmethoden. *Mathematische Annalen* **57**, 509-540.

**Summary:** In this paper, which is a condensation of his 1902 Göttingen doctoral dissertation with the same title, the author, apparently unaware of the still earlier work of Laplace and Fourier, states that the method of approximation which is best in the sense that the maximum error is as small as possible was first proposed by Poncelet and was systematically worked out by Chebyshev. He discusses both the general case of approximation to a continuous function over an interval and the special case of approximation to a finite set of points, with particular emphasis on the latter. Among other results, he shows that the best approximation to a set of n (>m) points by a function containing m constants to be fitted is identical with the best approximation to some subset of m+1 points.

**References:** Tchebychef (1854).

**Citations:** Haar (1918), de la Vallée Poussin (1919).

* Mitscherlich, Alfred (1903). Die Schwankungen der landwirtschaftlichen Reinerträge berechnet für einige Fruchtfolgen mit Hilfe der Fehlerwahrscheinlichkeitsrechnung. *Zeitschrift für die Gesamte Staatswissenschaft*, Ergänzungsheft **8**, vii + 120 pp. Verlag der H. Laupp'schen Buchhandlung, Tübingen.

**Summary:** On page 51 the author proposes the use of the midrange of a sample as an estimate of the population mean and one-eighth the sample range as an estimate of the probable error. He justifies the latter by the argument that (for a normal distribution) a value chosen at random differs from the mean m by four times the probable error r only seven times in a thousand. Hence he takes m+4r and m−4r, respectively, as approximations to the expected largest and smallest values, and their difference, 8r, as an approximation to the expected range of the sample, so that one-eighth the sample range is a reasonable estimate of the probable error r. As an example, he considers data (number of observations not stated) on the yield of rye, varying from 130 kg./hectare to 190 kg./hectare. He estimates the mean yield to be (130 + 190)/2 = 160 kg./hectare and the probable error to be (190−130)/8 = 7.5 kg./hectare.

**References:** Jordan (1877b), Rodewald (1898).

**Citations:** Czuber (1918, 1921), von Bortkiewicz (1922b).

* Saunder, S. A. (1903). Note on the use of Peirce's criterion for the rejection of doubtful observations. *Monthly Notices of the Royal Astronomical Society* **63**, 432-436.

**Summary:** On page 433 the author states: " . . . I found Peirce's criterion so useful [in an analysis of photographic measures of formations on the moon] that I thought it might be of interest to call attention to it, especially as I believe that the criterion is seldom if ever employed by the most experienced computers, at all events in this country [England]." He goes on to say (pp. 434-435): "The general question of the rejection of discordant observations has been discussed by Dr. Glaisher [(1873)], who strongly upholds De Morgan's [(1847)] view that when recognisable mistakes have been eliminated no observation should be rejected, but that when the first solution has been effected the conditional equations should be re-weighted in accordance with their residuals and solved again, the process being continued until the weights assumed for the last solution are sufficiently near to those given by that solution. How many solutions this would in general require I do not know, but it is clear that the process might be a very laborious one. In many cases its length would be practically prohibitive, and if we decide to give all the equations, either unit or zero weight, it is certain, as Dr. Glaisher points out, that we shall get a better result by excluding certain discordant observations than we should by including them. I believe that the practice amongst computers of experience is to rely almost entirely on their individual

judgment, taking into account the conditions of the observations, and drawing the line somewhere about those observations which give residuals of five times the probable error. But for those who, like myself, have a very limited practical acquaintance with least square solutions, difficult cases, like the one I have just quoted, must from time to time arise, and I cannot help thinking that others may be as glad as I have been to have recourse to a criterion established on some definite principle, which may be considered as an embodiment of that accumulated experience which we lack. The computation of the criterion is quite a short matter with the help of such tables as those given in Chauvenet [Gould's], but . . . in order to facilitate its application I have computed a short table which I append to this paper." Let $\kappa\epsilon$ be the critical residual, where $\epsilon$ is the root-mean-square error, m the number of conditional equations, $\mu$ the number of unknown quantities, and n the number of observations it is proposed to reject. On page 436 the author tabulates $\kappa$ to 2DP for $\mu$=3,4; n=1,2,3,4 and m=10(2)20(5)50(10)100.

**References:** De Morgan (1847), Peirce (1852), Gould (1855), Chauvenet (1863), Glaisher (1873).

**Citations:** Brunt (1917), Rider (1933).

* Edgeworth, F. Y. (1905). The law of error. *Transactions of the Cambridge Philosophical Society* **20**, 35-65, 113-141. (JFM 36, 305).

**Summary:** This paper contains little that is directly relevant to our subject; however, "the *median* and *percentiles* of a large set of observations" are mentioned in a footnote on page 120 as "relations other than summation by which the law of error is set up . . . though they cannot be classed as *functions* of the elements".

**References:** Laplace (1812), Todhunter (1865), Glaisher (1872), Edgeworth (1883a, 1885), Czuber (1891a), Pearson (1895).

**Citations:** Bowley (1901) [1907], Czuber (1903) [1908], Charlier (1910), Edgeworth (1911), Julin (1921), Keynes (1921), Rietz (1924), Bowley (1928), Cramér (1928), Shewhart (1931), Kendall (1943), Cramér (1946), Kendall (1946).

* Holmes, George K. (1905). Measurement of concentration of wealth. *Publications* [*Journal*] *of the American Statistical Association* **9** (71), 318-319.

**Summary:** In this note, the author comments favorably on the scheme of graphic presentation (Lorenz curve) proposed by Lorenz (1905), but disagrees with the criticism of his own measure of concentration of wealth [see Holmes (1892)] offered by Lorenz. The author writes (pp. 318-319): "It is in the very nature of the principle . . . that a doubling of the column for wealth owned will double the degree of inequality of distribution. This was in accordance with my theory, and was very plainly in my mind at the time my article . . . was written. It seems, however, that Mr. Lorenz has misunderstood my article to this extent. My hypothesis had in view such a result as the doubling of a degree of inequality by the doubling of the column of amount of wealth owned. . . . Concerning this Mr. Lorenz writes: 'The error in this measure lies in the fact that the distance between the medians varies not only with the degree of concentration (for it does this), but also with changes in the total wealth. Let us suppose that each individual's wealth in the above case is doubled. The two medians would now be $17 and $19.65, and their difference $2.65, showing an increase in concentration, but by hypothesis the relative position of the numbers has not changed.' If there is any hypothesis that the relative position of the numbers has not changed, it was and is not mine, nor is it a part of this scheme for measuring distribution. The contrary is true in both cases. A doubling of the degree of inequality which Mr. Lorenz obtains by multiplying the wealth column by 2 is precisely what is to be expected by hypothesis and arithmetical principle."

**Comments:** We see that the doubling of Holmes' measure of concentration when each individual's wealth is doubled is viewed as a virtue by Holmes but as a fault by Lorenz, because the former prefers an absolute measure of concentration and the latter a relative one.

**References:** Holmes (1892), Lorenz (1905).

**Citations:** Watkins (1905, 1908), Persons (1909), von Bortkiewicz (1931).

* Lorenz, M. O. (1905). Methods of measuring the concentration of wealth. *Publications [Journal] of the American Statistical Association* **9** (70), 209-219.

**Summary:** The author discusses various methods of measuring the concentration [or dispersion] of wealth. As part of a triple measure proposed by Holmes (1892), he considers the difference between the median of the amount of wealth and the wealth of the median owner of wealth. He raises the objection that this difference is doubled if the wealth of each individual is doubled, which represents no increase in concentration. A second measure discussed by the author is Bowley's quartile measure of relative variability, which he finds somewhat more satisfactory. A third measure, proposed by the author on page 217, is a graphical one—"plot along one axis cumulative per cents. of the population from poorest to richest, and along the other the per cent. of the total wealth held by these per cents. of the population." The smooth curve connecting the points plotted in this way has come to be known as a Lorenz curve. If the wealth is distributed equally [uniformly], the Lorenz curve is a straight line; the greater the concentration of wealth, the more the Lorenz curve deviates from a straight line.

**References:** Holmes (1892), Pareto (1897), Bowley (1901) [1902].

**Citations:** Holmes (1905), Watkins (1905, 1908), Persons (1909), Gini (1914a), Ricci (1916), Niceforo (1919), Dalton (1920), Gumbel (1928), d'Addario (1930), von Bortkiewicz (1931), Galvani (1931), d'Addario (1932), Galvani (1932), Yntema (1933), Pietra (1935a), d'Addario (1936), Gini (1936, 1939), Davis (1941).

* Watkins, G. P. (1905). Comment on the method of measuring concentration of wealth. *Publications [Journal] of the American Statistical Association* **9** (72), 349-355.

**Summary:** On page 349, the author comments as follows on the article by Lorenz (1905) and the note by Holmes (1905) on the method of measuring the concentration of wealth: "In considering this question, the fundamental point to be kept in mind is that concentration is a fact of relation. Hence *the quantities to be measured are relative*." On page 353, he elaborates on this point as follows: "Mr. Holmes' method is good as far as it goes. Where brevity is more important than completeness, it could hardly be improved upon. But his own misinterpretation conceals the merits of the method. The result should be taken, not as an absolute amount, but as a relative quantity. The *ratio* of the difference between the two medians to the absolute amount, or to the average, of wealth, remains the same when all the absolute amounts have been multiplied by the same number. Mr. Holmes, in his reply to Mr. Lorenz's criticism, fails to see this way out, and continues to interpret his criterion absolutely, trying to give concentration an unnatural and untrue meaning. Inequality of distribution is by its very nature a question of relations between distributees. The satisfactory use of any average, but perhaps especially of the median, supposes regular and symmetrical distribution of the quantities dealt with. The method is therefore not applicable without qualification . . . ." Other comments of the author include the following (pp. 352-353): "Continuity, as well as relativity, is an important requirement for any measure or expression of a series of numerous cases of quantitative change or difference. To meet this requirement, either such numerical classes as those just mentioned above [with equal and small range], or else a curve, should be used. Mr. Lorenz rightly prefers a curve, though he does not see the full possibilities of some numerical procedure. A satisfactory method must be adequately analytic. . . . Mr. Lorenz's method is a great improvement over the others he enumerates. It meets all the requirements of analysis, continuity, and relativity."

**References:** Holmes (1905), Lorenz (1905).

**Citations:** Watkins (1908), Persons (1909), Gini (1914a), von Bortkiewicz (1931).

* Charlier, C. V. L. (1906). Researches into the theory of probability. *Lunds Universitets Arsskrift* (N. F.) **1** (2) [*Kongl. Fysiografiska Sallskapets Handlingar* (N. F.) **16**], No. 5, 59 pp. (JFM 36, 305).

**Summary:** The author discusses the Hagen-Bessel theory that an error of observation may be considered as the sum of a great many small elementary errors, and applies it to frequency curves which are not, strictly speaking, error curves. As a consequence of this theory, he shows that a frequency curve may have one of two forms, which he calls Type A and Type B, and that no other forms can occur except those obtained by superposition (addition) of two or more curves of Types A and B. Charlier's Type A curve is applicable to characters which are unlimited in both directions, and his Type B curve to those (such as counts) which are limited in one direction. The author gives a detailed discussion of his Type A and Type B frequency curves. The only part directly relevant to our study is a discussion of the relative positions of mean, median, and mode. For his Type A curve, he shows that (neglecting moments above the fourth and terms of higher order), the median lies roughly one third of the way from the mean to the mode. He illustrates this fact by computing the mode (3.075), the median (3.359), and the mean (3.501) of the distribution of the frequency of glands in the right foreleg of 2000 female swine, to which he fits a Type A frequency curve with reasonable success, though a Type B curve is theoretically more appropriate and actually fits better at the discontinuous end. Lastly, he studies the problem, first considered by Pearson in 1894, of dissecting a frequency curve into components.

**References:** Laplace (1812), Pearson (1895).

**Citations:** Charlier (1910), Cramér (1928), Kendall (1943), Cramér (1946), Kendall (1946).

* Mansion, Paul (1906). Sur la méthode des moindres carrés dans le Nachlass de Gauss. *Annales de la Societé Scientifique de Bruxelles* **30** (1), 169-174. (JFM 36, 991).

**Summary:** The author summarizes important contributions to the history and the critique of the method of least squares contained in extracts of letters and of papers of Gauss published in the eighth volume of Gauss' *Werke* (Teubner, Leipzig, 1900). He makes brief mention of the theory of combination of observations, introduced by Laplace (1786) [see also Laplace (1793, 1799)], in which the largest error (in absolute value) is smaller than for any other system. He points out that Gauss (1809) criticized this method on the grounds that it uses for the final calculation of the unknowns only a number of equations of condition equal to the number of unknowns; the other equations are used only to decide the choice which one should make.

**References:** Laplace (1799), Legendre (1805), Gauss (1809), Laplace (1812), Gauss (1823).

**Citations:** Goedseels (1909), Mansion (1913).

* Slocum, C. E.; Hancock, E. L. (1906). *Text-Book on the Strength of Materials*. Ginn & Co., Boston. (JFM 37, 850—listing only). Second edition, 1911.

**Summary:** The only part of this book relevant to our subject is the following statement (page 15) based on the "weakest link" concept: "The tensile strength of long rods is affected in a way different from any of the preceding [effect of diameter and form of cross section]. Since no material is perfectly homogeneous, the longer the rod the greater the chance that a flaw will occur in it somewhere. If, then, by numerous tests of short pieces, it has been determined how much a material lacks of being homogeneous, the strength of a rod of this material of any given length can be calculated by means of the theory of probabilities. Such a theory has been worked out by Professor Chaplin [(1880, 1882)] and verified experimentally."

**References:** Chaplin (1880, 1882).

* Galton, Francis (1907a). One vote, one value. *Nature* **75**, 414. (JFM 38, 288).

**Summary:** The author suggests that if a jury has to assess damages or the council of a society has to decide on a sum of money suitable for some particular purpose, each member should independently form a judgment as to the amount and the median of these independent judgments should be taken, since the number of votes that it is too high is exactly balanced by the number of votes that it is too low, while every other estimate is condemned by the majority of voters as being either too high or too low. In particular, use of the arithmetic mean would give a voting power to "cranks" in proportion to their crankiness. For an odd number of votes, the median is the middlemost value; for an even number, it is taken as the arithmetic mean of the two middlemost.

**Citations:** Galton (1907b,c), Hooker (1907), Yule (1907).

* Galton, Francis (1907b). Vox populi. *Nature* **75,** 450-451. (JFM 38, 288).

**Summary:** At a stock show, 800 contestants each paid sixpence for stamped and numbered cards on which they entered their estimates of the dressed weight of a particular living ox, and those who guessed most successfully received prizes. After the completion of the competition, the author collected the cards, weeded out 13 as defective or illegible, and arranged the remaining 787 guesses in order of magnitude. He found that the median estimate was 1207 lbs., the first quartile 1162 lbs. and the third quartile 1236 lbs., so that the quartile deviation (probable error) was 37 lbs. The actual dressed weight proved to be 1198 lbs., so that the median estimate (vox populi) was in error by only 9 lbs. (0.8 percent).

**Reference:** Galton (1907a).

**Citations:** Galton (1907c), Hooker (1907), Yule (1907).

* Galton, Francis (1907c). The ballot-box. *Nature* **75,** 509-510. (JFM 38, 288).

**Summary:** The author answers the criticism by Hooker (1907) by saying that the latter missed the point of the author's earlier papers [Galton (1907a,b)], which was that the verdict given by the ballot box must be the median estimate. He defends the use of the median in the second of those papers on the grounds that the suppression of any one value in a series can only make a difference of one-half place in the median, but may make a great difference in the mean. He admits, however, that the mean would have given a result closer to the true value in the case in question (actually 1197 lbs. as compared with the 1196 lbs. computed by Hooker from the $5^{th}$, $10^{th}$, . . . , $95^{th}$ percentiles).

**References:** Galton (1907a,b), Hooker (1907).

* Hooker, R. H. (1907). Mean or median. *Nature* **75,** 487-488. (JFM 38, 288).

**Summary:** The author, while admitting that the median has certain advantages, such as ease of calculation and the elimination of crank judgments [Galton (1907a)], suggests that the arithmetic mean would be more appropriate in the case of the dressed weight of a live ox [Galton (1907b)]. Judging from the incomplete data given by Galton in the latter paper, he states that the mean would be approximately 1196 lbs., which is much closer to the true weight (1198 lbs.) than is the median (1207 lbs.). He closes by asking Galton to indicate what criteria should be used in deciding whether to use the mean or the median.

**References:** Galton (1907a,b).

**Citations:** Galton (1907c), Yule (1907).

* Kozák, Josef (1907). *Grundprobleme der Ausgleichungsrechnung nach der Methode der kleinsten Quadrate, Erster Band.* Carl Fromme, Wien und Leipzig. (JFM 37, 263-264; 38, 289).

**Summary:** The third section (pp. 49-77) deals with measures of precision, including the mean absolute error, the root-mean-square error, and the probable error, as well as the $m^{th}$ root of the mean of the $m^{th}$ powers of the absolute values of the errors, of which the mean absolute error and the root-mean-square error are special cases (m = 1 and 2, respectively), and the p-percent error limits, of which the probable error is a special case (p = 50). The eighth section (pp. 224-232) deals with the smallest and largest errors (in absolute value) in a series of observations. The author points out that both depend not only on the underlying distribution, but also on the number of observations, the expected value of the smallest error decreasing and that of the largest error increasing as the number of observations increases. The ninth section (pp. 233-245) deals with the rejection of outlying observations, which is related to the question of the maximum error. The author discusses in detail the criteria of Peirce, Chauvenet, and Stone, and refers also to the work of Bessel, Gould, Airy, Faye, and Bertrand, without citing specific publications.

**References:** Hagen (1837) [1882], Gerling (1843), Helmert (1872), Jordan (1877b) [1895], Czuber (1891a, 1903).

**Citations:** Cranz (1896) [1910], Kozák (1908-10).

* Rohne, H. (1907). Einige Anwendungen der Wahrscheinlichkeitsrechnung auf die Schiesslehre. *Artillerische Monatshefte* **1,** 232-236.

**Summary:** The author makes an empirical study of the ratio of the largest and median deviations (ganze und mittlere Streuung), $A_n$ and $s_{50}$, from the arithmetic mean as a function of sample size. He compares the results with theoretical ones computed from the equation

$$n \cdot (2/\pi) \int_{\frac{A_n\ 0.6744}{s_{50}\sqrt{2}}}^{\infty} e^{-t^2} dt = 1,$$

which he tabulates for various values of n [3, 5(5) 30, 40, 50, 100, 500, 1000, 2000]. He proposes the use of this ratio in detecting the presence of outlying observations, as indicated by an observed value of this ratio much greater than the tabulated value, and applies the results to data on artillery shots.

**Citations:** Cranz (1896) [1910], von Eberhard (1938).

* Vogeler, R. (1907). Der Maximalfehler und die amtlichen Fehlergrenzen; ferner Vergleichung einer Reihe züfalliger Ereignisse mit dem Fehlergesetz. *Zeitschrift für Vermessungswesen* **36,** 129-143. (JFM 38, 282).

**Summary:** The author gives an expository account of the controversy between Jordan and Helmert concerning the maximum error. He recalls that Jordan insisted that an error greater than three times the root-mean-square error m should be regarded as a gross blunder and the corresponding observation discarded; since this is not in accord with the Gaussian law of error, which imposes no upper limit on the size of an error, he developed a new error law which is consistent with his hypothesis. Helmert, on the other hand, pointed out that Jordan's law is a low-order approximation to the Gaussian law and that the maximum error depends not only on the error law, but on the number of observations; for a hundred or so observations, 3m might serve very well as an estimate of the maximum error to be expected, but this value is almost sure to be exceeded if the number of observations is increased to several thousand. The author agrees with Helmert's view, giving both theoretical and empirical evidence to support it. He points out that the results of Blümcke (1897, 1898, 1901), while mathematically interesting, are

irrelevant to the question of the maximum error and that of whether the Gaussian law of error should be replaced by a different law.

**References:** Hagen (1837), Helmert (1872 [1907], 1877b), Jordan (1877b [1895, 1904]), Jordan et al. (1879), Jordan (1890), Blümcke (1897, 1898, 1901).

**Citation:** Wellisch (1909).

* Yule, G. Udny (1907). Mean or median. *Nature* **75,** 534. (JFM 38, 288).

**Summary:** The author comments on two papers by Galton (1907a,b). He agrees that Galton's solution of the problem considered in the first of these two papers, that of the amount of money to be voted by a jury or a committee, is correct, since any amount greater than the median would be defeated by majority vote. The problem of averaging a series of estimates in order to arrive at objective truth, as in the case of the dressed weight of the ox discussed in Galton's second paper, appears to the author to be on a different footing. The answer depends on whether one wishes the result to be least erroneous as a rule or least subject to sampling fluctuation and on the distribution of the underlying population. On the latter point, he cites the work of Edgeworth (1885, 1886a, 1887d), who showed that, while the probable error of the median is about 25% greater than that of the mean for samples from a normal population, it is less than that of the mean for sufficiently leptokurtic populations.

**References:** Edgeworth (1885, 1886a, 1887d), Galton (1907a,b), Hooker (1907).

* Kozák, Josef (1908-10). *Grundprobleme der Ausgleichungrechnung nach der Methode der kleinsten Quadrate, Zweiter Band.* Carl Fromme, Wien und Leipzig. Erster Teil, 1908 (JFM 39, 291); Zweiter Teil, 1910 (JFM 41, 275 — listing only).

**Summary:** This volume is subtitled "Theorie des Schiesswesens auf Grundlage der Wahrscheinlichkeitsrechnung und Ferlertheorie." A portion of the sixth section (pp. 229-247) deals with measures of accuracy or precision. The author discusses the same measures of precision as in the third section of Volume 1 [Kozák (1907)], computed now from the observations. He mentions the possibility of determining the probable error by ordering the observations and counting off half the absolute errors, but states that this method is rather inaccurate, especially for a small number of observations, and recommends instead computing the mean absolute error and multiplying by 0.845. In the tenth section, one portion (pp. 555-568) deals with the probable errors of the measures of dispersion discussed earlier and of the median absolute error, while another (pp. 568-581) deals with the rejection of observations. For a set of n observations, the author considers the following special cases of Stone's criterion, which rejects on the average 1 out of m observations when the errors follow a normal distribution: $m = n$, $m = 2n$ (Chauvenet's criterion), and $m = n^2$ (Vallier's criterion).

**References:** Hagen (1837) [1882], Gerling (1843), Helmert (1872) [1907], Jordan (1877b) [1895], Bertrand (1889), Czuber (1891a), Vallier (1894), Cranz (1896), Czuber (1903) [1908], Kozák (1907), Wellisch (1909).

**Citations:** Cranz (1896) [1910], Keynes (1921).

* Mazzuoli, A. (1908). Sulla probabilità di colpire. *Rivista Marittima* **41,** 5-8.

**Summary:** The author proposes the following criterion for the rejection of outliers: Given n observations or measurements of the same quantity, reject any observation which differs from the mean by an amount greater than that which would be expected to occur once in n observations. He tabulates both empirical and theoretical values (which agree fairly well) for the multiple of the probable error which warrants rejection of an observation if it deviates from the mean by more than that amount, for n = 3(1)6(2)16, 19. The theoretical values are apparently based on the assumption of normality.

**Citation:** Cranz (1896) [1910].

* Watkins, G. P. (1908). An interpretation of certain statistical evidence of concentration of wealth. *Publications [Journal] of the American Statistical Association* **11** (81), 27-55.

**Summary:** The author attempts to assess evidence of concentration of wealth contained in probate statistics for various time periods in Massachusetts, the United Kingdom and France. He makes passing mention of the papers by Holmes (1905), Lorenz (1905), and Watkins (1905). He notes that the Lorenz curve is of less practical value than it would appear to be from the results of its application to hypothetical data by Lorenz, because for the actual data used by the author "the curves resemble a capital L with enormously elongated arms and so little thickness of the different parts that the eye can form no judgment of quantitative relations." To overcome this difficulty, he proposes plotting the logarithms of the numbers instead of the numbers themselves. He finds that the resulting logarithmic curves approximate a straight line, which corresponds to an adiabatic curve (general hyperbola) for the original curve. He suggests measuring concentration by the slope of the logarithmic curves, the steeper of two curves being the one which expresses the greater concentration.

**References:** Pareto (1897), Holmes (1905), Lorenz (1905), Watkins (1905).

**Citations:** Persons (1909), Watkins (1909), Gini (1914a), von Bortkiewicz (1931).

* Bohlmann, G. (1909). Die Grundbegriffe der Wahrscheinlichkeitsrechnung in ihrer Anwendung auf die Lebensversicherung. *Atti del IV Congresso Internazionale dei Matematici* (Roma, 1908) **3,** 244-278.

**Summary:** The author writes (p. 266, compiler's translation): "An individual that has reached the age x will be denoted by (x). . . . w will be a limiting age (Grenzalter) that no one outlives. One can either choose it according to experience or let it tend to infinity. Axiom $V_I$. If e is a given positive number, then for all values of x, y and such that $e \leq x \leq y \leq w$, the probability that (x) survives to age y will be measured by a number p(x, y) which depends upon x and y. Axiom VI. The probability p(x, w) that x survives to age w is 0."

**Comments:** The concept of the limiting (oldest) age plays an important role in actuarial theory. We shall see later that it has been the subject of controversy between those who hold that there is a fixed upper limit on the length of human life and those who take the more reasonable view that the oldest age is simply the largest order statistic of a sample drawn from the entire human race, which tends to increase with the size of the sample.

**Reference:** Poincaré (1896).

**Citations:** Keynes (1921), Insolera (1933, 1935, 1937-38).

* Dunkel, Otto (1909). Generalized geometric means and algebraic equations. *Annals of Mathematics* (2) **11,** 21-32. (JFM 40, 124).

**Summary:** Given n positive numbers $m_1, m_2, m_3, \ldots, m_n$ arranged in ascending order, the author defines a mean $y = [(m_1^x + m_2^x + m_3^x + \ldots + m_n^x)/n]^{1/x}$ which is a continuous function of the real variable x for all values of x except $x = 0$. He shows that $\lim_{x\to 0} y = (m_1 m_2 m_3 \ldots m_n)^{1/m}$ [the geometric mean], $\lim_{x\to -\infty} y = m_1$ [the smallest value], and $\lim_{x\to +\infty} y = m_n$ [the largest value]. He also shows that $\Sigma m_i/n \geq (\Sigma m_i m_j / {}_nC_2)^{1/2} \geq (\Sigma m_i m_j m_k / {}_nC_3)^{1/3} \geq \ldots \geq (m_1 m_2 m_3 \ldots m_n)^{1/n}$, with the equality signs holding only when $m_1 = m_2 = m_3 = \ldots = m_n$, the members of the inequality being combination means [see Fechner (1874)], of which the first and last are the arithmetic and geometric means, respectively. Either of the above two series of means may be considered as generalizations of the geometric mean.

**Citations:** Dodd (1922), Galvani (1927, 1931), Cisbani (1938), Gini (1938), Dodd (1940).

* *Gini, Corrado (1909). Il diverso accrescimento delle classi sociali e la concentrazione della ricchezza. Giornale degli Economisti* (2) **38,** 27-83.

**Summary:** Let N be the number of persons in a population who have an income (Fr. revenu, Ital. redditi) greater than r and let A be the aggregate amount of their income. Given two sets, $(N_0, r_0, A_0)$ and $(N_1, r_1, A_1)$, of corresponding values of N, r and A, the author fits a Pareto distribution with parameters K and $\alpha$ to N as a function of r by setting $\log N_0 = \log K - \alpha \log r_0$, $\log N_1 = \log K - \alpha \log r_1$ and solving for K and $\alpha$; likewise, he fits a Pareto distribution with parameters H and $\beta$ to A as a function of r by setting $\log A_0 = \log H - \beta \log r_0$, $\log A_1 = \log H - \beta \log r_1$ and solving for H and $\beta$. If $N_0 < N_1$, so that $A_0 < A_1$, then $\alpha \geq \beta$, with equality holding if and only if all incomes are equal. Let $\delta = \alpha/\beta \geq 1$, so that $N_0/N_1 = (A_0/A_1)^{\delta}$. The author states that $\delta$ seems to him to be an optimum *index of concentration* of income above a certain limit.

**Note:** If we consider, not the distribution of income for an entire population, but that of income (or of some other quantity) for a sample of size n, with ordered values $x_1 \leq x_2 \leq \ldots \leq x_n$, then we have, for $N_0 = m < n$ and $N_1 = n$, $A_0 = \Sigma_{i=n-m+1}^{n} x_i$ and $A_1 = \Sigma_{i=1}^{n} x_i$, so that $(\Sigma_{i=n-m+1}^{n} x_i / \Sigma_{i=1}^{n} x_i)^{\delta} = m/n$. It appears, however, that the index of concentration $\delta$ of a sample can be expected to be approximately constant (subject only to sampling variation) only if the sample has come from a Pareto population; otherwise, it will depend upon m.

**References:** Galton (1869, 1889), Pareto (1897).

**Citations:** Gini (1910), Furlan (1911), Gini (1912, 1914a), Savorgnan (1915), Gini (1926), d'Addario (1930), Yntema (1933), Castellano (1935), Pietra (1935a), Gini (1936, 1939).

* Goedseels, P. J. E. (1909). *Théorie des Erreurs d'Observation.* (Third edition). Charles Peeters, Louvain and Gauthier-Villars, Paris. (JFM 40, 288-289); fourth edition, 1914 (JFM 45, 345-347).

**Summary:** Given (1) a system of n equations in p unknowns $(n > p)$, $F_i(x, y, z, u, \ldots) = r_i (i = 1, 2, \ldots, n)$ and (2) the intervals $(A_i, B_i)$ containing the respective residues $r_i$, one wishes to determine, for each of the unknowns x, y, z, u, . . . , for example for x, the smallest interval (I, S) containing that unknown, i.e. an interval such that for every value of x less than I or greater than S, one or more of the residues $r_i$ lie outside the given intervals. After linearizing the given equations (if they are not already linear) by setting $x = x_0 + h$, $y = y_0 + k$, $z = z_0 + \ell$, . . . , expanding in the form $f(x, y, z, \ldots) = f(x_0, y_0, z_0, \ldots) + h\, df/dx_0 + k\, df/dy_0 + \ell\, df/dy_0 + \ldots + R$ and suppressing the remainder R, the author gives a general solution and a numerical example. He calls this method the *most approximative method.* If all observations are equally trustworthy and if positive and negative errors are equally likely, then $A_1 = A_2 = \ldots = A_n = A$, $B_1 = B_2 = \ldots = B_n = B$, and $-A = B = M$ (say). Consider a series of equations in a single unknown of the form $x = m$, having the same approximation M, and suppose the equations are arranged in order of increasing values of m: $x = m_1 + r_1$, $x = m_2 + r_2, \ldots, x = m_n + r_n$. Then the most approximative value is the midrange, $(m_1 + m_n)/2$, and the approximation of this value is the difference, $M - (m_n - m_1)/2$, between the given approximation, M, and the semirange, $(m_n - m_1)/2$. When $M - (m_n - n_1)/2 = 0$, the midrange

$(m_1+m_n)/2$ is the *exact* value of the unknown. When $M-(m_n-m_1)/2<0$, the data are absurd. Now suppose one does not know the limits of the errors. One then seeks the *minimum approximation*, V, which one can assign to the errors $r_i$ and the corresponding values of the unknowns x, y, . . . , or, what amounts to the same thing, the values which it is necessary to attribute to the unknowns in order that the largest residue (in absolute value) shall be as close as possible to zero. He gives a numerical example for the case $p = 2$, considering the approximation M as an additional unknown, and determining the smallest interval (I, S) which contains it, and thus the smallest value, $V = I$, which one can assign to it. Then he discusses methods not based on order statistics, including the method of least squares and the empirical methods of Tobie Mayer (1750) and Cauchy (1837). In summary, he states that he prefers the most approximative method when the limits of error are known and the minimum approximation otherwise, especially in very important questions, even though the calculations become quite laborious for $p > 2$; in questions of lesser importance, the method of least squares or even one of the empirical methods may be used to save labor.

**Note:** The preface to the third edition would lead the reader to believe that the first (autograph) edition (1898) and the second (first printed) edition (1902) (JFM 33, 247; 34, 261), neither of which the compiler has seen, contain little or nothing on order statistics. However [see the author's report on a paper by de la Vallée Poussin (1911)], the second edition does give the minimum approximation and the corresponding value of the unknown for the case $p = 1$.

**References:** Mayer (1750), Legendre (1805), Gauss (1809, 1823), Cauchy (1837), Bertrand (1889), Mansion (1906), de la Vallée Poussin (1909).

**Citations:** de la Vallée Poussin (1909), Goedseels (1910, 1911), de la Vallée Poussin (1911), Tits (1912), Whittaker & Robinson (1924), Goedseels (1925), Alliaume (1927a), Bruen (1938).

* Persons, W. M. (1909). The variability in the distribution of wealth and income. *Quarterly Journal of Economics* **23** (3), 416-449.

**Summary:** The author notes that there is no generally accepted method of measuring the concentration of wealth. He compares the methods of Lorenz, Pareto and Watkins and one based on the coefficient of variation. He notes the same deficiency as did Watkins (1908) for the Lorenz curve. Pareto's method is based on the assumption that the data follow the Pareto distribution $N = A\,x^{-\alpha}$ or, in logarithmic form, $\log N = \log A - \alpha \log x$. The author determines the theoretical frequencies of classes in two communities of 1000 with $\alpha = 2$ and $\alpha = 3$ respectively and computes the median, quartiles and Bowley's measure of dispersion, $(Q_1 - Q_2)/(Q_1 + Q_2)$, for each. He points out that Watkins' method is a variation of Pareto's. Since Pareto's method is meant to be applied only to cases in which the points lie along a straight line whose slope $\alpha$ can be computed, but Watkins applies his method however the points be situated and offers no numerical measure of variability, the latter method is subjective and its meaning is unclear. After considering the advantages and disadvantages of the various methods, the author recommends the one based on the coefficient of variation, which in no way depends upon order statistics.

**References:** Pareto (1897), Holmes (1905), Lorenz (1905), Watkins (1905, 1908).

**Citations:** Watkins (1909), Gini (1910, 1914a), Ricci (1916), Gini (1917), Winkler (1924), d'Addario (1932), Yntema (1933), Pietra (1935a), Bresciani-Turroni (1937), Gini (1939).

* de la Vallée Poussin, Ch. J. (1909). L'approximation minimum d'un système d'équations (résumé). *Annales de la Société Scientifique de Bruxelles* **33** (1), 173.

**Summary:** A system (1) $a_i x + b_i y + \ldots + \ell_i u - m_i = 0$ $(i = 1, 2, \ldots, m)$ of m linear equations in n unknowns $(m > n)$ is in general incompatible. We shall substitute for it the following: (2) $a_i x + b y + \ldots + \ell_i u - m_i = r_i$ $(i = 1, 2, \ldots, m)$, where we give the name residues to the quantities $r_i$. With Goedseels, we call the minimum approximation of the system (1) the smallest value which one can assign to the largest residue (considered in absolute value) of the system (2) in order that this system (2) shall be compatible. The author states the following theorems concerning the minimum approximation, which assume that any n of the first members of equations (1) form a system of linearly independent expressions: I. The values of the unknowns which provide the minimum approximation M of the system (1) give at least n+1 residues attaining this limit M in absolute value. II. The minimum approximation and the corresponding values of the unknowns, for a system of n+1 equations in n unknowns, are obtained by general formulas. III. If $m > n+1$, the minimum approximation of a system of m equations in n unknowns is that of a certain system of n+1 equations which are part of the proposed system. One can deduce from these theorems an iterative procedure for determining the minimum approximation. It is stated that the complete memoir will appear elsewhere [see de la Vallée Poussin (1911)].

**Reference:** Goedseels (1909).

**Citations:** Goedseels (1909), de la Vallée Poussin (1911).

* Watkins, G. P. (1909). The measurement of concentration of wealth. *Quarterly Journal of Economics* **24** (1), 161-179; rejoinder by W. M. Persons, 180-190.

**Summary:** The author states at the outset that this paper is intended primarily as a reply to the paper by Persons (1909), but that it contains some further development of the points under consideration. He objects to the use by Persons of the coefficient of variation (the ratio of the standard deviation to the arithmetic mean) because the distribution of wealth is far from the normal distribution, for which the arithmetic mean and the standard deviation are appropriate measures of central tendency and of dispersion, respectively. He suggests that the mode or the median would be a more appropriate measure of the former and the average deviation of the latter. He defends his logarithmic curve against the objections raised by Persons that "no numerical measure is offered" and "the comparison of slopes of the curves has to be made entirely by eye" by saying that the direction of the difference between slopes is important for comparison and is usually obvious, while the degree of the difference is less important than Persons claims. He also considers the Lorenz curve and raises objections to its use, especially for truncated data. In conclusion, he states (p. 178): "All the objections raised by Dr. Persons to the use of logarithmic curves, as enumerated by him, are met in the foregoing discussion. The meaning of the test *is* evident to one who is familiar with curves and with the significance of logarithms. The eye is put to no severe strain to tell whether two lines whose position may be shifted right or left at will are parallel or not. The modification of the curve by the use of averages for the size of estates meets the objection with regard to the largest estates. . . . " In his rejoinder, Persons states that his argument for the adoption of the coefficient of variation was not based on *a priori* reasons, such as similarity between the distribution of wealth and the normal curve of distribution, but on practical ones (how it would work), the same justification used by Pareto and by Watkins for their methods. He discusses the Lorenz curve and the Pareto and Watkins methods based upon approximate straight lines. He questions whether the relationship is adequately represented by any straight line, but if one is to be used he favors Pareto's method of finding the single straight line which best fits all the points to Watkins' method of connecting adjacent points by straight lines. In conclusion, he states (p. 190): "There is a need for a method of interpreting wealth statistics that does not allow the subjective element to enter. The coefficient of variability is such a measure. It does not tell the whole story concerning the distribution of wealth, but simply enables one to answer the single question, is the concentration among one set of estates at one period greater or less than the concentration among another set of estates at another period?"

**References:** Watkins (1908), Persons (1909).

**Citations:** Gini (1914a), Ricci (1916).

* Wellisch, Siegmund (1909). *Theorie und Praxis der Ausgleichungsrechnung, Erster Band: Elemente der Ausgleichungsrechnung.* Carl Fromme, Wien-Leipzig. (JFM 40, 287-288).

**Summary:** In Section 19 (pp. 77-81), the author gives formulas for estimating the probable error from the median of the absolute errors and for the probable error of such an estimate. In Section 20 (pp. 81-84), he gives a numerical example of the use of this method along with others based on the square of the mean square root, the mean first power, and the square root of the mean square of the absolute errors. In Section 27 (pp. 105-108), he discusses the estimation of the probable error from differences of all pairs of observations, referring to the work of von Andrae (1869), Jordan (no specific reference), and Helmert (1876a). In Section 28 (pp. 108-110), he compares the accuracy of this and other methods, and in Section 29 (pp. 111-112), he gives a numerical example of their use. In Section 40 (pp. 149-153), which is entitled "Ausscheidung von Beobachtungen" (Rejection of Observations), the author gives a discussion of the pros and cons of rejecting discordant observations, with a history of the various methods that have been proposed. The author himself favors a criterion which would, in effect, reject on the average exactly one among n observations, by setting $\Theta(t_n) = (n-1)/n$ for a sample of size n, where $\Theta(t) = (2/\sqrt{\pi}) \int_0^t e^{-t^2} dt$, and rejecting all observations which differ from the mean by more than $t_n\sqrt{2}\sigma$, where $\sigma$ is the population standard deviation (if known) or its root-mean-square estimate from the sample. This is the criterion of Stone (1868) with m = n.

**References:** Anonymous [Svanberg] (1821), Muncke (1825), Hagen (1837), Bessel & Baeyer (1838), Gerling (1843), De Morgan (1847), Peirce (1852), Gould (1855), Airy (1856), Winlock (1856), Chauvenet (1863), von Andrae (1869), Glaisher (1873), Stone (1873b), Helmert (1876a, 1877b), Jordan (1877a), Newcomb (1886), Lehmann-Filhes (1887), Bertrand (1888b), Faye (1888), Czuber (1891a), Vogeler (1907).

**Citations:** Kozák (1908-10), Wellisch (1913).

* Charlier, C.V.L. (1910). *Grunddragen af den Matematiska Statistiken.* Statsvetenskaplig Tidskrifts Expedition, Lund. Second edition, *Vorlesungen über die Grundzüge der Mathematischen Statistik,* Verlag Scientia, Lund, 1920. English translation, *Elements of Mathematical Statistics,* by J. A. Greenwood, Cambridge, Mass., 1947.

**Summary:** The preface to the first edition contains comments on the development of mathematical statistics, of which the following are of special interest: "The *Théorie analytique des probabilités* of Laplace [(1812)] is doubtless the most significant work that has appeared in the realm of mathematical statistics. Unfortunately, excepting Poisson's elegant development of some theorems of Laplace, the many points of attack found in the work of Laplace remained as good as unobserved. Only in very recent years have we opened our eyes to the great collection of undeveloped basic theorems which are entombed in the great work of Laplace. Responsibility for the stagnation which set in at that time in the development of mathematical statistics rests principally upon Gauss. This great mathematician believed himself able to prove that fluctuations in the elements of a statistical series—he concerned himself chiefly with series of astronomical and geodetic observations—follow strictly the simple law called after him the Gaussian law of error. Where deviations appeared, he believed that he could attribute them solely to the small number of observations. He stated—on the basis of an erroneous mathematical proof—that the deviations would vanish if only the number of observations were sufficiently large. This theorem pervaded all mathematical statistics of the nineteenth century like an article of faith, and the method of least squares, based on the Gaussian law of error, was, and often still is, considered a definitive solution of the problem of strict scientific treatment of series of observations." (pp. 1-2 of Greenwood's translation). It is stated by von Bortkiewicz (1922b, footnote on p. 198) that the author (second edition, footnote on p. 16), like Czuber (1918) and Yule (1911), considered six times the

standard deviation as a sort of normal value of the range. This footnote is not found in Greenwood's translation, unless the reference is to the footnote on page 14, which would make the ratio of range to standard deviation five instead of six. Moreover, the author's awareness that this ratio depends on the sample size is clearly shown by his Table 6 (Sec. 10, p. 18 of Greenwood's translation), in which he gives the limits, either side of the mean, beyond which an average of one element of a statistical series with N = 10(10)100(100)1000, 10000, 100000 elements will fall. Chapters XI and XII deal with Charlier Type A and Type B frequency curves, respectively.

**References:** Laplace (1812), Hagen (1837), Galton (1889), Pearson (1895), Edgeworth (1905), Charlier (1906).

**Citations:** Czuber (1921), Keynes (1921), von Bortkiewicz (1922b), Crum (1923), Dodd (1923), Rietz (1924), Cramér (1928), von Mises (1931), Boldrini (1942), Cramér (1946).

* Gini, Corrado (1910). Indici di concentrazione e di dipendenza. *Società Italiana per il Progresso delle Science, Atti del 3ma Riunione*, pp. 453-469 (summary). *Biblioteca dell' Economista* (5) **20** (1922), 1-151 (complete); errata, 341.

**Summary:** Let $a_i$ be the value of some characteristic A (e.g., income or wealth) of the $i^{th}$ of n individuals ordered according to the magnitude of that characteristic, so that $a_1 \leq a_2 \leq \ldots \leq a_n$. Then the mean value of that characteristic (income, say) for the highest m of the n individuals is greater than or equal to that for all n individuals; i.e., $\Sigma_{i=n-m+1}^{n} a_i/m \geq \Sigma_{i=1}^{n} a_i/n$. This inequality can be written in the equivalent form $A_m \equiv \Sigma_{i=n-m+1}^{n} a_i / \Sigma_{i=1}^{n} a_i \geq m/n$. In either case, the inequality is strict unless $a_i$ is the same for all i. If the characteristic A is income, then a value of $A_m$ which greatly exceeds m/n indicates that income is heavily concentrated in the hands of the m individuals with the highest incomes. The author defines the *index of concentration* $\delta$ as the exponent of the power to which $A_m$ must be raised to make it equal to m/n, i.e. the value of $\delta$ such that $A_m^{\delta} = m/n$. Obviously, $\delta \geq 1$; $\delta = 1$ corresponds to the case of no concentration of income (all incomes equal), and the larger the value of $\delta$, the greater the concentration of income in the hands of the m individuals with the highest incomes. For a given n, of course, $\delta$ may vary with m. {See note on paper by Gini (1909).} Now let $b'_i$ be the value of a second characteristic B (wealth, say) of the individual whose income is $a_i$. While the values of $a_i$ are ordered ($a_x > a_{x-1}$, x = 2, 3, . . . , n), the values of $b'_i$ are not necessarily ordered; we may have $b'_x \gtrless b'_{x-1}$. Hence, we may have $B'_m \equiv \Sigma_{i=n-m+1}^{n} b'_i / \Sigma_{i=1}^{n} b'_i \gtrless m/n$. If $B'_m > m/n$, characteristic B is also concentrated in the hands of the m individuals having the highest value of characteristic A (which is almost certainly true if A is income and B is wealth); if $B'_m = m/n$, there is no relation between the concentrations of A and B; if $B'_m < m/n$, B is concentrated in the hands of the n − m individuals having the lowest values of A (which is likely to be true if A is income or wealth and B is the number of children). Put differently, $B'_m > m/n$ implies a positive relation between A and B; $B'_m = m/n$, no relation; and $B'_m < m/n$, a negative relation. If the relation between A and B is positive, we may have $A_m \geq B'_m$. If $A_m = B'_m$, the relation is perfect, i.e. characteristics A and B are equally concentrated; if $A_m > B'_m$, A is more concentrated than B; if $A_m < B'_m$, A is less concentrated than B. The *index of dependence* of B from A is defined as the value of $\rho$ such that $(B'_m)^{\rho} = m/n$. The *index of independence* of B from A is defined as the value of $\tau$ such that $(A_m)^{\tau} = B'_m$. The index $\delta$ of concentration of A is related to the indices $\rho$ and $\tau$ by the equation $\tau = \delta/\rho$. If the individuals are ordered according to their values of characteristic B, so that $b_1 \leq b_2 \leq \ldots \leq b_n$ and if $a'_i$ is the value of characteristic A for the individual whose B-value is $b_i$, one can determine an index of concentration of B, an index of dependence of A from B, and an index of independence of A from B. If the indices of dependence and independence of A from B are respectively the same as those of B from A, the relations between A and B are said to be reversible; otherwise, irreversible. Examples using real data and references are given in the complete (1922) version.

**References:** Pareto (1897), Gini (1909), Persons (1909).

**Citations:** Furlan (1911), Mortara (1911), Gini (1912), Porru (1912), Gini (1914a), Savorgnan (1915), Ricci (1916), Gini (1917), Niceforo (1919), Gini (1926), Saibante (1928), d'Addario (1930), von Bortkiewicz (1931), Yntema (1933), Castellano (1935), Gini (1936), Bresciani-Turroni (1939), Gini (1939), Pizzetti (1941b).

* Goedseels, P. J. Éd. (1910). Application de la théorie des erreurs de l'auteur à la compensation des coordonnées des sommets dans les levés topographiques. *Annales de la Société Scientifique de Bruxelles* **34** (2), 257-287. (JFM 41, 270).

**Summary:** The author summarizes the results given in his book [Goedseels (1909)] on the most approximative method, the method of minimum approximation, and the method of least squares, and applies all three methods to a numerical example involving the compensation of the coordinates of the vertices in a topographic survey. He insists that the method of least squares is not as good as the other two methods, even in the case of normally distributed errors, for which it gives the most probable values, since these values are inadmissible if they lie outside the interval (I,S) of the most approximative method. In the numerical example, he first ascertains that the maximum error $\epsilon$ specified by the observer is admissible, i.e. that it is not less than the minimum approximation m. Then he proceeds to compensate the z-coordinates of the data points by the most approximative method and by the method of least squares. He rejects the results of the latter method as inadmissible, since the z-coordinate it gives for one point lies outside the interval given by the most approximative method. He shows that in this particular instance one can obtain admissible results by the method of least squares if he assumes that the probable errors of observation are not constant but proportional to the second members of the equations. But recomputation under that assumption is more laborious than the most approximative method, and moreover it does not always yield admissible results. Finally, the author ignores the maximum error stated by the observer and compensates the z-coordinates by the method of minimum approximation. The resulting values lie within the intervals given by the most approximative method, and the author shows that this must always be true if any admissible value $e \geq m$ is designated as $\epsilon$ by the observer.

**Reference:** Goedseels (1909).

* Edgeworth, F. Y. (1911). Probability. *Encyclopaedia Britannica*, 11th edition, Vol. 22, pp. 399ff.; 13th edition (1926), Vol. 22, pp. 376-403.

**Summary:** In the introductory paragraphs (p. 376) the author writes: "The doctrine of averages and of the deviations therefrom technically called 'errors' is distinguished from the other portion of the calculus [*of probabilities*] by the peculiar difficulty of its method. The paths struck out by Laplace and Gauss have hardly yet been completed and made quite secure. The doctrine is also distinguished by the importance of its applications. The theory of errors enables the physicist so to combine discrepant observations as to obtain the best measurement. It may abridge the labour of the statistician by the use of samples. It may assist the statistician in testing the validity of inductions. It promises to be of special service to him in perfecting the logical method of concomitant variations; especially in investigating the laws of heredity. . . . The kinetics of free molecules (gases) forms another important branch of science which involves the theory of errors. . . . In Part I probability and expectation will be considered apart from the peculiar difficulties incident to errors or deviation from averages. . . . Part II is devoted to averages and the deviations therefrom, or more generally that grouping of statistics which may be called a law of frequency. . . . " Our interest lies in Part II (pp. 390-403). The author defines and discusses various averages (arithmetic, geometric, harmonic, and quadratic means; mode and median); the normal law of error and various "proofs" of it; other laws of error or laws of frequency, including the generalized law of error (Edgeworth series) proposed by the author [Edgeworth (1905)] and the Pearson system of frequency curves [K. Pearson (1895)] as well as the laws of Bernoulli, Cauchy, Poisson and Pareto; measures of dispersion (including the standard deviation and the mean absolute deviation); linear and nonlinear regression; the method of least squares, a modification also making use of mean cubes, and Laplace's Method of Situation, both in its original form [Laplace (1799)] and as modified and extended by

the author [Edgeworth (1888)]; normal and abnormal correlation; and the treatment of discordant observations.

**Note:** All references to specific pages are to those in the 13th (1926) edition.

**References:** Laplace (1799, 1812), Gauss (1823), Quetelet (1835, 1846), DeMorgan (1847), Herschel (1850), Airy (1861), Todhunter (1865), Venn (1866), Whitworth (1867), Glaisher (1872), Galton (1875), Merriman (1877), Edgeworth (1885, 1886a), Newcomb (1886), Edgeworth (1887a,d, 1888), Bertrand (1889), Galton (1889), Czuber (1891a), K. Pearson (1895), Poincaré (1896), Pareto (1897), Whitworth (1897), Edgeworth (1898), Czuber (1899), Sheppard (1899a,b), Bowley (1901), K. Pearson (1902), Czuber (1903), Kapteyn (1903), Edgeworth (1905).

**Citations:** King (1912), Keynes (1921), Rietz (1924), Bowley (1928), Shewhart (1931), Bruen (1938).

* Furlan, V. (1911). Neue Literatur zur Einkommensverteilung in Italien. *Jahrbücher für Nationalökonomie und Statistik* **97** [(3) **42**], 241-255.

**Summary:** The author summarizes the results given by Gini (1909, 1910) on distributions of income and on Gini's index of concentration $\delta$. Under the tacit assumption that incomes follow the Pareto law $y = Ax^{-\alpha}$, he proves that $\delta = \alpha/(\alpha-1)$ [or $\alpha = \delta/(\delta-1)$]. He notes, however, that for certain places and times (e.g. Hamburg, 1883-1899), the value of $\delta$ computed directly from the data is markedly smaller than that computed from the formula $\delta = \alpha/(\alpha-1)$. He points out that Gini (1910) sought to explain the discrepancy on the basis of assessment of the higher incomes at a greater percentage of their true value than the lower incomes. Conversely, if the lower incomes are assessed at a greater percentage of their true value than the higher ones, the discrepancy between $\delta$ and $\alpha/(\alpha-1)$ will be in the opposite direction.

**Comments:** The compiler suspects that discrepancies between $\delta$ and $\alpha/(\alpha-1)$ may be due to failure of the true incomes to follow the Pareto law, as much as or more than to assessment of high and low incomes at different percentages of their true values.

**References:** Pareto (1897), Gini (1909, 1910).

**Citations:** Czuber (1914), Gini (1914a), Savorgnan (1915), Ricci (1916), von Bortkiewicz (1931), Castellano (1935), Gini (1939).

* Goedseels, P. J. E. (1911). Simplifications de la méthode la plus approximative et de l'approximation minima. Etalonnage des lunettes stadimétriques. *Annales de la Société Scientifique de Bruxelles* **35** (1), 351-368. (JFM 42, 254).

**Summary:** After a brief description of the two methods, the most approximative method and the method of minimum approximation, considered in the third edition of his book on the theory of errors [Goedseels (1909)], the author calls attention to an interesting study of the method of minimum approximation, including a simplification of that method, proposed by de la Vallée Poussin (1911). He proposes, in turn, two other simplifications of both methods. These simplifications apply only when there are only one or two unknowns, of which at least one is positive. The latter is really no restriction at all, since the data can be transformed so that at least one unknown is positive. The restriction to one or two unknowns is less serious than one might think, since one always proceeds by successive elimination of the unknowns, and the number of them is always eventually less than three; moreover it is in the final stages, where this condition is satisfied, that the methods in question become most complicated. In an abstract of this paper reprinted in a footnote on pp. 351-352, de la Vallée Poussin writes: "The method of minimum approximation utilizes generally in the definitive solution only a small number of the proposed relations, but the difficulty is to discern them. The work of Goedseels has as its object the rapid elimination in the course of the calculations of the useless relations which enter in, by means of the following remark of which the evidence is immediate: If a *positive* unknown $\lambda$ satisfies a series of inequalities of the form (1) or (2): (1) $a_1\lambda + b_1 \leq f$, $a_2\lambda + b_2 \leq f$, . . .; (2) $a_1\lambda + b_1 \geq \phi$, $a_2\lambda + b_2 \geq \phi$, . . . any one of the inequalities of the series can be neglected

in the presence of another of which the two coefficients a and b are respectively equal or larger in case (1), equal or smaller in case (2). This remark [applicable also to the most approximative method] makes feasible the determination of the minimum approximation of a series of ten observations performed . . . in order to calibrate a stadimetric telescope. The calculations are even simpler than those of the method of least squares, over which they also have the immense advantage of furnishing information of a precise nature. The author indicates, in addition, . . . a process of successive substitutions or of interpolation which permits avoiding with advantage the formation of all the inequalities to which the systematic method of elimination leads. . . . " (Compiler's translation).

**References:** Goedseels (1909), de la Vallée Poussin (1911).

**Citations:** Tits (1912), Dodd (1922), Bruen (1938).

* Keynes, J. M. (1911). The principal averages and the laws of error which lead to them. *Journal of the Royal Statistical Society* **74**, 322-331.

**Summary:** The author begins (p. 322) by stating the problem in the following words: "We are given a series of measurements, or observations, or estimates of the true value of a given quantity; and we wish to determine what function of these measurements will yield us the *most probable* value of the quantity, on the basis of this evidence. . . . Corresponding to each law of error which we might assume, there is some function of the measurements which represents the most probable value of the quantity. The object of the following investigation is to discover what laws of error, if we assume them, correspond to each of the simple types of average, and to discover this by means of a systematic method." Using the principles of inverse probability and of the multiplication of independent probabilities, he proceeds to determine the most general laws of error leading to the arithmetic mean, the geometric mean, the harmonic mean, and the median; he states that he does not think it is possible to find by his method a law of error which leads to the mode. The normal law and Laplace's first law are special cases of the most general laws leading respectively to the arithmetic mean and the median, and the only ones which satisfy the additional restriction that positive and negative errors of the same magnitude be equally probable. The lognormal law is a special case of the most general law leading to the geometric mean. There are no symmetric laws which lead to either the geometric or the harmonic mean. The author points out that Laplace did not notice that his first law of error led to the median because, instead of finding the most probable value, he sought the value which the true value is as likely to fall short of as to exceed. He also states that Fechner was the first to introduce the median into use. He makes use of the property of the median (which he notes was known to Fechner) that it is the value from which deviations of the observations must be taken in order to minimize the sum of their absolute values.

**References:** Laplace (1774), Gauss (1809?), Laplace (1812), Fechner (1874?), Kapteyn (1903).

**Citations:** Keynes (1921), Rietz (1924), Bruen (1938), Kendall (1946).

* Mortara, Giorgio (1911). Note di economia induttiva: Sulla distribuzione dei redditi. *Giornale degli Economisti e Rivista di Statistica* **42,** 455-471.

**Summary:** The author discusses distributions of income and various statistical measures that can be applied to them, including several that make use of ordered observations. As examples, he tabulates average incomes of the first, second, . . ., tenth thousandths, hundredths and tenths (from above) of persons in Saxony in 1888 and 1908, and their ratios to the general mean in the year under consideration, as well as average salaries, in 1882 and 1910, for the top tenth, second tenth, . . ., lowest tenth of officials and of magistrates in Italy and their ratios to the respective general means. In connection with a discussion of Gini's index of concentration, he points out that, given two sets of n values each arranged in decreasing order, $a_1 > a_2 > \ldots > a_n$ and $b_1 > b_2 > \ldots > b_n$, with means a and b respectively, a sufficient (but not necessary) condition that $\Sigma_{i=1}^{k} a_i/ka > \Sigma_{i=1}^{k} b_i/kb$ is that $(a_i/a) > (b_i/b)$ for $i = 1,2, \ldots, k$ and $(a_i/a) <$

$(b_i/b)$ for $i = k+1, k+2, \ldots, n$. Like Furlan (1911), the author proves that if Pareto's law holds, $\alpha = \delta/\delta-1$), where $\delta$ is Gini's index of concentration and $\alpha$ is one of the parameters (sometimes called an index of inequality) in Pareto's law. The author points out that the index $\delta$ has two important advantages over the index $\alpha$: (a) it is adapted equally well to incomplete (censored or truncated) income data as to complete data and (b) its range of validity is greater [it does not depend on the assumption that the data obey Pareto's law].

**Note:** Gini's index of concentration and other statistical measures making use of ordered observations are applied by the author to finite populations, but they are equally applicable to samples, in which case the ordered observations are the sample order statistics.

**Reference:** Gini (1910).

**Citations:** Gini (1914a), Ricci (1916), d'Addario (1930), Castellano (1935), Gini (1939).

* de la Vallée Poussin, Ch. J. (1911). Sur la méthode de l'approximation minimum. *Annales de la Société Scientifique de Bruxelles* **35** (2), 1-16; rapport par E. Goedseels, **35** (1), 65-67. (JFM 42, 255).

**Summary:** The author begins with the following introduction (pp. 1-3): "Let us consider a system of n linear equations in m unknowns (n>m): (1) $a_ix + b_iy + \ldots + \ell_iu - m_i = 0$ $(i=1, 2, \ldots, n)$. This system is in general incompatible. We will substitute for it the following: (2) $a_ix + b_iy + \ldots + \ell_iu + m_i = r_i$ $(i=1, 2, \ldots, n)$, where we give to the quantities $r_i$ the name *residues*, and we call, with Goedseels [(1909)] the *minimum approximation* of the system (1) the smallest value which one can assign to the largest residue of the system (2), considered in absolute value, for which this system (2) is compatible. The existence of this approximation has been established implicitly by the procedure for determination explained by Goedseels in his book. Moreover it is easy to see that this minimum approximation should exist, since the largest residue, without regard to sign, is a positive quantity and has, consequently, a finite lower limit. It is in addition a continuous function of the variables x, y, . . ., u. By consequence, the lower limit is necessarily attained, at least for a system of finite values of the variables, since infinite values make all the residues infinite, with the possible exception only of $m-1$ when one assumes the determinants (a, b, . . ., $\ell$) different from zero. The problem having as object the calculation of the minimum approximation corresponding to a system (1) was stated and solved . . . by Laplace [(1799), Book III, no. 39]. But the method followed in this solution involves more laborious calculations than one could use in the current practice of sciences of observation. Six years after Laplace, . . . Legendre [(1805)] posed the same problem. He stated that he did not know an easy method to resolve the question and proposed to substitute for it the method of least squares. The method called the minimum approximation by Goedseels involves calculations simpler than those of Laplace and better disposed for calculators. But these calculations are still much too long in most cases. We are therefore going to revive the question and expound with precision several theorems which can facilitate, in practice, the solution of a system of equations (1) by the method of minimum approximation. We assume that none of the determinants (a, b, . . ., $\ell$) is equal to zero." (Compiler's translation). The author states and proves the following theorems: (1) The values X, Y, Z, . . . of the unknowns which give the minimum approximation M of the system (1) give at least (m+1) residues equal to $\pm$M. (2) If the number n of equations exceeds the number m of unknowns by more than one, the minimum approximation M of the set of equations is equal to the minimum approximation corresponding to one of the groups formed by combining the n equations (m+1) at a time in all possible ways. (3) The minimum approximation M of a series (1) of n linear equations in m unknowns is equal to the largest minimum approximation $M_1$, $M_2$, . . . corresponding to the various groups which one can form by taking the equations (1) m+1 at a time.

**References:** Laplace (1799), Legendre (1805), Goedseels (1909), de la Vallée Poussin (1909).

**Citations:** Goedseels (1911), Tits (1912), Mansion (1913), Dodd (1922), Whittaker & Robinson (1924), Bruen (1938).

* Yule, G. Udny (1911). *An Introduction to the Theory of Statistics.* Charles Griffin and Co., Ltd., London. (JFM 42, 263—listing only); review, *Nature* **87,** 345. Second edition, 1912; third edition, 1916; fourth edition, 1917; fifth edition, 1919; sixth edition, 1922 (JFM 48, 616—listing only); seventh edition, 1924 (JFM 50, 355—$\ell$.o.); eighth edition, 1927 (JFM 53, 521—$\ell$.o.); ninth edition, 1929 (JFM 55, 939—$\ell$.o.), tenth edition, 1932; eleventh edition (with M. G. Kendall), 1937 (JFM 63, 1128); twelfth edition, 1940 (JFM 67, 472—$\ell$.o.); thirteenth edition, 1944; fourteenth edition, 1950.

**Summary:** Chapter 5 (of the fourteenth edition—see note below) deals with averages, Chapter 6 with measures of dispersion, and Chapter 7 with measures of skewness and kurtosis. In each case one or more of the measures is based on order statistics. The median (Mi) and its relation to the arithmetic mean (M) and the mode (Mo) are discussed on pp. 111-118, the range on p. 125, the mean deviation about the median on pp. 138-140, the quartiles ($Q_1$, $Q_3$) and the quartile deviation [$Q = (Q_3 - Q_1)/2$] on pp. 140-143, and Pearson's measure of skewness [(Mean − Mode)/Standard Deviation $\doteq$ 3(Mean − Median)/Standard Deviation] and another measure of skewness based on order statistics [$(Q_1 + Q_3 - 2Mi)/2Q$] on pp. 160-163. The standard errors of quantiles [median, quartiles, deciles, etc.] and of the semi-interquartile range are discussed in Chapter 18, as is the correlation between the errors in two quantiles. Let $x_p$ be the p quantile and $y_p$ the corresponding probability density, and $\sigma$ be the standard deviation of the underlying population. Then $\sqrt{\text{var}(x_p)} = \sigma_{x_p} = (\sigma/y_p)\sqrt{pq/n}$. The correlation of the errors of two quantiles $x_{p_1}$ and $x_{p_2}$ is $\sqrt{p_2q_1/q_2p_1}$, where $q_i = 1 - p_i$ $(i = 1, 2)$. The standard error of the semi-interquartile range (quartile deviation) in a normal distribution is $0.78672\ \sigma/\sqrt{n}$.

**Note:** All references in the above summary to specific chapters and pages are to the fourteenth edition. Not having access to earlier editions, the compiler is unable to state in what editions the various ideas first made their appearance and how they evolved, though this would be interesting and informative.

**Citations:** Bowley (1901) [1920], Gini (1912), King (1912), Edgeworth (1913), von Bortkiewicz (1915), Gini (1915a,b), Secrist (1917), Niceforo (1919), Persons (1919), Czuber (1921), Hall (1921), Jones (1921), Julin (1921), Keynes (1921), von Bortkiewicz (1922b), Crum (1923), Edgeworth (1923), Rietz (1924), Goodrich (1927), Bowley (1928), Horst (1931), Shewhart (1931), Student (1931), Dodge (1933), Julin (1933), Foster (1934), Pearson (1935), Snedecor (1937), Bruen (1938), Gini (1939), Weibull (1939b), Arley & Buch (1940), Simon (1941), Thompson (1941), Beard (1942), Boldrini (1942), Evans (1942), Wilks (1943), Cramér (1946), Hoeffding (1948), Benson (1949), Walsh (1949a).

* Dettori, Giovanni (1912). Contributo allo studio della variabilità dei prezzi. *Studi Economico-Giurdici della Reale Università di Cagliari* **4** (1a), 1-77.

**Summary:** The author discusses four measures of variability: (1) the simple mean deviation from the arithmetic mean, ${}^1S_A$; (2) the quadratic mean deviation from the arithmetic mean (standard deviation), ${}^2S_A$; (3) the simple mean deviation from the median, ${}^1S_M$; and (4) Gini's mean difference, $\triangle$. The last two of these are based on order statistics. He calculates these four measures for the prices of various commodities over different periods of time in different countries (78 sets of data in all) and their ratios to the arithmetic mean (measures of relative variability). He compares the values of the other measures with the corresponding values (for the same data) of Gini's mean difference. He reports that, for the 78 sets of data, $\triangle/{}^1S_A$ varies from 1.08 to 1.56, $\triangle/{}^2S_A$ varies from 0.60 to 1.36, and $\triangle/{}^1S_M$ varies from 1.19 to 1.92. He states that the mean difference provides the precise measure of absolute variability of phenomena and should be used for economic data. However, he shows that the other three measures are closely related to the mean difference. As a measure of the degree of relationship, he uses Gini's index of cograduation (rank correlation coefficient) rather than the product-moment correlation coefficient.

**Reference:** Gini (1912).

**Citations:** Gini (1912), Czuber (1914), Gini (1914a), Pietra (1915), Gini (1916b), Niceforo (1919), Gini

(1932), Castellano (1935), Pietra (1935a), Gini (1939).

* Fisher, R. A. (1912). On an absolute criterion for fitting frequency curves. *Messenger of Mathematics* (2) **41,** 155-160. (JFM 42, 302).

**Summary:** The author considers the method of least squares and the method of moments, but rejects both of them in favor of what is essentially the method of maximum likelihood, though he does not use that name for it. He points out (p.160) that the quantity which he calls the likelihood in later works is "a relative probability only, suitable to compare point with point, but incapable of being interpreted as a probability distribution over a region, or of giving any estimate of absolute probability".

**Comments:** This paper contains nothing directly relevant to a study of order statistics, but it is of interest because it contains the author's first statement of the principle of maximum likelihood. Fisher was not the first to use this principle [see Lambert (1760), D. Bernoulli (1778), and Gauss (1809)], as many statisticians have apparently been led to believe. Nevertheless, the impetus which he gave to it is responsible for its central role in the modern theory of statistical estimation, of which he may be said to be the founder. In the sequel, we shall encounter many instances in which it is employed, especially in estimation from censored samples.

**References:** Chauvenet (1863).

**Citations:** Fisher (1922), Cramér (1946), Kendall (1946).

* Gini, Corrado (1912). Variabilità e mutabilità, contributo allo studio delle distribuzioni e delle relazioni statistiche. *Studi Economico-Giuridici della R. Università di Cagliari* **3** (2), i-iii, 3-159. Review by W. Lexis, *Jahrbücher für Nationalökonomie und Statistik* **103** [(3) **48**] (1914), 403-404.

**Summary:** The author discusses in considerable detail the mean difference which has come to be called by his name [even though he points out that it goes back to the work of Jordan (1869, 1872), von Andrae (1869, 1872), and Helmert (1876a)]. Let $a_1, a_2, \ldots, a_n$ be n quantities arranged in ascending order [the order statistics of a sample of size n]. The author shows that the arithmetic mean of the $n(n-1)$ differences (without repetition) of the n quantities $a_i$ ($i = 1, 2, \ldots, n$) is given by $\triangle = \{2/[n(n-1)]\} \sum_{i=1}^{(n+1)/2} (n+1-2i)(a_{n-i+1}-a_i)$, while that of the $n^2$ differences with repetition (including n zero differences) is given by $\triangle_R = (2/n^2)\sum_{i=1}^{n+1)/2}(n+1-2i)(a_{n-i+1}-a_i)$, so that $\triangle_R = [(n-1)/n]\triangle$. Equivalently, one may write $\triangle = \{1/[n(n-1)]\} \sum_{i=1}^{n} d_{i,n-i+1}|a_i-a_{n-i+1}|$ and $\triangle_R = (1/n^2)\sum_{i=1}^{n} d_{i,n-i+1}|a_i - a_{n-i+1}|$ where $d_{i,n-i+1}=|n+1-2i|$. The author also discusses other measures of variability and their relations to each other and to the mean difference. Included are the $m^{th}$ root of the mean of the $m^{th}$ powers of the absolute values of the deviations from the arithmetic mean [of which the mean deviation (from the mean) and the standard deviation are special cases ($m = 1$ and $m = 2$ respectively)], the mean deviation from the median, the root-mean-square difference, and the quartile deviation. The author shows that the root-mean-square deviation is a minimum when deviations are taken from the arithmetic mean, but that the mean (absolute) deviation is a minimum when deviations are taken from the median. He attributes the latter result to Laplace (1812) [second supplement (1818), pp. 42-43 (pp. 572-573 of the third edition)].

**Comments:** The result concerning the median and the mean deviation which the author attributes to Laplace is not stated explicitly by Laplace, though for n odd it follows as a special case ($p_i = 1, i = 1, \ldots, n$) of his inequalities (c) on page 573 of the third (1820) edition. As we have already seen, this result is stated explicitly and proved by various writers at least as far back as Glaisher (1872) and Helmert (1874). The result concerning the mean and the standard deviation goes back at least as far as Gauss (1809).

**References:** Laplace (1812) [1818], von Andrae (1869), Jordan (1869), von Andrae (1872), Jordan (1872), Galton (1875), Helmert (1876a,b), Gini (1909, 1910), Yule (1911), Dettori (1912), Porru (1912).

**Citations:** Bowley (1901) [1920], Dettori (1912), Porru (1912), Czuber (1914), Gini (1914a,b, 1915a,b), Pietra (1915), Savorgnan (1915), Gini (1916b), Ricci (1916), Weinberg (1916), Gini (1917, 1918), Niceforo (1919), Dalton (1920), de Pietri-Tonelli (1920), Julin (1921), Savorgnan (1921), Pietra (1925), Gini 1926), Gumbel (1928), Gini & Galvani (1929), de Gleria (1929), de Finetti & Paciello (1930), Gini (1930), de Gleria (1930), March (1930), von Bortkiewicz (1931), de Finetti (1931a), Galvani (1931), Gini (1932), Pietra (1932a), Yntema (1933), Anderson (1935), Castellano (1935), Pietra (1935a,b), Wold (1935), Gini (1936), Bresciani-Turroni (1939), Gini (1939), Pizzetti (1941a), Kendall (1943, 1946).

* King, Willford I. (1912). *The Elements of Statistical Method.* The Macmillan Company, New York-London. Second edition, 1919; third edition, 1924.

**Summary:** In Chapter XII (pp. 121-140), the author discusses types and averages, including the mode Z, the median M, the arithmetic average or mean a, the weighted arithmetic average, and the geometric average. Concerning the median he writes (pp. 127-132): "If a number of similar objects are placed side by side in order of their size, they are said to be **arrayed.** . . . If any group of objects is thus arrayed, the middle one is known as the median item. . . . The median may also be defined as that item whose size corresponds most closely to the size of all the other items in the array. Stated in mathematical language this means that, **when all deviations are considered positive,** the sum of the deviations from the median is a minimum. . . . On the whole, the median is one of the most valuable types for practical use, and for studies such as wages, distribution of wealth, etc., is often decidedly superior to either the mode or the arithmetic average." In Chapter XIII (pp. 141-158), he discusses measures of dispersion, including the range, the average deviation $\delta$, the standard deviation $\sigma$, and the quartile deviation. He dismisses the range with the statement (p. 143) that it is too indefinite to be used as a practical measure of dispersion. He notes that the average deviation may be taken from the arithmetic average, from the median, or from the mode; in his example (p. 146), he takes it from the median, which makes it a minimum. In a group of n items $m_i (i = 1, 2, \ldots, n)$, he defines the median M as the $(n+1)/2$ item, the first quartile $Q_1$ as the $(n+1)/4$ item, the third quartile $Q_3$ as the $3(n+1)/4$ item, and other quantiles (deciles and percentiles) analogously; he defines the quartile deviation as $(Q_3-Q_1)/2$. On page 155 he states: "The quartile deviation is . . . useless when it is desired to give weight to the extremes in which respect it is exactly the opposite of the standard deviation; the average deviation occupying the intermediate and, for general purposes, superior position." On pp. 156-158, he discusses the Lorenz curve. In Chapter XIV (pp. 159-166), the author discusses various measures and coefficients of skewness, including $(a-Z)/\delta$, $(a-M)/\delta$, $(Q_3+Q_1-2M)/(Q_3-Q_1)$, $[\Sigma(m_i-a)^3/n]^{1/3}/\sigma$ and $[\Sigma(m_i-a)^3/n]^{1/3}/\delta$.

**Note:** All references to specific chapters and pages are to those of the 1923 printing of the second edition.

**References:** Bowley (1901), Edgeworth (1911), Yule (1911).

**Citations:** Dodd (1914), Gini (1914a), Secrist (1917), Julin (1921), Rietz (1924), Eells (1930), Yang (1933), Gini (1939).

* Newcomb, Simon (1912). Researches on the motion of the moon, Part II. The mean motion of the moon and other astronomical elements derived from observations of eclipses and occultations extending from the period of the Babylonians until A.D. 1908. *Astronomical Papers*, Vol. 9, pp. 1-249, U.S. Government Printing Office, Washington.

**Summary:** The portion of this posthumous work (completed only a month before the author's death in 1909) that is relevant to our study is contained in Section 53 ((pp. 212-214) on weights of the equations, in which the author states: "In solving for the mean longitude we have made little distinction of weights except to reject abnormally discordant observations. . . . The familiar method is to reject all observations, the residuals of which are so large as to indicate some abnormal sources of error, and to retain the others with unmodified weight. That this method is not the best one is very evident from the fact that it is

necessary to assign some limit to the value of an error which can be regarded as normal, and that this limit is necessarily a matter of judgment. The final result then becomes a discontinuous function of the residual of the rejected observation, the continuity being broken at the point regarded as the limit of normal error. A simple example will make the case clear. If we have three observed results a,b,c, of which the mean is to be taken, and if c be the result which may be abnormal, then so long as c is retained we shall have Mean = (1/3)(a + b + c); the mean will then continually increase with c. When c passes the normal limit the mean changes per saltum to (1/2)(a+b). If $c_0$ be the limit, we have two values differing by the amount (1/6)(2$c_0$−a−b), of which we choose either one or the other. As the limit of abnormality is necessarily doubtful, being very largely a matter of judgment, it follows that if c is near the limit $c_0$, we shall be in doubt which of the two values are to be adopted. The laws of probability show that the best result is a weighted mean of the two, the weights being proportional to their probabilities. This involves making a weight to be assigned to an observation depend upon its deviation from the mean of the other observations, a proceeding which seems to violate the logical rules of procedure in combining observations. But the method is logically rigorous if properly applied. The subject was treated by the writer in a former publication [Newcomb (1886)] where a method was shown of making the weight a function of the residual error of the observation. . . . So little is gained by aiming at complete rigor of method that almost any modification which will prevent the incongruity of changing the weight per saltum from 1 to 0 at a certain point will do. Almost any reasonable law of diminution will answer the purpose. In taking a simple mean, an easy method which will lead to a result practically as good as the most rigorous treatment is this: We premise that up to a certain limit $e_0$ the weight of an observation should not be diminished on account of its discordance. The question will arise as to the law of diminution of the weight when the residual error exceeds $e_0$. In this case let us put for △ the excess of the error above the limit $e_0$. Then I propose to determine the weight by the condition $w=e_0/(e_0+\triangle)$. This will lead to practically the same result as if we substituted for the actually observed quantity another quantity corresponding to the residual $e_0$ [i.e., if we Winsorized]. . . . We shall still have to regard observations as quite abnormal, and reject them entirely [i.e., trim] when the residuals are so large as to show some gross error. . . . "

**Reference:** Newcomb (1886).

* Porru, Emanuele (1912). La concentrazione della ricchezza nelle diverse regioni d'Italia. *Studi Economico-Giurdici della Reale Università di Cagliari* **4** (1), 79-115. Review by Gustavo del Vecchio, *Rivista Italiana di Sociologia* **16,** 721-722.

**Summary:** The author compares the formulas of Pareto (1897) and Gini (1910) for the distribution of income and the associated measures of concentration, $\alpha$ and $\delta$, respectively. Pareto's formula is $\log N_x = \log H - \alpha \log X$, where $N_x$ is the number of persons with income greater than X, and $\alpha$ and H are constants (to be determined from the data). Gini's formula is $\log N_x = \delta \log A_x$ − log K, where $N_x$ is the same as in Pareto's formula, $A_x$ is the total amount of the $N_x$ incomes, and $\delta$ and K are constants (to be determined from the data). The author points out that, theoretically, increases in the value of $\delta$ should correspond with decreases in the value of $\alpha$, and vice versa, but in practice this is not always true. He states that the index of concentration ($\delta$) of Gini is a better measure of concentration than Pareto's index $\alpha$.

**References:** Pareto (1897), Gini (1910, 1912).

**Citations:** Gini (1912), Czuber (1914), Gini (1914a), Savorgnan (1915), Gini (1916b, 1917), Niceforo (1919), Dalton (1920), d'Addario (1934a), Castellano (1935), Gini (1936, 1939).

* Tits, L. (1912). Nouvelle simplification de la méthode la plus approximative et de la méthode de l'approximation minimum. *Annales de la Société Scientifique de Bruxelles* **36** (2), 256-263. (JFM 42, 301).

**Summary:** The author states and proves three theorems which embody further simplifications, beyond those of de la Vallée Poussin (1911) and Goedseels (1911), of the most approximative method and the

method of minimum approximation formulated by Goedseels (1909). Consider linear expressions in two or more variables of the form $a_ix + b_iy + c_iz + \ldots + \ell_iu - m_i$; denote these expressions by $(e_i)$. Denote by $\triangle$ the determinant $(a_1, b_2, \ldots, \ell_n, m_{n+1})$ and by $M_i$ the minor of any element $m_i$ in this determinant. Then the following theorems hold. Theorem I (1) If n variables x,y,z, . . ., u are assumed to satisfy n+1 inequalities of the form $(e_i) \leq 0 (i = 1,2, \ldots, n+1)$; if moreover the fraction $-\triangle/M_k$ and all the fractions $M_s/M_k$ corresponding to a certain minor $M_k$ are zero or negative, all the systems (x,y,z, . . .,u) which satisfy the inequalities $(e_s) \leq 0$ other than $(e_k) \leq 0$ satisfy the latter also, and one can therefore suppress $(e_k) \leq 0$ from the series of n+1 inequalities $(e_i) \leq 0$ without changing the systems (x,y,z, . . .,u) defined by these inequalities. (2) The same proposition is true for inequalities of the form $(e_i) \geq 0$ if the fraction $+\triangle/M_k$ and the fractions $M_s/M_k$ are zero or negative. Theorem II (1) If two variables (x,y) are assumed to satisfy the three inequalities $ax + b_iy \leq c_i (i = 1,2,3)$ in which a is a positive constant and the coefficients b are arranged in increasing order; if, moreover, one has $(c_3 - c_2)/(b_3 - b_2) \leq (c_2 - c_1)/(b_2 - b_1)$, one can, without changing the solutions of the proposed system, suppress the intermediate equation $[ax + b_2y \leq c_2]$. (2) The same proposition is true for inequalities of the form: $ax + b_iy \geq c_i (i = 1,2,3)$ in which a is a positive constant, the coefficients b are arranged in increasing order, and where one has also $(c_3 - c_2)/(b_3 - b_2) \geq (c_2 - c_1)/(b_2 - b_1)$. Theorem III (1) If two variables x and y are assumed to satisfy inequalities of the form: $ax + b_iy \leq c_i (i = 1,2, \ldots, n)$ in which a is a positive constant and the coefficients b are arranged in increasing order; if, moreover, one has, for each value of the index: $(c_{k+1} - c_k)/(b_{k+1} - b_k) \leq (c_k - c_{k-1})/(b_k - b_{k-1})$, one can, without changing the solutions of the proposed system, suppress all the relations included between the first and the last. (2) The same proposition is true for inequalities of the form: $ax + b_iy \geq c_i (i = 1,2, \ldots, n)$ except that the last stated condition is replaced by the following: $(c_{k+1} - c_k)/(b_{k+1} - b_k) \geq (c_k - c_{k-1})/(b_k - b_{k-1})$. The author uses these theorems to simplify the solution of a numerical example given by Goedseels (1911).

**References:** Goedseels (1909, 1911), de la Vallée Poussin (1911).

* Dodd, Edward L. (1913). The probability of the arithmetic mean compared with that of certain other functions of the measurements. *Annals of Mathematics* (2) **14,** 186-198; abstract, *Bulletin of the American Mathematical Society* (2) **19** (1912), 54-55. (JFM 44, 269).

**Summary:** The author points out that Czuber (1891a) recounted many of the attempts that have been made to relate the principle of the arithmetic mean as the most probable value with the Gaussian probability law, but quoted from Bertrand (1889), p. 180, an example to show that this law and principle are not strictly compatible. The author states that one of the objects of this paper is to show this incompatibility by other means. He points out that, strictly speaking, the Gaussian law has no most probable error, since each of the infinite number of possible errors has the same probability, zero. He then offers the following definition (pp. 187-188) to facilitate the discussion of the relative magnitude of probabilities: "The probability of $f_1$ will be said to be greater than that of $f_2$, if the probability that the error of $f_1$ will lie in the interval $-\alpha$ to $+\alpha$ is greater than the probability that the error of $f_2$ will lie in the same interval, for all positive values of $\alpha$ less than some $\alpha'$, — in other words, if the probability that $f_1$ will differ from a [the true value] by less than $\alpha$ is greater than that $f_2$ will differ from a by less than $\alpha$, when $\alpha < \alpha'$." He shows that, under the Gaussian probability law, the quadratic mean (root-mean-square) of two measurements has a greater probability than their arithmetic mean, provided the product of the precision constant $h = 1/\sigma\sqrt{2}$ by the true value a is less than 2, where $\sigma$ is the population standard deviation. Under the same law, he shows that the probability of bm is less than that of m (the arithmetic mean) if b is a constant greater than unity, but that there exist positive values of the constant b for which the probability of bm is greater than that of m. He also shows (p. 196) that "the probability $[P'_\alpha]$ of the median of an odd number, $2\nu + 1$, of measurements is less than that $[P_\alpha]$ of the arithmetric mean—under the Gaussian law—and if $\nu$ is made sufficiently large, and then $\alpha$ taken small enough, the ratio of $P'_\alpha$ to $P_\alpha$ can be made as near $\sqrt{2}/\sqrt{\pi} = .7979$ as we please." In a footnote, however, he remarks: "This does not necessarily discredit the use of the median in economic, biological or other investigations. Only Gaussian distributions are being considered in this article."

**Remarks:** The only portion of this paper directly relevant to our subject is that dealing with the median. The other results mentioned above, however, are of indirect interest, because they discredit the universal use of the arithmetic mean, even for Gaussian distributions, and Gauss' (1809) first "proof" of the method of least squares, thus opening the way for alternatives based on order statistics.

**References:** Bertrand (1889), Czuber (1891a), Poincaré (1896), Czuber (1903).

**Citations:** Dodd (1914, 1922).

* Edgeworth, F. Y. (1913). On the use of the theory of probabilities in statistics relating to society. *Journal of the Royal Statistical Society* **76** (2), 165-193.

**Summary:** This paper (the presidential address delivered before the Royal Statistical Society, December 17, 1912) deals only incidentally with order statistics. On page 183, the author mentions confirmation of Pearson's rule "as to the relation which commonly prevails (in skew groups) between the different kinds of average—the mode, the median, and the mean." On pp. 189-190, he illustrates "the genesis of the law of error . . . by linear functions of digits taken at random" [the seventh decimal place of logarithms of successive integers from 101 to 1300]. He obtains 48 statistics by subtracting 112.5 (25 times 4.5) from the sums of 25 digits, arranges them in order from smallest to largest, and computes their median, quartiles, deciles, and mean deviations (from zero), which he compares with the theoretical values, finding that the agreement, when the probable error is taken into account, is quite good. Agreement is (as one would expect) not quite so good for 48 statistics obtained by subtracting 72 (16 times 4.5) from the sums of 16 digits.

**References:** Laplace (1812), Cournot (1843), De Morgan (1847), Venn (1866), Edgeworth (1885), Pearson (1895), Bowley (1901), Yule (1911).

**Citations:** von Bortkiewicz (1915), Bowley (1928).

* Goodwin, H. M. (1913). *Elements of the Precision of Measurements and Graphical Methods*. McGraw-Hill Book Company, New York-London.

**Summary:** Part I deals with the precision of measurements. On pp. 20-21 the author gives the following discussion of rejection of observations: "In a series of measurements taken under similar conditions, it not unfrequently happens that an observation will differ quite widely from others in the series, and the tendency to regard such an observation as erroneous and to reject it is great, particularly among beginners. If such an observation obviously contains a *mistake*, as, for example, the recording of a wrong number, the recording [of] the wrong scale division, the incorrect adding up of weights, etc., it may, of course, be legitimately rejected. If, however, no mistake is apparent, the observation should never be rejected without the most scrupulously unbiassed judgment on the part of the observer or the application of some mathematical criterion for the rejection of doubtful observations. For the experienced observer the former procedure is preferable, even though several mathematical criteria, Peirce's, Chauvenet's, etc., have been deduced, which are very satisfactory when the number of observations considered is large. In most physical work the number of observations is not very great, however, and one widely discordant from the others has an undue weight on the value of the mean. It is frequently better to reject such an observation, even though it contains no apparent mistake. A good criterion to follow in such cases is the following:—Compute the mean and the average deviation a.d., omitting the doubtful observation. Compute also the deviation, d, of the doubtful observation from the mean. If $d \geq 4$a.d., reject the observation, since it can be shown that the probability of the occurrence of an observation whose deviation is four times the average deviation is only one in a thousand. An error of this unusual magnitude is called a *Huge* Error."

* Mansion, Paul (1913). Sur les recherches de Laplace relatives à la théorie des erreurs. *Annales de la Société Scientifique de Bruxelles* **37** (2), 107-117; rapport par E. Goedseels, **37** (1), 67-79. (JFM 44, 268). Reprinted in *Mathesis* **33** [(4) **3**], Supplement 4, 11 pp. (JFM 44, 278, 1116—listing only).

**Summary:** The author summarizes the work of various authors on three methods applied to the theory of errors: (1) Method of the minimum largest error (minimum approximation) [minimum sum of $(2k)^{th}$ powers of errors as $k \to \infty$] — Laplace (1799, Book III, no. 39), Prony (1804, pp. xviii-xxi), Gauss (1809, no. 186), Laplace (1812, no. 24, pp. 343-345), and Mansion (1906); (2) Method of situation [minimum sum of absolute values of errors (first powers), which occurs when measured from the sample median, but if the Boscovich restriction that the sum of their values taken with regard to sign must be zero is added, the errors must be measured from the sample mean]—Laplace (1799, Book III, no. 40), Prony (1804, pp. xxi-xxiv), Gauss (1809, no. 186), and Laplace (1812—second supplement, 1818, pp. 30-50); (3) Method of least squares [minimum sum of squares of errors]—various writings of Laplace, Gauss, and others. The author gives several examples [data taken from Hagen (1837, pp. 192-195)] of the application of the three methods in the case of one unknown, also one example in the case of two unknowns. In the latter case, as in the cases of three or more unknowns, he points out that the results are indeterminate unless one specifies the form of the relations, which is apparently the reason that Laplace (1799) considered only equations of the form $x + ay - b = 0$. Even in the case of a single variable, the method of situation is indeterminate [does not have a unique solution] when the number of observations is even, since the result may be either of the two middle ones or any value between them. On the other hand, an order-preserving or order-reversing transformation does not affect this method, but does affect the other two. The author reaches the following conclusion (p 116): "Laplace and Gauss both insisted upon the difficulties presented by the theory of errors: each method depends critically upon more or less arbitrary assumptions. The simplest obviously consists of considering only equations of the form $x = a_1, x = a_2, \ldots, x = a_n, a_1 \leqslant a_2 \leqslant \ldots \leqslant a_n; y = b_1, y = b_2, \ldots, y = b_n, b_1 \leqslant b_2 \leqslant \ldots \leqslant b_n$ given directly by the observations or deduced indirectly from equations in several unknowns, solved in all possible ways, where one has eliminated all questionable values, i.e. those very different from the others. If one wants the minimum approximation, one takes as the final value the mean of the extreme observations [the midrange]; if one wants to make the sum of the absolute values of the errors or residues a minimum, one chooses the median value (or one of the median values) as the final value. Finally, one takes that final value equal to the mean of all the a's if one wishes to make the sum of squares of the errors a minimum. This last method alone utilizes equally all the observed data. In each of the methods, one can evaluate the approximation obtained." (Compiler's translation).

**References:** Boscovich (1757), Laplace (1799), Prony (1804), Gauss (1809), Laplace (1812), Hagen (1837), Boudin (1865), Mansion (1906), de la Vallée Poussin (1911).

**Citations:** Dodd (1922), Alliaume (1927a).

* Wellisch, S. (1913). Ein interessanter Fall der Ausscheidung von Beobachtungen. *Österreichische Zeitschrift für Vermessungswesen* **11,** 142-146. (JFM 44, 271).

**Summary:** The author notes that numerous authors, including Gauss, Hagen, Bessel, Baeyer, Gerling, Faye, Bertrand, Czuber and himself [Wellisch (1909)], have written on the subject of discarding discordant observations in order to improve the arithmetic mean. He considers the example of Stampfer (1839), who discarded three out of nine observations in order to obtain what he considered the best value of the ratio of the length of the Vienna fathom to that of the meter: $1.8966657 \pm 0.0000037$. He points out that correct calculation gives 1 Vienna fathom $= 1.8966652 \pm 0.0000034$m. for the six observations retained by Stampfer as compared with $1.8966274 \pm 0.0000150$m. for all nine observations. He attempts to explain the difference between the six observations retained by Stampfer and the three rejected by him [the one (1.896614) due to Liesganig and the two (1.896562 and 1.896522) obtained by Prony by comparison with the French standard half toise] on the basis of the temperature coefficients of expansion of the metals (iron and platinum) used for the standard bars.

**References:** Stampfer (1839), Wellisch (1909).

* Contarino, Francesco (1914). Media aritmetica, media probabile e media più probabile. *Atti dell'Accademia Pontaniana* (Naples) **44,** 1-143. (JFM 45, 372—listing only).

**Summary:** The author discusses the arithmetic mean and several other means, none of which are based on order statistics. The only part of this paper which is relevant to our study is his discussion of discordant observations (pp. 41-43). In particular he mentions the criterion of Peirce and the views of Faye, Airy, Bessel, Poincaré and Bertrand on rejection of outliers.

**References:** Laplace (1812), Gauss (1823), Encke (1832-34), Hagen (1837), Chauvenet (1863), Stone (1873a), Bertrand (1889), Poincaré (1896), Czuber (1899).

* Czuber, Emanuel (1914). Beitrag zur Theorie statistischer Reihen. *Versicherungswissenschaftlichen Mitteilungen* (N. F.) **9** (2), page nos. unknown. Reviews by Gaetano Pietra, *Rivista Italiana di Sociologia* **19** (1915), 410-422; by Corrado Gini, *Deutsches Statistisches Zentralblatt* **7** (1915), (1), 20-24.

**Summary:** In his review, Pietra points out that the author [Czuber], in this paper, of which the compiler has been unable to obtain a copy, presents a critical analysis of the work of Gini (1912), in two parts. In the first part Czuber deals with the determination of more appropriate parameters for the measurement of the variability of quantitative characteristics and of the heterogeneity of qualitative characteristics. In the second he takes up the formulas of Pareto and Gini for the distribution of incomes, which he applies to the distribution of incomes in Austria in 1904 and 1911, and gives a critique of the results. The review by Pietra deals primarily with certain salient points in the first part. On page 411 Pietra states: "For the study of the variability of characteristics, from among the simple and quadratic mean deviations from the arithmetic mean, the simple mean deviation from the median, and Gini's mean difference, Czuber decides at once for the mean difference because, he asserts, 'the diversity of the magnitudes is indicated in a more natural way by their differences, since with the deviations from an intermediate value, whether it be the arithmetic mean or the median, an extraneous element comes into the field'." (Compiler's translation). The reviewer [Pietra] holds, however, that this criterion cannot be assumed in an absolute sense, and proceeds to consider the question from an analytic point of view. He notes that the author [Czuber] uses $\Delta/a$, where $\Delta$ is the mean difference and a is the range, as a measure of relative variability. He notes also that Czuber proposes a method of computing $\Delta$ which he [Czuber] claims is faster than that used by Gini. The procedure is as follows: Suppose one wishes to compute the mean difference $\Delta$ of n ordered quantities, $a_1 \leq a_2 \leq \ldots \leq a_n$. Let $s_1 = a_1, s_2 = a_1 + a_2, \ldots, s_n = a_1 + a_2 + \ldots + a_n$; $s'_2 = a_n, s'_2 = a_n + a_{n-1}, \ldots, s'_n = a_n + \ldots + a_2 + a_1$; $S = s_1 + s_2 + \ldots + s_n$ and $S' = s'_1 + s'_2 + \ldots + s'_n$. Then $\Delta = 2(S' - S)/[n(n-1)]$. Czuber also considers the mean difference with repetition, $\Delta_R = (n-1)\Delta/n$, and the ratios $\Delta/2A$ and $\Delta_R/2A$ (Gini's ratio of concentration), where A is the arithmetic mean, and points out that the two ratios are asymptotically equal as $n \to \infty$. In the second part, Czuber compares the index $\alpha$ of Pareto and the index $\delta$ of Gini for concentration of income (wealth, etc.), and states a preference for the former. Pietra quotes at length from Gini's review in rebuttal.

**Note:** The compiler has been unable to obtain a copy of this publication, and hence has had to rely on the reviews by Pietra and Gini for information as to its contents.

**References:** Furlan (1911), Dettori (1912), Gini (1912), Porru (1912), Gini (1914a), [Pietra (1915), Savorgnan (1915)].

**Citations:** Gini (1915a), Pietra (1915), Savorgnan (1915), Ricci (1916), Gini (1918), Niceforo (1919), de Gleria (1929), de Finetti & Paciello (1930), de Finetti (1931a), Gini (1932), Pietra (1932a), Anderson (1935), Castellano (1935), Gini (1936, 1939), Pizzetti (1940), Davis (1941), Pizzetti (1941a).

* Dodd, Edward L. (1914). The Error Risk of the Median Compared with that of the Arithmetic Mean. *Bulletin of the University of Texas, Scientific Series*, No. 27; abstract, *Bulletin of the American Mathematical Society* **19** (1913), 500. (JFM 44, 269).

**Summary:** ". . . Dr. Dodd shows . . . that under the Gaussian law the error risk of the arithmetic mean [of an odd number of measurements] is less than that of the median. The median is a function of the measurements which lacks certain partial derivatives at some points, and it cannot be given a Taylor development about a point with all coordinates equal. Thus, it is not one of the functions considered by Czuber [(1903), p. 276] in his treatment of error risk, 'Fehlerrisiko'. The median, however has found considerable favor among biologists and economists; and in certain non-Gaussian distributions is probably superior to the arithmetic mean." (Abstract, p. 500). The author closes (p. 11) with the following observations: "The theorem of this paper does not recommend the use of the arithmetic mean in preference to the median under all circumstances; but merely when it is assumed that the distribution is a Gaussian distribution. But, in view of this theorem, if there is any reason for supposing—in a particular case—that the median gives a more satisfactory result than the arithmetic mean, this is also a reason for supposing that the distribution is not Gaussian. . . ."

**References:** Bertrand (1889), Czuber (1903), King (1912), Dodd (1913).

* Fuller, Weston E. (1914). Flood flows. *Transactions of the American Society of Civil Engineers* **77**, 564-617; discussion, 618-694.

**Summary:** The author begins (p. 564) with the statement: "The determination of the magnitude of the maximum flood that may be expected to occur on a river is of importance in many problems connected with engineering works." On pp. 566-567 he summarizes his results as follows: "Nomenclature.—Q = the greatest average rate of flow for 24 consecutive hours during a period of years, in cubic feet per second; Q(Max.) = the maximum rate of discharge of a flood, in cubic feet per second; Q(Ave.) = the average yearly flood [average of maximum 24-hour average flood for each year], in cubic feet per second; T = the number of years in the period considered; A = the catchment area of the river, in square miles; C = a coefficient which is constant for the river at the point of observation [tabled elsewhere in the paper]. Formula Proposed . . .: $Q = CA^{0.8}(1+0.8 \log T)$, . . . . $Q(\text{Max}) = Q(1+2A^{-0.3})$ . . . ." On pp. 577-578 the author states: "The second largest flood is here taken as representing the size of that one which will probably come in one-third of the total period. . . . In the same way, the third flood is taken as the most probable one for a period equal to one-fifth of the total time, and so on." This corresponds to plotting position $(2m-1)/2n$ for the $m^{th}$ order statistic of a sample of size n; another method, based on plotting position $(m-1)/n$, is given. In the discussion, Arthur E. Morgan questions some of the assumptions made by the author, citing data for the 1913 flood on the Miami River at Dayton, Ohio among others. Allen Hazen writes (pp. 626-628): "This is a most important paper, because as far as the writer knows, it is the first attempt to apply the principles of probabilities to the flood problem. . . . In another paper to be presented to the Society [Hazen (1914)], the writer has described a graphical method of representing data of the same general character as those used by the author, based on the probability curve, or, as it is otherwise called, the normal law of error. A new kind of cross-section paper is made, in which the spacing of the lines in one direction is computed from tables of the probability curve, so that figures representing the summation of that curve plotted on it fall in a straight line. . . . The figures for maximum flood flows of the same stream through a series of years clearly bear a relation to the normal law of error, but they do not follow it exactly. When plotted on probability paper they form what is called a skew curve. The variations downward are more numerous than the variations upward, but are smaller in magnitude. It is interesting to note that if the logarithms of the numbers representing the several floods are used instead of the floods themselves, the agreement with the normal curve of error is closer. In other words, the variations up and down from the mean follow a geometric rather than an arithmetical ratio. . . ." Robert E. Horton corrects the author's statement (p. 610) that "as far as the writer knows, the first suggestion of the use of the element of time in a formula for flood flows was made by Allen Hazen . . .

in 1910." He points out that both this idea and the application of the theory of probability to flood flows were suggested to him in 1896 by George W. Rafter, and that he made extensive use of both during the years between 1896 and 1908, including numerous plots on logarithmic (lognormal probability) paper.

**Citations:** Hall (1921), Foster (1924), Jarvis (1926), Goodrich (19 ), Foster (1936), Grassberger (1936), Saville (1936), Gumbel (1938b), Kimball (1938), Creager (1939), Gumbel (1939c,f, 1941a, 1943c, 1945b).

* Gini, Corrado (1914a). Sulla misura della concentrazione e della variabilità dei caratteri. *Atti del R. Istituto Veneto di Scienze, Lettere ed Arti* **73** (2), 1203-1248. Reviews by C. Bresciani-Turroni, *Giornale degli Economisti e Rivista di Statistica* (3) **52** (1916), 68-70; by Gaetano Pietra, *Rivista Italiana di Sociologia* **18** (1914), 583-587.

**Summary:** Given n ordered values [e.g. the order statistics of a sample of size n from any population (specified or unspecified)], $a_1 \leq a_2 \leq \ldots \leq a_n$. Let $\Sigma_{k=1}^{i} a_k = A_i (i = 1,2,\ldots,n)$, $A_i/A_n = q_i$ and $i/n = p_i$. Then the *ratio of concentration* R, which the author defines by the equation

$$R = \Sigma_{i=1}^{n-1} (p_i - q_i) / \Sigma_{i=1}^{n-1} p_i,$$

increases with increasing concentration, taking the values 0 and 1 for minimum and for maximum concentration, respectively. A plot of $q_i$ against $p_i$ is called the *curve of concentration* or Lorenz curve [see Lorenz (1905), King (1912)]. The author studies various other measures of concentration (relative variability), one of them being the ratio of the mean difference [Gini (1912)] to twice the arithmetic mean. He shows that this ratio is equal to the ratio of concentration R. Other measures of concentration studied involve ratios of various measures of variability, including the mean deviation from the median, to various measures of central tendency, including the median. On page 1236 the author remarks that no matter what type of measures of variability (mean deviation, root-mean-square deviation, etc.) one chooses, the deviations can be measured either from the arithmetic mean or from the median.

**References:** Lorenz (1905), Watkins (1905, 1908), Gini (1909), Persons (1909), Watkins (1909), Gini (1910), Furlan (1911), Mortara (1911), Dettori (1912), Gini (1912), King (1912), Porru (1912).

**Citations:** Czuber (1914), Gini (1915a), Pietra (1915), Savorgnan (1915), Ricci (1916), Gini (1917), Vinci (1918), Niceforo (1919), Amoroso (1925), Pietra (1925), Gini (1926), Saibante (1926), Gumbel (1928), Saibante (1928), d'Addario (1930), de Finetti & Paciello (1930), Gini (1930), de Gleria (1930), von Bortkiewicz (1931), de Finetti (1931a), Galvani (1931), d'Addario (1932), Galvani (1932), Gini (1932), Pietra (1932a,b), Castellano (1933a), Yntema (1933), Castellano (1935), Pietra (1935a), Wold (1935), Gini (1936), Nair (1936), Bresciani-Turroni (1937), Gini (1939), de Vergottini (1939, 1940), Pizzetti (1941a,b), Hoeffding (1948).

* Gini, Corrado (1914b). Di una misura della dissomiglianza tra due gruppi di quantità e delle sue applicazioni allo studio delle relazioni statistiche. *Atti del Reale Istituto Veneto di Scienze, Lettere ed Arti* **74** [(8)**17**] (2), 185-213.

**Summary:** The author discusses the index of dissimilarity of two groups and its relation to the mean difference with repetition. Given two groups $\alpha$ and $\beta$ of n quantities each, $a_1 \leq a_2 \leq \ldots \leq a_n$ and $b_1 \leq b_2 \leq \ldots \leq b_n$, $a_k$ and $b_k$ ($k = 1,2,\ldots,n$) are said to be *cograduate*. If n pairs are formed by taking for each pair one quantity from the group $\alpha$ and one from the group $\beta$ (without replacement), the author proves that the sum of the n differences between paired quantities is a minimum if each pair consists of two cograduate quantities. The *index of dissimilarity* between the two groups is defined to be $I = \Sigma_{k=1}^{n} |a_k - b_k|/n$. The

author extends the definition to the case of groups containing unequal numbers of quantities. Suppose a group of n quantities is divided into s partial groups. If $n_k$ ($k=1,2,\ldots,s$) is the number of quantities in the $k^{th}$ partial group and $I_k$ is its index of dissimilarity with the total group, then the (weighted) mean index of dissimilarity is $I = \Sigma_{k=1}^{s} n_k I_k/n$. The author shows that $I \leq \triangle_R$, where $\triangle_R$ is the mean difference with repetition of the total group, and gives a necessary and sufficient condition that $I = \triangle_R$.

**Note:** If each group is a sample from some population (specified or unspecified), the ordered quantities in a group are the order statistics of a sample. For two samples, each of size n, the $k^{th}$ order statistics ($k=1,2,\ldots,n$) are cograduate.

**Reference:** Gini (1912).

**Citations:** Gini (1915a,b, 1916a), Pietra (1925), Gini (1926), Gini & Galvani (1929).

* Hazen, Allen (1914). Storage to be provided in impounding reservoirs for municipal water supply. *Transactions of the American Society of Civil Engineers* **77,** 1539-1640; discussion, 1641-1669.

**Summary:** The relevant portion of this paper is that dealing with the use of probability paper. On pp. 1549-1550 the author writes: "The practical difficulty with . . . plotting [on ordinary graph paper] is the great curvature of the lines showing the required storage. This difficulty is so great as to make the method unsatisfactory in most cases; but it has been removed by using paper ruled with lines spaced in accordance with a probability curve, or, as it is otherwise called, the normal law of error. The spacing of the lines for this paper was computed from figures taken from probability curve tables, and arranged so that the line which represents the summation of the probability curve, when plotted on it, is straight. . . . In plotting observations on this paper, either of two methods may be used. [In the first method, applicable when data are plentiful, the data are grouped into classes.] . . . In the second method, the whole space is divided into as many vertical strips as there are terms, and the figure for each term, arranged in its order of magnitude, is plotted at the percentage which corresponds to the middle of its strip. . . . The decimal position of the $m^{th}$ term in a series of n terms is found to be $P = (2m-1)/2n$." The author plots storage data for various reservoirs on normal probability paper. In the discussion, in response to a request from H. T. Cory, the author gives more details (pp. 1665-1668) concerning the construction of probability paper, both arithmetic and logarithmic. On pp. 1666-1668 he states: "Logarithmic probability paper has been found helpful where the degree of skewness in the curve is very great. For the flow and storage data used in the paper, the arithmetic probability paper has proved more satisfactory. For some other investigations the logarithmic probability paper has seemed to be better." An example of the use of logarithmic probability paper is given in Fig. 41 on p. 1667.

**Citations:** Hall (1921), Foster (1924), Goodrich (1927), Hazen (1930), Dodge (1933), Foster (1934), Grassberger (1936), Lane & Lei (1949).

* Walker, Gilbert T. (1914). Correlation in seasonal variations of weather, III. On the criterion for the reality of relationships or periodicities. *Memoirs of the Indian Meteorological Department* **21** (9), 13-15.

**Summary:** The author criticizes the use, in testing the significance of the largest of a number of measures of the same kind, of the same criterion that would be applied to a case that had not been selected for its magnitude. He works out approximate tests for the significance of (1) the largest of m correlation coefficients, all based on samples of size n, and (2) the largest of m periodicities. He tabulates the ratios A and B of the probable values of the greatest of m accidental correlation coefficients and amplitudes of periodicities, respectively, to the probable value of a single one, for

m = 2(1)10(2)40(5)100(10)200(20)500(50)1000, assuming that the former are normally and the latter exponentially distributed.

**Reference:** Chauvenet (1863).

**Citations:** Fisher (1929), Davis (1941), Kendall (1946), Hartley (1949).

* von Bortkiewicz, L. (1915). Über die Zeitfolge zufälliger Ereignisse. *Bulletin de l'Institut International de Statistique* **20** (2), 30-111; discussion, **20** (1), 87-92.

**Summary:** The author shows that if the occurrence of a series of events is random, the time intervals (gaps) between events are exponentially distributed. As an example, he considers intervals between deaths of members of the International Statistical Institute during the 13-year period 1890-1902. There were 65 deaths during these years, and hence 64 intervals between them. The time between the beginning of 1890 and the first death is combined with the time between the last death and the end of 1902 to form a 65th interval. The average number of deaths per year was 5, and the number of years in which 0,1,2, . . . deaths occurred should theoretically follow a Poisson distribution with parameter 5. Comparisons of the empirical results with theory are made in both cases (intervals with the exponential distribution and frequencies with the Poisson). The discussion includes a summary by the author, comments by Fahlbeck and Tschuprow and a rejoinder by the author.

**Comments:** The intervals (in days) between deaths are arranged in ascending order, but only for the purpose of grouping them into classes, so the question of order statistics does not really enter at this point. However, the times of occurrence (measured from the start of 1890) are essentially order statistics of a censored sample, if one neglects the fact that some persons entered the sample (became members of ISI) and others presumably left it for reasons other than death during the period of interest (1890-1902).

**References:** Yule (1911), Edgeworth (1913).

**Citations:** Sukhatme (1936), Greenwood (1946), Kendall (1946), Moran (1947).

* Gini, Corrado (1915a). Indici di omofilia e di rassomiglianza e loro relazione col coefficiente di correlazione e con gli indici di attrazione. *Atti del Reale Istituto Veneto di Scienze, Lettere ed Arti* **74** [(8) **17**] (2), 583-610.

**Summary:** In section 1, the author defines cograduate and contragraduate quantities and states three theorems; the first definition and the first theorem were given in an earlier paper [Gini (1914b)], but the others are new. Let $\alpha$ and $\beta$ be two groups of n ordered quantities each, $a_1 \leq a_2 \leq \ldots \leq a_n$ and $b_1 \leq b_2 \leq \ldots \leq b_n$. Then $a_k$ and $b_k (k = 1,2, \ldots, n)$ are called *cograduate quantities* and $a_k$ and $b_{n-k+1}$ are called *contragraduate quantities*. Theorem 1(2). If n pairs are formed, one quantity in each pair coming from $\alpha$ and one from $\beta$, the sum of the absolute values of the differences of paired quantities is a minimum (maximum) if each pair is made up of cograduate (contragraduate) quantities. Theorem 3. If each quantity in one group is associated with each quantity in the other group, the arithmetic mean of the $n^2$ differences of pairs of associated values is given by $\Delta_{\alpha\beta} = 2\Delta_{(\alpha+\beta)} - (1/2)(\Delta_\alpha + \Delta_\beta)$, where $\Delta_\alpha$, $\Delta_\beta$ and $\Delta_{(\alpha+\beta)}$ denote respectively the mean differences between the n quantities in group $\alpha$, between the n quantities in group $\beta$ and between the 2n quantities in the group $(\alpha+\beta)$ obtained by combining the quantities in group $\alpha$ with those in group $\beta$. In section 2, the author defines the *index of homophily between the intensities of two characters* as $O = (M - M_0)/(M_1 - M_0)$ or $(M - M_0)/(M_0 - M_2)$, according as $M_1 - M_0 < 0$ or $> 0$, where M is the sum of the absolute values of the n observed differences of pairs of associated quantities, each quantity in one group being associated with one and only one in the other group, $M_1$ is the minimum possible value of M (given by Theorem 1), $M_2$ is the maximum possible value of M (given by Theorem 2) and $M_0 = n\Delta_{\alpha\beta}$, where $\Delta_{\alpha\beta}$ is given by Theorem 3. In section 3, the author defines the *index of homophily between the*

*deviations of two characters* as $\omega=(\mu-\mu_0)/(\mu_1-\mu_0)$ or $(\mu-\mu_0)/(\mu_0-\mu_2)$, according as $\mu-\mu_0<0$ or $>0$, where the $\mu$'s have the same meaning with respect to deviations of the n quantities from their arithmetic mean as have the M's with respect to the quantities themselves. In section 4, he defines the *variation of A in the $i^{th}$ case* as $v_{ai}=\ell_{ai}/\lambda_a$, where $\lambda_a$ denotes an average of the absolute values of n deviations $\ell_{ai}$, and gives a similar definition for the *variation of B in the $k^{th}$ case*, $v_{bk}$. The averages most often used to measure the intensity of the deviations are the arithmetic mean, the quadratic mean and the median, which lead to the simple mean deviation, the root-mean-square deviation and the probable deviation. The author defines the *index of homophily between the variations of two characters* as $o=(m-m_0/(m_1-m_0)$ or $(m-m_0)/(m_0-m_2)$, according as $m-m_0<0$ or $>0$, where the m's have the same meaning with respect to variations as have the $\mu$'s with respect to deviations. In section 5, the author discusses applications of the indices of homophily, in section 6 he distinguishes between the index of homophily between variations and the coefficient of correlation, and in section 7 he discusses their respective fields of application. The remaining sections deal with an extension of the theory to the case of qualitative data, which does not appear to be relevant to our subject.

**References:** Yule (1911) [1912], Gini (1912), Czuber (1914), Gini (1914a,b).

**Citations:** Gini (1915b, 1916a,b), Weinberg (1916), Gini (1918), Pietra (1925), Gini (1926), Gini & Galvani (1929).

* Gini, Corrado (1915b). Nuovi contributi alla teoria delle relazioni statistiche. *Atti del Reale Istituto Veneto di Scienze, Lettere ed Arti* **74** [(8) **17**] (2), 1903-1942.

**Summary:** In section 1, the author reviews what he has done in two earlier papers [Gini 1914b, 1915a)] and previews the present one. In section 2, he proves theorems concerning sums of squares of the n differences of pairs of quantities, one of each pair coming from group $\alpha$ and one from group $\beta$, each group being of size n, which are analogous to Theorems 1 and 2 of the previous paper [Gini (1915a)] concerning sums of their absolute values. In section 3, he proves the following theorem: If each of the n quantities in a group $\alpha$ is coupled with each of the n quantities in a group $\beta$, the arithmetic mean of the squares of the $n^2$ differences between the coupled quantities is given by $({}^2\Delta_{\alpha\beta})^2=({}^2S_\alpha)^2+({}^2S_\beta)^2+(A_\alpha-A_\beta)^2$, where ${}^2\Delta_{\alpha\beta}$ denotes the quadratic mean of the $n^2$ differences and ${}^2S_\alpha$ and ${}^2S_\beta$ the root-mean-square deviations from their respective arithmetic means, $A_\alpha$ and $A_\beta$, of the n quantities in group $\alpha$ and in group $\beta$, respectively. He also corrects an error in Theorem 3 of the previous paper, where the mean differences $\Delta_\alpha$,$\Delta_\beta$ and $\Delta_{(\alpha+\beta)}$ should be replaced by the mean differences with repetition $\Delta_{R\alpha}$,$\Delta_{R\beta}$ and $\Delta_{R(\alpha+\beta)}$, respectively, and proves the analogous theorem for quadratic mean differences: $({}^2\Delta_{\alpha\beta})^2=2({}^2\Delta_{R(\alpha+\beta)})^2-(1/2)({}^2\Delta_{R\alpha})^2-(1/2)({}^2\Delta_{R\beta})^2$. In section 4, the author discusses groups of deviations and groups of variations, and gives results for the maximum, minimum and probable value of sums of squares of the differences between deviations and between variations analogous to those for sums of their absolute values given in the previous paper. In section 5, he defines the concept of corresponding quantities, deviations and variations. In section 6, he gives formulas for quadratic indices of homophily analogous to those for simple indices given in the previous paper. In section 7, he reviews the simple index of dissimilarity of groups of quantities given in an earlier paper [Gini (1914b)] and extends it to groups of deviations and of variations. In section 8, he gives formulas for the analogous quadratic indices between quantities, between deviations and between variations. In section 9, he gives formulas for inverse quadratic indices of dissimilarity between deviations and between variations, which differ from the direct quadratic indices in that they are defined as quadratic means of sums of contragraduate deviations or variations rather than of differences of cograduate ones. In section 10, he gives a relation between the quadratic index of homophily between deviations and the coefficient of correlation, and discusses its consequences. The remaining sections deal with measures applicable to qualitative data, and do not appear to be relevant to our subject. In the appendix, the author discusses the whole concept of indices of relation; in particular, he compares his indices with others, not based on ordered values, proposed by Mortara.

**References:** Yule (1911) [1912], Gini (1912, 1914b, 1915a).

**Citations:** Gini (1916a,b), Pietra (1925), Gini (1926), Gini & Galvani (1929).

* Mantel, W. (1915). Toevallige verdeeling van eene rechte lijn. *Nieuw Archief voor Wiskunde* (2) **11,** 394-413. (JFM 45, 362-363).

**Summary:** The author states and solves ten problems. The first, second and eighth problems deal with the evaluation of certain multiple integrals. The results are applied to the solution of the remaining seven problems, all of which deal with the random division of a unit interval into n segments by $n-1$ points chosen at random. The solution of problem 4 is a generalization of results given by Halphen (1873) and by van den Berg (1891). Several numerical examples are worked out.

**References:** Halphen (1873), van den Berg (1891).

**Citation:** Bottema (1925).

* Pietra, Gaetano (1915). Delle relazioni tra gli indici di variabilità. Note I, II. *Atti del Reale Istituto Veneto di Scienze, Lettere ed Arti* **74** (2), 775-792, 793-804.

**Summary:** The author studies the relations among four measures of variability previously discussed by Gini (1912) and others: (1) mean deviation from the arithmetic mean; (2) mean deviation from the median; (3) mean difference; and (4) root-mean-squared deviation from the arithmetic mean. Section 1 (pp. 775-786) of the first note deals with relations among the two mean deviations and the mean difference, which the author denotes by $'S_A$, $'S_M$ and $\triangle$ respectively, as well as the mean difference with repetition, $\triangle_R = [(n-1)/n]\triangle$, and Gini's concentration ratio, $R = \triangle/2A$ or $R_R = \triangle_R/2A$, where A is the arithmetic mean. Section 2 (pp. 787-792) deals with relations between the root-mean-squared deviation from the arithmetic mean, $^2S_A$, and the other measures, especially the mean deviation from the arithmetic mean. Among other results, the author presents the well-known inequality $'S_M \leq 'S_A$ and the less well-known (presumably new) inequality $C' \leq {}^2S_A/2A \leq (1/2)\sqrt{C'(nC'+1)}$, where $'S_A/2A = C'$ (constant). The relations involving the mean deviation are geometric in form. In the second note, the author makes a further study of the four measures of variability considered in the first note, with emphasis on the behavior of and limits on $'S_M/2A$, $\triangle/2A$, and $^2S_A/2A$ as functions of $'S_A/2A$ and the sample size n.

**References:** Dettori (1912), Gini (1912), Czuber (1914), Gini (1914a).

**Citations:** Czuber (1914), Ricci (1916), Gini (1926), d'Addario (1930), de Gleria (1930), de Finetti (1931a), Galvani (1931), d'Addario (1932), Gini (1932), Pietra (1932a,b), Castellano (1933a), Anderson (1935), Castellano (1935), Pietra (1935a), Gini (1936), Bresciani-Turroni (1939), Davis (1941), Livada (1942).

* Savorgnan, Franco (1915). Intorno all'approssimazione di alcuni indici della distribuzione dei redditi. *Atti del Reale Istituto Veneto di Scienze, Lettere ed Arti* **74** [(8) **17**] (2), 837-893.

**Summary:** The author reviews in considerable detail the work of Gini (1909, 1910, 1912), Furlan (1911), Czuber (1914) and others on Gini's index of concentration $\delta$ and its relation to the parameter $\alpha$ of the Pareto distribution, sometimes called Pareto's index. He re-examines Czuber's data on incomes in Austria during the period 1903-1911. Unlike Czuber, he prefers Gini's index to Pareto's on theoretical grounds. He asserts, however, that in most cases the approximate relation [$\delta \doteq \alpha/(\alpha-1)$ or $\alpha \doteq \delta/(\delta-1)$] is sufficiently good so that in practice it matters little which is used. He discusses the ratio of concentration

proposed by Gini (1914a) and Gini's claim that it is valid regardless of the distribution of the character being studied (income, wealth, etc.). He also discusses various measures of dispersion, including the mean deviation from the arithmetic mean $'S_A$, the mean deviation from the median $'S_M$, the mean difference $\Delta$, and the range a ["the extent of the field of variation, resulting from the difference between the extreme values" (p. 846, compiler's translation)]. He notes that Gini (1912), Czuber (1914) and Lexis [in his 1914 review of the work of Gini (1912)] all prefer the mean difference.

**References:** Pareto (1897), Gini (1909, 1910), Furlan (1911), Gini (1912), Porru (1912), Czuber (1914), Gini (1914a).

**Citations:** Czuber (1914), Gini (1916b), Ricci (1916), Gini (1917), Vinci (1918), Niceforo (1919), Gini (1926), von Bortkiewicz (1931), Gini (1932), Pietra (1932a), Castellano (1935), Gini (1936, 1939).

* Trachtenberg, M.I. (1915). A note on a property of the median. *Journal of the Royal Statistical Society* **78,** 454.

**Summary:** The author gives the following proof of the well-known minimum deviation property of the median: "The sum of the positive [absolute] deviations of two quantities $X_1$ and $X_2$ from a third X is *constant and a minimum* when the third is *any point between them*, the value of the constant being the difference between $X_1$ and $X_2$ themselves. If, then, $x_1, x_2, \ldots, x_{n-1}$ are a series of quantities, ranged in order of magnitude, divide them up into *enveloping pairs* $x_1, x_{n-1}; x_2, x_{n-2}; \ldots x_r, x_{n-r}; \ldots$. Then it follows from the above that any point M *within all the pairs* makes the sum of the deviations of each pair, and therefore the sum of the deviations of all the quantities, a minimum. Thus, if the number of quantities be even, any point in the region between the two middle quantities gives a minimum deviation-sum, and if it be odd, the middle quantity gives the minimum, because it is zero distance from itself and gives a minimum for each enveloping pair. The property is thus established."

**Citations:** Rietz (1924), Bruen (1938).

* Gini, Corrado (1916a). Sul criterio di concordanza tra due caratteri. *Atti del Reale Instituto Veneto di Scienze, Lettere ed Arti* **75** [(8) **18**] (2), 309-331.

**Summary:** The author restates the criterion of concordance (homophily), which he now calls criterion $\alpha$, proposed in an earlier paper [Gini (1915a)]. He shows by examples that if the intensities of two characters A and B are concordant according to criterion $\alpha$ (i.e., if $M_{AB} - M_{AB,0} < 0$, in the notation of the earlier paper), the intensities of A and $\beta$, where B and $\beta$ are contrary to each other (i.e., their deviations from some value, e.g. the arithmetic mean, are negatives of each other), may be concordant, discordant or neither. In order to circumvent this anomaly, he proposes criterion $\beta$: A and B are said to be concordant or discordant according as $M_{AB} - M_{AB,0} \lesseqgtr M_{A\beta} - M_{A\beta,0}$. But according to this criterion, if A and B (or $\beta$) are concordant, $\alpha$ and B (or $\beta$), where A and $\alpha$ are contrary to each other, may be concordant, discordant or neither. To overcome this difficulty, the author proposes criterion $\gamma$: A and B are said to be concordant or discordant according as $M_{AB} + M_{\alpha\beta} - M_{AB,0} - M_{\alpha\beta,0} \lesseqgtr M_{\alpha B} + M_{A\beta} - M_{\alpha B,0} - M_{A\beta,0}$. Similar criteria are given for deviations and for variations, with the M's replaced by $\mu$'s and m's as in the earlier paper. The author shows that criteria $\beta$ and $\gamma$ coincide in every case for deviations and for intensities (modalities) provided the arithmetic means of A and B are equal and those of $\alpha$ and $\beta$ are equal. When the two criteria coincide, $\beta$ is less laborious. It can be simplified whenever the distribution curves of the two characters are symmetric. The result is criterion $\delta$: The modalities, deviations or variations of two characters are concordant or discordant according as the discordance between them is less than or greater than that between the modality, deviation or variation of one of the characters and the contrary of the other. The author points out that the criteria $\alpha$, $\beta$ and $\gamma$ coincide for deviations and variations provided the distribution curves are symmetric and for modalities provided the distribution curves are symmetric, the arithmetic means of A and B are equal and those of $\alpha$ and $\beta$ are equal. In such cases, it is most convenient to use the

criterion δ. The author defines quadratic measures [see Gini (1915b)], based on squares of the differences between modalities, deviations and variations, analogous to criteria $\alpha$, $\beta$, $\gamma$, $\delta$, and shows that all four coincide, so that one can always use $\alpha$ (the simplest). He also discusses an extension of the concept of complementary characters necessary to make possible in every case the application of some of the criteria of concordance, and lack of connection (or independence) and lack of concordance (or indifference) between two characters.

**References:** Gini (1914b, 1915a,b).

**Citations:** Gini (1916b), Pietra (1925), Gini (1926).

* Gini, Corrado (1916b). Indici di concordanza. *Atti del Reale Istituto Veneto di Scienze, Lettere ed Arti* **75** [(8) **18**] (2), 1419-1461.

**Summary:** The author reviews the criteria for the existence of concordance or discordance discussed in an earlier paper [Gini (1916a)] and their respective areas of application, then turns to extension of these criteria to measures of the intensity of concordance or discordance. The simple indices of concordance or discordance for modalities (intensities) based on criterion $\alpha$ are just the simple indices of homophily proposed in another earlier paper [Gini (1915a)], viz. $O_\alpha=(M_{AB}-M_{AB.0})/(M_{AB.1}-M_{AB.0})$ or $O_\alpha=-(M_{AB}-M_{AB.0})/(M_{AB.2}-M_{AB.0})$ according as $M_{AB}\lesseqgtr M_{AB.0}$, where the M's are as defined in the earlier paper, the subscript 1 (2) designating the relative minimum (maximum) determined by pairing cograduate (contragraduate) quantities. Analogous indices based on the other criteria are given by $O_\delta=(M_{AB}-M_{A\beta})/(M_{AB.1}-M_{A\beta.2})$ or $O_\delta=-(M_{AB}-M_{A\beta})/(M_{AB.2}-M_{A\beta.1})$ according as $M_{AB}\lesseqgtr M_{A\beta}$, $O_\beta=(M_{AB}-M_{AB.0}-M_{A\beta}+M_{A\beta.0})/(M_{AB.1}-M_{AB.0}-M_{A\beta.2}+M_{A\beta.0})$ or $O_\beta=-(M_{AB}-M_{AB.0}-M_{A\beta}+M_{A\beta.0})/(M_{AB.2}-M_{AB.0}-M_{A\beta.1}+M_{A\beta.0})$ according as $(M_{AB}-M_{AB.0})\lesseqgtr(M_{A\beta}-M_{A\beta.0})$ and $O_\gamma=(M_{AB}+M_{\alpha\beta}-M_{AB.0}-M_{\alpha\beta.0}-M_{A\beta}-M_{\alpha B}+M_{A\beta.0}+M_{\alpha B.0})/(M_{AB.1}+M_{\alpha\beta.1}-M_{AB.0}-M_{\alpha\beta.0}-M_{A\beta.2}-M_{\alpha B.2}+M_{A\beta.0}+M_{\alpha B.0})$ or $O_\gamma=-(M_{AB}+M_{\alpha\beta}-M_{AB.0}-M_{\alpha\beta.0}-M_{A\beta}-M_{\alpha B}+M_{A\beta.0}+M_{\alpha B.0})/(M_{AB.2}+M_{\alpha\beta.2}-M_{AB.0}-M_{\alpha\beta.0}-M_{A\beta.1}-M_{\alpha B.1}+M_{A\beta.0}+M_{\alpha B.0})$ according as $(M_{AB}+M_{\alpha\beta}-M_{AB.0}-M_{\alpha\beta.0})\lesseqgtr(M_{A\beta}+M_{\alpha B}-M_{A\beta.0}-M_{\alpha B.0})$. The same formulas, with O and M replaced by $\omega$ and $\mu$ or by o and m, hold for simple indices of homophily between deviations or between variations. Analogous formulas are also given for quadratic indices of homophily [see Gini (1915b)]. If the relative minima and maxima in the above formulas and the analogous ones for quadratic indices are replaced by absolute minima and maxima $M_{AB.1}=0$, $\mu_{AB.1}=0$, $m_{AB.1}=0$, $M_{AB.2}=n\,{}^1S_{(A+B+\alpha+\beta)}$, $\mu_{AB.2}=n({}^1S_A+{}^1S_B)$, $m_{AB.2}=2n$, ${}^2M_{AB.1}=0$, ${}^2\mu_{AB.1}=0$, ${}^2m_{AB.1}=0$, ${}^2M_{AB.2}=4n({}^2S_{(A+B+\alpha+\beta)})^2$, ${}^2\mu_{AB.2}=2n({}^2S_A)^2+2n({}^2S_B)^2$, ${}^2m_{AB.2}=4n$, the notation being that of the earlier papers, the indices of homophily become indices of correlation. If the discordance between two quantities is measured by the absolute value of their difference, we have $M_{AB.0}=n\triangle_{R(A+B+\alpha+\beta)}$, $\mu_{AB.0}=n\triangle_{R(\ell_A+\ell_B+\ell_\alpha+\ell_\beta)}$, $m_{AB.0}=n\triangle_{(v_A+v_B+v_\alpha+v_\beta)}$, ${}^2M_{AB.0}=n({}^2\triangle_{R(A+B+\alpha+\beta)})^2=2n({}^2S_{A+B+\alpha+\beta)})^2=n({}^2S_A)^2+n({}^2S_B)^2+(n/2)(D_A^2+D_B^2+D_\alpha^2+D_\beta^2)$, ${}^2\mu_{AB.0}=n({}^2\triangle_{R(\ell_A+\ell_B+\ell_\alpha+\ell_\beta)})^2=n({}^2S_A)^2+n({}^2S_B)^2$, ${}^2m_{AB.0}=n({}^2\triangle_{R(v_A+v_B+v_\alpha+v_\beta)})^2=2n$, where $\triangle_R$ and ${}^2\triangle_R$ denote mean differences and quadratic mean differences (with repetition); $D_A$, $D_B$, $D_\alpha$, $D_\beta$ are the differences from the arithmetic mean of the n intensities of the characters A, B, $\alpha$ and $\beta$ respectively and the remaining notation is that of the earlier papers. In his summary, the author reviews the above indices of concordance between the values of two quantitative characters along with others (not relevant to this study) for qualitative characters and for ranks.

**References:** Dettori (1912), Gini (1912), Porru (1912), Gini (1915a,b), Savorgnan (1915), Gini (1916a).

**Citations:** Gini (1917), Pietra (1925), Gini (1926), Watkins (1933), Kendall (1946).

* Ricci, Umberto (1916). L'indice di variabilità e la curve dei redditi. *Giornale degli Economisti* (3) **53**, 177-228; review by Gaetano Pietra, *Rivista Italiana di Sociologia* **21** (1917), 308, 310-319.

**Summary:** The author states that the number of indices of variability is infinite. In parts I and II, he discusses the simple mean deviation (from the arithmetic mean, not from the median, which he barely mentions) and the mean difference, respectively. He also mentions the quadratic mean deviation from the arithmetic mean (the standard deviation) and the quadratic mean difference, and points out that Gini has shown that the latter is equal to $\sqrt{2}$ times the former. In addition to these measures of absolute variability, he discusses measures of relative variability, which are obtained by dividing one of the absolute measures by a measure of central tendency, e.g. the arithmetic mean. He discusses the Lorenz curve of concentration of wealth. He defines the ratio of concentration [Gini (1912)] as the mean difference divided by twice the arithmetic mean. In Part III, he relates the indices of variability and the ratio of concentration to the distribution of income. This part is based largely on the work of Pareto (1897) and of Gini (1910, 1914a).

**References:** Pareto (1897), Czuber (1903), Lorenz (1905), Persons (1909), Watkins (1909), Gini (1910), Furlan (1911), Mortara (1911), Gini (1912), Czuber (1914), Gini (1914a), Pietra (1915), Savorgnan (1915).

**Citations:** Niceforo (1919), Dalton (1920), de Pietri-Tonelli (1920), Gini (1926), d'Addario (1930), von Bortkiewicz (1931), d'Addario (1932), Gini (1932), Pietra (1932a,b), Yntema (1933), Dominedò (1934), Castellano (1935), Pietra (1935a), Bresciani-Turroni (1937, 1939), Gini (1939), Davis (1941).

* Spillman, W. J.; Tolley, H. R.; Reed, W. G. (1916). The average interval curve and its application to meteorological phenomena. *U.S. Monthly Weather Review* **44,** 197-200.

**Summary:** The relevant portion of this paper is summarized as follows in the authors' own words (pp. 197-199): "It is not difficult when the mean and standard deviation of the extreme winter minima are known for a given locality, to calculate the average frequency with which temperatures lower than any assigned limit will occur or the temperature limit that will be exceeded once on the average in a given number of years. However, the labor involved is considerable, especially when the calculations must be made for several stations, and more especially when several temperatures or intervals are concerned. In making these calculations for a large number of stations it occurred to the senior author of this paper to shorten the labor by constructing a curve with average intervals between successive occurrences of a minimum lower than a given temperature, T, as ordinates and with the departure of T from the mean as abscissae, T being expressed in terms of the standard deviation as the unit. Accordingly he constructed such a curve . . . . The equation of this curve was derived . . . [under the assumption] that the frequency curve for the extreme winter minima is normal. . . . There are three important limitations in the application of the method here outlined to meteorological phenomena. . . . [1] The periods for which data are available give so few values of the variable that the resulting values of the mean and the standard deviation are not as reliable as it is desirable that they should be. . . . [2] The frequency distribution of many meteorological phenomena cannot be accurately represented by a normal frequency curve. . . . [3] When a given event occurs once on the average, say, in 10 years, this does not mean that it will occur at regular intervals of 10 years."

**Citations:** Tolley (1916), Gumbel (1942b).

* Tolley, H. R. (1916). Frequency curves of climatic phenomena. *U.S. Monthly Weather Review* **44,** 634-642.

**Summary:** The author points out that the usefulness of the average and standard deviation of a set of data is somewhat dependent on symmetry, in which case mean, median, and mode all coincide and

one can usually assume that the distribution is normal, so that one knows what proportion of cases fall above (or below) $\mu + c\sigma$ for any value of c. He cites evidence, however, that rainfall is not symmetrically distributed, with somewhat more cases falling below the mean than above the mean, which is associated with positive skewness. He discusses the methods proposed by Pearson and others for dealing with skew frequency curves. He also discusses (page 639) the return period: "The reciprocal of the probability of the occurrence of an event may be called the *average interval* between such occurrences." He plots skew frequency curves with skewness $k = -1.2(0.6)1.2$, showing the relative positions of mean, median, and mode, tabulates the areas under such curves for $x/\sigma = -3.00(0.05)3.00$ and $k = -1.4(0.2)1.4$, and applies his results to the calculation of values below which the winter temperature will fall once in ten years at selected U.S. weather stations.

**References:** Merriman (1884), Pearson (1895), Spillman, Tolley & Reed (1916).

**Citations:** Secrist (1917), Hall (1921), Gumbel (1942b).

* Weinberg, W. (1916). Über Korrelationsmessung. *Deutsches Statistisches Zentralblatt* **8** (6), 145-158.

**Summary:** The author compares the conventional measures of variability (simple mean deviation and quadratic mean deviation) with the simple mean difference and the quadratic mean difference proposed by Gini (1912). In a footnote he points out that the simple mean difference dates back to the work of Jordan (1869). He states (incorrectly) that the variance (the square of the quadratic mean deviation or standard deviation) is equal to *twice* the square of the quadratic mean difference (with repetition); actually the former is equal to *half* the latter. He also compares the conventional Pearsonian correlation coefficient with the index of homophily defined by Gini (1915a), and discusses the use of measures of correlation in medicine, biology, anthropology and genetics.

**References:** Jordan (1869), Galton (1889), Czuber (1891a, 1899, 1903), Gini (1912, [1915a]).

**Citations:** von Bortkiewicz (1931), Castellano (1935), Pizzetti (1941a).

* Becker, L. (1917). The arithmetical mean and the "middle" value of certain meteorological observations. *Proceedings of the Royal Society of Edinburgh* **37**, 210-214. (JFM 46, 789).

**Summary:** The author begins (page 210) with the statements: "The arithmetical mean of similar meteorological observations is usually regarded as their representative value. The object of this paper is to show that in a certain case this assumption is not borne out by observations." He continues (pp. 210-211): "The investigation refers to the maximum temperature in the shade as observed at Glasgow Observatory in the forty-eight years 1868 to 1916. For each of the 73 periods of 5 days the arithmetical mean of the (240) maximum temperatures observed in the forty-eight years was calculated, and the average maximum temperature, $t_0$, was interpolated from these means for each day of the year. Let $M(\tau)$ designate the number of days of same date on which the temperature lies within the limits $t_0 + \tau \pm 1/2$ degree. For each day of the year $M(\tau)$ was counted, $\tau$ being successively all the positive and negative integers. These numbers change from day to day in a regular way for the same value of $\tau$, and no appreciable error is committed by combining the numbers computed on all the days of the month and ascribing the result to the middle of the month. Table I [p. 212] contains the $M(\tau)$ per thousand days. . . . If there is only one constant cause to an effect, the arithmetical mean coincides with the "middle" value [median], i.e., that value which occupies a middle position when the figures expressing the effect are arranged in order of magnitude, and the positive and negative departures from the arithmetical mean are equally probable. None of these properties are shown by the actual countings given in Table I, and it follows that the individual values are not grouped about only one value. . . . Under these circumstances the best representative value of such or similar data is the middle value $t'_0$, for there are even chances of an observation lying above or below this value. Let $t'_0 = t_0 + m$. The zero point from which $\tau$ is counted

must be changed by m so as to give equal sums on both sides of the zero. m is interpolated from the figures in Table I, and its values are given in the bottom line of the table. According to these values of m the middle value lies in winter up to 0.5°F. above the arithmetical mean, and in the summer up to 0.7°F. below the arithmetical mean. As to the grouping of the individual temperatures about the middle value, let $N(\tau)$ designate the number of days of the same date in the forty-eight years on which the maximum temperature lies within the limit[s] $t'_0 + \tau - 1/2$ and $t'_0 + \tau + 1/2$, $t'_0$ being the middle value of the maximum temperature on that date. Hence $N(\tau) = M(\tau + m)$, and $N(\tau)$ can be obtained from Table I by interpolation. The result is compiled in Table II [p. 213]. In the winter months the temperatures are scattered about 8°F. further to the negative side of the middle value than towards the positive side, and from March to October the extremes of the high temperatures lie much further from the middle value than the extreme low temperatures do, the scattering in September extending 10°F. further to the positive side than to the negative side. Hence the curves representing $N(\tau)$ with abscissae $\tau$ are steep for positive $\tau$ in winter and for negative $\tau$ in summer. When all the months are taken together the $N(\tau)$ curve becomes symmetrical. . . ."

**Reference:** Chauvenet (1863).

* Brunt, David (1917). *The Combination of Observations*. Cambridge University Press, Cambridge, England. (JFM 46, 799, 1495—listing only). Second edition, 1931.

**Summary:** Chapter II (see note below) deals with the law of error. Reference is made to the work of Gauss, Todhunter, and Glaisher. One section (pp. 27-28) deals with the median and deduces the law of error [Laplace's first, which the author writes in the form $f(u) = (\ell^2/2)e^{-\ell^2|u|}$] which results from the assumption that the median is the most probable value. Chapter VIII deals with the rejection of observations. Besides the criterion of Wright & Hayford, which he appears to favor, the writer discusses also those of Peirce and Chauvenet and refers to the work of Gould, Airy, and Saunder. He also states (p. 131): "Bessel opposed the rejection of any observation unless the observer was satisfied that the external conditions produced some unusual source of error not present in the other observations of the series."

**Note:** All specific references to chapters and pages are to the second edition, the only one which the compiler has seen.

**References:** Gauss (1809, 1823), Peirce (1852), Gould (1855), Airy (1856), Chauvenet (1863), Todhunter (1865), Glaisher (1872), Helmert (1872), Jordan (1877b), Wright [& Hayford] (1884) [1906], Saunder (1903).

**Citations:** Rietz (1924), Goodrich (1927), Lidstone (1930), Shewhart (1931), Rider (1933), Pearson (1935), Bruen (1938), Jeffreys (1938), Kendall (1946), Hartley (1949).

* Gini, Corrado (1917). Delle relazioni tra le intensità cograduate di due caratteri. *Atti del Reale Istituto Veneto di Scienze, Lettere ed Arti* **76** [(8) **19**] (2), 1147-1185.

**Summary:** The author distinguishes between the concepts of *corresponding* intensities and *cograduate* intensities. The *cograduation* between two characters is said to be *maximum* or *perfect* when the corresponding intensities of the two characters are always cograduate. When, on the other hand, the corresponding intensities are always contragraduate, it is said that there is *maximum* or *perfect contragraduation* between the two characters. It follows that when there is perfect cograduation between two characters, the measure of the relation between corresponding intensities of the two characters is identical with the measure of the relation between cograduate intensities of the same characters. On the other hand, if there is perfect contragraduation between the two characters, the measure of the relation between their corresponding intensities coincides with the measure of the

relation between the cograduate intensities of one of the two characters and the contrary of the other. The author points out that various writers, including Quetelet (1835) [1869] have confused corresponding intensities and cograduate intensities. Much of this paper deals with the relation between income and wealth of individuals and the analysis of data on this relation for various places and times, based in part on the Pareto distribution. The author plots the logarithm of income against the logarithm of the cograduate intensity of wealth, and calls the resulting curve the *curve of cograduation*. He finds, however, that the plot of the logarithm of income vs. the square of the logarithm of the cograduate intensity of wealth is more nearly a straight line. After fitting Pareto distributions $\log N_x = K - \alpha \log x$ and $\log N_y = h - \alpha_1 (\log y)^2$, where $N_x$ is the number of persons having income greater than x and $N_y$ is the number having wealth greater than y, he sets $N_x = N_y$ and obtains $\log x = (K-h)/\alpha + (\alpha_1/\alpha)(\log y)^2$. He calls this the *equation of cograduation* and $\alpha_1/\alpha$ the *constant of cograduation*. He studies the form of the curve of cograduation and the relations among constants of cograduation between income and wealth, indices of concentration of income and wealth, and mean values of income and wealth for various cities and countries at various times.

**References:** Quetelet (1835) [1869], Galton (1889), Pareto (1897), Persons (1909), Gini (1910, 1912), Porru (1912), Gini (1914a), Savorgnan (1915), Gini (1916b).

**Citations:** Pietra (1925), Gini (1926), d'Addario (1930).

* Secrist, Horace (1917). *An Introduction to Statistical Methods*. The Macmillan Company, New York. (JFM 46, 799—listing only). Second edition, 1925. (JFM 52, 415—listing only).

**Summary:** Chapter VIII (pp. 234-293) of this book deals with averages as types. The author discusses the arithmetic mean, the median, and the mode and gives numerical examples of their computation from grouped and ungrouped data. Concerning the median, he writes (p. 255): "While it is generally called an average it is more accurately a measure of partition or distribution." By analogy with the median M, which is the $(n+1)/2^{th}$ item in an ordered sample of size n, he defines other measures of partition, especially the lower quartile $Q_1$ [the $(n+1)/4^{th}$ item] and the upper quartile $Q_3$ [the $3(n+1)/4^{th}$ item], interpolating between successive ordered values when necessary. He discusses the choice of average; in summary (p. 290), he states: "The average best suited for a particular use depends upon the purpose one has in mind. . . . No single average will suffice for all purposes." In Chapter XI (pp. 377-424), he discusses various measures of dispersion [range, average deviation (from mean, median, or mode; minimum when taken from the median), quartile deviation $(Q_3 - Q_1)/2$, standard deviation $\sigma$, and probable error] and of skewness [$(\text{Mean-Mode})/\sigma \doteq 3(\text{Mean-Median})/\sigma$ and $(Q_1 + Q_3 - 2M)/(Q_3 - Q_1)$].

**References:** Jevons (1874), Wright (1884) [Wright & Hayford (1906)], Bowley (1901), Yule (1911), King (1912), Tolley (1916).

**Citations:** Rietz (1924), Yang (1933).

* Czuber, E. (1918). Die Anwendung der Wahrscheinlichkeitsrechnung auf Fragen der Landwirtschaft. *Zeitschrift für das Landwirtschaftliche Versuchswesen in Oesterreich* **21**, 1-100.

**Summary:** On page 36, the author states: "One will find in most cases that the range of a series of observations is approximately six times the standard deviation; a strong deviation from this relation can be taken as a sign of considerable irregularity in the distribution." (Compiler's translation). On this and the next twelve pages, he illustrates the use of this rule of thumb by computing the standard deviation and the range of nine series of observations ranging in number from 100 to 1200. For the eight series ranging in number from 100 to 500, the range exceeds six times the standard deviation only once, and then only slightly. For the one series of 1200 observations, the range (3.10) exceeds six times the standard deviation (2.46) by a considerable amount. The author attributes this excess to asymmetry of

the series of observations, making no mention of the effect of sample size on the ratio of range to standard deviation.

**References:** Czuber (1903), Mitscherlich (1903).

**Citation:** von Bortkiewicz (1922b).

* Frank, Philipp (1918). Über die Fortpflanzungsgeschwindigkeit der Diffusion. *Physikalischer Zeitschrift* **19,** 516-520.

**Summary:** Consider the Gaussian error integral $\Phi(x) = (2/\sqrt{\pi}) \int_0^x e^{-x^2} dx$ and the function $\Psi(x) = 1 - \Phi(x) = (2/\sqrt{\pi}) \int_x^\infty e^{-x^2} dx$ derived from it. In order to determine the largest value to be expected in a sample of size N from a normal (Gaussian) population, the author sets $\Psi(x) = 1/N$. For very large values of the argument, the following approximation holds: $\Psi(x) = (e^{-x^2}/x\sqrt{\pi})(1 - 1/2x^2 + \ldots)$. As a first approximation, he then sets $e^{-x^2}/x\sqrt{\pi} = 1/N$, and obtains $x = \sqrt{\ln N}$.

**Citations:** von Mises (1923), Gumbel (1935-36).

* Gini, Corrado (1918). Di una estensione del concetto di scostamento medio e di alcune applicazioni alla misura delle variabilità dei caratteri qualitativi. *Atti del Reale Istituto Veneto di Scienze, Lettere ed Arti* **77** [(8) **20**] (2), 397-461.

**Summary:** The author summarizes the defining formulas for the simple and quadratic mean deviations from the arithmetic mean, ${}^1S_A = \Sigma_{i=1}^n |a_i - A|/n$ and ${}^2S_A = \sqrt{\Sigma_{i=1}^n (a_i - A)^2/n}$; the simple and quadratic mean differences, $\Delta = \Sigma_{i=1}^n \Sigma_{j=1}^n |a_i - a_j|/n(n-1)$ and ${}^2\Delta = \sqrt{\Sigma_{i=1}^n \Sigma_{j=1}^n (a_i - a_j)^2/n(n-1)}$; and the simple and quadratic mean differences with repetition, $\Delta_R = \Sigma_{i=1}^n \Sigma_{j=1}^n |a_i - a_j|/n^2$ and ${}^2\Delta_R = \sqrt{\Sigma_{i=1}^n \Sigma_{j=1}^n (a_i - a_j)^2/n}$, of a series of n terms $a_i (i = 1, 2, \ldots, n)$ with arithmetic mean $A = \Sigma_{i=1}^n a_i/n$. He notes that ${}^2\Delta_R = {}^2S_A\sqrt{2}$. He recalls that in an earlier paper [Gini (1912)] he proposed the use in some cases of the mean difference as an index of variability and that some authors [e.g. Czuber (1914)] have advocated its use in every case, since the computation of the arithmetic mean introduces an extraneous element into the operation of finding a measure of variability. He shows that ${}^1S_A$ and ${}^2S_A$ can be computed without finding the arithmetic mean. Let us call the *deviation of a character A in the case i with respect to the case j* the quantity, positive or negative, $d_{ij}$ which measures the diversity of the modalities which the character A presents in cases i and j. The absolute value $|d_{ij}|$ of this quantity is called the *difference of A between cases i and j*. The *deviation of A in the case i* is the arithmetic mean of the $(n-1)$ deviations of the case i with respect to the other $(n-1)$ cases, which is given by $S_i = \Sigma_{i=1}^n d_{ij}/(n-1)$. The arithmetic mean of the n absolute values of $S_i$, which is given by ${}^1S = \Sigma_{i=1}^n |S_i|/n = \Sigma_{i=1}^n |\Sigma_{j=1}^n d_{ij}|/n(n-1)$, is called the *(simple) mean deviation of the series*. With repetition, these formulas become $S_{R,i} = \Sigma_{j=1}^n d_{ij}/n$ and ${}^1S_R = \Sigma_{i=1}^n |S_{R,i}|/n = \Sigma_{i=1}^n |\Sigma_{j=1}^n d_{ij}|/n^2$. The quadratic mean of the n values of $S_i$, which is given by ${}^2S = \sqrt{\Sigma_{i=1}^n S_i^2/n} = \sqrt{\Sigma_{i=1}^n (\Sigma_{j=1}^n d_{ij})^2/n(n-1)^2}$ is called the *quadratic mean deviation of the series*. With repetition, this formula becomes ${}^2S_R = \sqrt{\Sigma_{i=1}^n S_{R,i}^2/n} = \sqrt{\Sigma_{i=1}^n (\Sigma_{j=1}^n d_{ij})^2/n^3}$. The *difference of A in the case i* is the arithmetic mean of the $(n-1)$ differences between the case i and the other $(n-1)$ cases, which is given by $\Delta_i = \Sigma_{j=1}^n |d_{ij}|/(n-1)$. The arithmetic mean of the n values of $\Delta_i$, which is given by $\Delta = \Sigma_{i=1}^n \Delta_i/n = \Sigma_{i=1}^n \Sigma_{j=1}^n |d_{ij}|/n(n-1)$, is called the *(simple) mean difference of the series*. With repetition, these formulas become $\Delta_{R,i} = \Sigma_{j=1}^n |d_{ij}|/n$ and $\Delta_R = \Sigma_{i=1}^n \Delta_{R,i}/n = \Sigma_{i=1}^n \Sigma_{j=1}^n |d_{ij}|/n^2$. *The quadratic difference of A in the case i* is the quadratic mean of the $(n-1)$ differences between the case i and the other $(n-1)$ cases, which is given by ${}^2\Delta_i = \sqrt{\Sigma_{j=1}^n d_{i,j}^2/(n-1)}$. The quadratic mean of the n values of ${}^2\Delta_i$, which is given by ${}^2\Delta = \sqrt{\Sigma_{i=1}^n ({}^2\Delta_i)^2/n} = \sqrt{\Sigma_{i=1}^n \Sigma_{j=1}^n d_{ij}^2/n(n-1)}$, is called the *quadratic mean difference of the series*. With repetition, these formulas become ${}^2\Delta_{R,i} = \sqrt{\Sigma_{j=1}^n d_{ij}^2/n}$ and ${}^2\Delta_R = \sqrt{\Sigma_{i=1}^n ({}^2\Delta_{R,i})^2/n} = \sqrt{\Sigma_{i=1}^n \Sigma_{j=1}^n d_{ij}^2/n^2}$. If A is a quantitative character, we set $d_{ij} = a_i - a_j$ and then ${}^1S_R = {}^1S_A$ and ${}^2S_R = {}^2S_A$. The part of this paper dealing with extension of the concept of the mean difference to qualitative characters is not directly relevant to our study.

**References:** Gini (1912), Czuber (1914), Gini (1915a).

**Citations:** Pietra (1925), Gini & Galvani (1929), Castellano (1935), Pietra (1935b), Wold (1935), Gini (1936, 1939), Davis (1941), Pizzetti (1941a).

* Haar, Alfred (1918). Die Minkowskische Geometrie und die Annäherung an stetige Funktionen. *Mathematische Annalen* **78,** 294-311.

**Summary:** The author states the condition which has come to be called the Haar condition, and shows that it guarantees uniqueness of the Chebyshev (minimax) solution of m ($> n$) linear equations in n unknowns, and hence of the hyperplane in Euclidean n-space whose maximum deviation from m ($> n$) points is a minimum. The condition is that the determinant of the coefficients of each subset of n of the m equations be of rank n.

**Comments:** Although the name of Haar is associated with the condition stated above, priority actually belongs to de la Vallée Poussin (1911), who stated the condition and used it in proving certain theorems concerning the method of minimum approximation (his name for the minimax method).

**Reference:** Kirchberger (1903).

* Vinci, Felice (1918). La concentrazione dei capitali delle nostre società ordinarie per azioni. *Rivista della Società Commerciali* **8** (3), 195-207.

**Summary:** The author studies variations of the concentration of capital in time and in various groups of society. In doing so, he makes use of percentiles, deciles, quartiles and medians, as well as the ratio of concentration of Gini (1914a). He adds nothing new to the theory of these measures.

**References:** Gini (1914a), Savorgnan (1915).

**Citations:** Saibante (1926), Gini (1930, 1936, 1939), Davis (1941).

* Niceforo, Alfredo (1919). *La Misura della Vita.* Fratelli Bocca Editori, Milano-Torino-Roma.

**Summary:** In Chapter II (pp. 68-111) the author discusses various measures of central tendency, including the arithmetic, geometric and harmonic means (M, G and H, respectively), the mode (T), and the median. Along with the median, he considers the quartiles ($Q_1$ = lower quartile, $Q_2$ = median, $Q_3$ = upper quartile) and the deciles, also a measure of skewness [$(Q_1 + Q_3 - 2Q_2)/Q$, where $Q = (Q_3 - Q_1)/2$] which is twice Bowley's. In Chapter III (pp. 112-152), he treats various measures of variability (absolute and relative), including the range (W), the quartile deviation (Q), Bowley's quartile measure of relative variability [$(Q_3 - Q_1)/(Q_3 + Q_1)$], the simple mean deviation (s) [from the arithmetic mean or from the median], the quadratic mean deviation from the arithmetic mean ($\sigma$) [standard deviation], the mean difference ($\Delta$), and the coefficient of variation ($M/\sigma$), also Gini's ratio of concentration (R). He notes that, for symmetric or mildly skewed distributions, $Q \doteq 2\sigma/3$ and $s \doteq 4\sigma/5$ (for a normal distribution, $Q = 0.6745\sigma$ and $s = 0.7979\sigma$). He also considers higher moments, especially the third and fourth moments about the mean ($\mu_3$ and $\mu_4$) and the associated measures of skewness and kurtosis ($\beta_1 = \mu_3^2/\mu_2^3$ and $\beta_2 = \mu_4/\mu_2^2$, where $\mu_2 = \sigma^2$) and the Pearson system of frequency curves. In Chapter V (pp. 188-230), he deals with the distribution of wealth. He discusses the Lorenz curve, also Pareto's law and the associated index ($\alpha$), which he compares with Gini's index of concentration ($\delta$). In the second appendix to Chapter VII (pp. 320-329), he gives formulas for computing the various measures of central tendency and variability, together with examples of their use, and in Chapter VIII (pp. 330-362), he compares the various measures and gives standard errors for several of them, assuming an underlying normal distribution.

**References:** Quetelet (1835), Cournot (1843), Quetelet (1846), Merriman (1884), Galton (1889), Pareto (1897), Bowley (1901), Lorenz (1905), Gini (1910), Yule (1911), Dettori (1912), Gini (1912), Porru (1912), Czuber (1914), Gini (1914a), Savorgnan (1915), Ricci (1916).

**Citation:** Castellano (1935).

* Persons, Warren M. (1919). Indices of business conditions. *Review of Economic Statistics* **1,** 5-107.

**Summary:** The relevant portion of this long paper deals with the use of the median, which the author prefers to any other average, in determining the indices of seasonal variation from a series of link relatives for a particular month. The author writes (page 26): "Having established the fact of seasonal variation, the next step is to measure such variation. Such measurement will be based upon the link relatives. If the arithmetic average of the link relatives be taken, the erratic items, the magnitudes of which are not dependent upon the seasonal element, have an undue influence on the size of such average. Since our data are not numerous, however, we think it best not to neglect entirely the extreme relatives. If we had a large number of items in the frequency table and if these items had a clearly marked class of concentration, the measurement corresponding to that class, the mode, might well be taken as the seasonal relative. As we have but few items the mode is not clearly defined, and the measurement of the middle item of each array, the median, is most satisfactory. The extreme items have no more influence in determining the position of the median than do the less extreme items. Of course, where the items are very closely grouped, the medians, arithmetic averages, geometric means, and modes would approximately coincide. . . ."

**References:** Bowley (1901), Yule (1911).

**Citations:** Bowley (1901) [1920], Crum (1923).

* de la Vallée Poussin, Ch. J. (1919). *Leçons sur l'Approximation des Fonctions d'une Variable Réelle.* Gauthier-Villars et Cie., Paris.

**Summary:** In chapter VI (pp. 74-92) the author discusses the polynomial of minimum approximation or the best approximation of degree n to a continuous function f(x) in an interval (a,b) or on a finite set of $n + 2$ points, E: $X_0 < X_1 < X_2 < \ldots < X_n < X_{n+1}$ of such an interval. He proves the following theorem: There exists one and only one polynomial P of degree $\leq n$ which is of minimum approximation on E. This polynomial provides deviations of the same absolute value and of opposite signs at two consecutive points of E and this property characterizes it. The only exception is if the representation is exact and the approximation zero. He also proves the following theorem: The best approximation of degree n of a continuous function f(x) in an interval (a, b) is that of a set of $n + 2$ points of this interval, chosen so that this best approximation will be as great as possible. A similar theorem holds if f(x) is defined only at $m > n + 2$ points, and provides a method of determining the best approximation (in the Chebyshev sense of minimizing the maximum absolute error) of degree $\leq n$ to a set of $m > n + 2$ points.

**Reference:** Kirchberger (1903).

**Citation:** Jackson (1924).

* Campbell, Norman (1920). The adjustment of observations. I. *Philosophical Magazine* (6) **39** (2), 177-194.

**Summary:** For the adjustment of observations in problems of importance in physics, the author advocates the use of the Method of Averages, which he calls the Method of Zero Sum, since it depends on the principle that the sum of a large random collection of errors is zero. He does not claim that this method is completely valid theoretically, but maintains that it is no less valid than the Method of Least

Squares. Accordingly, he says, the burden of proof rests on those who continue to use a method (least squares) which is neither practically convenient nor theoretically unobjectionable. He admits that the method of averages does not give a unique result, but depends on how the observational equations are grouped; however, he insists that the apparent uniqueness of the solution by least squares is misleading, since the result is subject to error. He proposes ordering the observational equations in increasing (or decreasing) order of the one of the measured quantities which is free from error or least subject to error; then, if there are N equations and n magnitudes, with $N = pn + q$ where p and q are integers and $q < n$, forming the first normal equation by adding the first p ordered equations, the second by adding the second p ordered equations, and so on until $n - q$ normal equations have been formed, and the other q normal equations by adding $p + 1$ ordered equations for each.

**Reference:** Merriman (1884).

**Citation:** Deming (1943).

* Dalton, Hugh (1920). The measurement of the inequality of incomes. *The Economic Journal* **30,** 348-361; reprinted as appendix to second edition (1925) of author's book, *Some Aspects of the Inequality of Incomes in Modern Communities,* Routledge & Kegan Paul Ltd., London (first edition, 1920; fifth impression, 1949).

**Summary:** The author considers the relative merits of various measures of variability (dispersion), when applied to distributions of income. He enunciates the principle of transfers, which states that any transfer of income from the richer to the poorer of two individuals, provided it is not so large as to reverse their relative positions, will diminish inequality, and should therefore diminish any good measure of variability. By this criterion, the standard deviation, whether absolute or relative, and the mean difference, whether absolute or relative, are good measures; Bowley's quartile measure, $(Q_3-Q_1)/(Q_3+Q_1)$, is an indifferent measure, and the mean deviation, whether absolute or relative, is a bad measure, since it is insensitive to any transfer between individuals on the same side of the mean. The author states that it can be shown [Gini (1912)] that Gini's mean difference is equal to the weighted arithmetic mean of deviations from the median, the weights being proportionate to the number of incomes, increased by one, which are intermediate between the median and the income whose deviation is being considered. The author also discusses Pareto's measure of inequality, which is valid only for a Pareto distribution, and a measure based on the Lorenz curve (in which percentages of total income are measured along the x-axis and the minimum percentages of the total number of income-receivers who receive various percentages of the total income are measured along the y-axis). He points out that Ricci (1916) has shown that the area between the Lorenz curve and the line of equal distribution is equal to half the relative mean difference. He notes that proportionate additions to incomes increase absolute mean deviation, absolute standard deviation, and absolute mean difference but leave the corresponding relative measures and Bowley's quartile measure unchanged, whereas equal additions to income leave the absolute measures unchanged but diminish the relative measures (including Bowley's) and proportionate additions to persons leave all these measures unchanged. He concludes that the behavior of the four relative measures is better than that of the three absolute measures, which he dismisses from further consideration. He then ranks the relative measures in order of merit as follows: 1-2 (tie). Relative standard deviation and relative mean difference. 3. Bowley's quartile measure. 4. Relative mean deviation.

**References:** Pareto (1897), Bowley (1901), Lorenz (1905), Gini (1912), Porru (1912), Ricci (1916).

**Citations:** von Bortkiewicz (1931), Gini (1932), Castellano (1933b), Yntema (1933), Castellano (1935), Bresciani-Turroni (1937), de Vergottini (1940), Davis (1941), Pizzetti (1941a).

* Daniell, P. J. (1920). Observations weighted according to order. *American Journal of Mathematics* **42,** 222-236. (JFM 47, 491-492).

**Summary:** In his introduction (p. 222), the author writes: "When a series of measurements of some quantity are made, two particular quantities require to be calculated expressing respectively the norm and the deviation. For the norm the mean or the median is used while there are three measures of dispersion, the standard or root-mean-square deviation, the mean numerical deviation and the quartile deviation. The question is as to which of these are the more accurate under a general law. Moreover if we choose for our norm the mean or average it appears occasionally profitable to discard one or several extreme measures. Whether, or in what cases, this is legitimate is discussed by Poincaré but no general conclusions are obtained. Besides such a discard-average we might invent others in which weights might be assigned to the measures according to their order. In fact the ordinary average or mean, the median, the discard-average, the numerical deviation (from the median, which makes it minimum), and the quartile deviation can all be regarded as calculated by a process in which the measures are multiplied by factors which are functions of order. It is the general purpose of this paper to obtain a formula for the mean square deviation of any such expression. This formula may then be used to measure the relative accuracies of all such expressions. Certain particular types are discussed and their accuracies calculated in percentages. . . . In our applications to special types, . . . we shall only consider cases in which the theoretical distribution is symmetrical, and this for logical reasons. It is useless to compare the relative merits of the various kinds of average, for example, the mean and the median, unless they all tend to coincide when n [the number of observations] increases indefinitely. If there is a lack of symmetry both the mean and the median are necessary, or at least valuable, indications of the nature of the distribution. Indeed, in practice, their difference is sometimes regarded as a measure of lack of symmetry." The mathematical analysis is too lengthy to be reproduced here. The following numerical values are given (p. 236) of the accuracies of the various types when the law is assumed to be normal: *Norms:* Median (M), 63.7%; Quartile average $[(Q_1 + M + Q_3)/3]$, 86.0%; Quartile discard average (outer quartiles discarded), 83.7%; Mean, 100%. *Dispersion:* Quartile deviation, 36.7%; Outer sextile deviation, 46.8%, [Mean] numerical deviation, 87.6%; Discard deviation (inner quartiles discarded), 96.3%; Standard deviation, 100%. For two extreme Pearson symmetric forms, $p(t) = (2/\pi)(1 + t^2)^{-2}$ and $p(t) = (3/4)(1-t^2)(t^2 \leqslant 1)$, which the author calls supernormal and subnormal, respectively, he gives the following relative accuracies: 162% and 45% for the median, 200% and 63% for the quartile discard average, and 34% and 118% for the numerical deviation.

**Comments:** When the author writes (incorrectly) of discarding the outer (or inner) quartiles, he means the lower 25% and the upper 25% (or the middle 50%) of the observations.

**Reference:** Poincaré (1896) [1912].

**Citation:** Dodd (1922).

* Fisher, R. A. (1920). A mathematical examination of the methods of determining the accuracy of an observation by the mean error, and by the mean square error. *Monthly Notices of the Royal Astronomical Society* **80,** 758-770. Reprinted as Paper 2 in *Contributions to Mathematical Statistics*, John Wiley & Sons, Inc., New York, 1950.

**Summary:** For the normal distribution, with pdf $\exp[-(x-m)^2/2\sigma^2]/\sigma\sqrt{2\pi}$, where m is the true mean and $\sigma$ the true standard deviation, the sample mean, mean error (adjusted so as to be asymptotically unbiased) and mean square error are defined by the equations $\bar{x} = S(x)/n$, $\sigma_1 = \sqrt{\pi/2}\,S(|x - \bar{x}|)/n$ and $\sigma_2^2 = S(x - \bar{x})^2/n$. Both $\sigma_1$ and $\sigma_2$ are biased estimates of $\sigma$, with expected values $E(\sigma_1) = \sqrt{(n-1)/n}\,\sigma$ and $E(\sigma_2) = \sqrt{2/n}\,\{[(n-2)/2]!/[(n-3)/2]!\}\,\sigma \doteq (1-3/4n)\sigma$. In both cases, the correction for bias is small relative to the standard error. The standard error of $\sigma_1$ is $(\sigma/\sqrt{n})\,\sqrt{(\pi-2)/2}$ when n is large and that of $\sigma_2$ is $\sigma/\sqrt{2n}$, so that of $\sigma_1$ is $\sqrt{\pi-2}$ times that of $\sigma_2$. The author proves that the standard error of $\sigma_2$ is smaller than that

of $\sigma_p$ (defined by analogy with $\sigma_1$ and $\sigma_2$) for any value of $p \neq 2$ [this was shown by Gauss (1816), of which the author was unaware—see note preceding reprint], and that in fact $\sigma_2$ contains all the information about $\sigma$ which the sample provides (i.e., $\sigma_2$ is a sufficient statistic for $\sigma$). This unique superiority of $\sigma_2$ holds only for the normal distribution, while $\sigma_1$ enjoys the same superiority for the double exponential distribution, with pdf $\exp[-|x-m|\sqrt{2}/\sigma]/\sigma\sqrt{2}$. The author states that, when n is large, it does not seem unreasonable to use $\sigma_1$ when sampling from distributions which resemble the double exponential rather than the normal. He suggests using the value of $\beta_2$ [$\alpha_4$] (the ratio of the fourth moment to the square of the second mement) as a test statistic to distinguish between the two cases; if it is near 3 (its value for the normal distribution), use $\sigma_2$, but if it is near 6 (its value for the double exponential distribution), use $\sigma_1$. However, it should be remembered that the factor $\sqrt{\pi/2}$ in the formula for $\sigma_1$ is derived under the assumption of a normal distribution; for a double exponential distribution, it should be replaced by $\sqrt{2}$, which is about 12% larger.

**Comments:** In computing the mean error, the author uses deviations from the arithmetic mean rather than the median, from which their sum is a minimum. For samples from a symmetric distribution, both the sample mean and the sample median are unbiased estimators of the population mean (or median), and the practical difference in the results is small (zero for $n = 2$, a maximum for some small value of n ($>2$), and tending to zero again as $n \to \infty$).

**Citations:** Fisher (1922), Whittaker & Robinson (1924), Davies & Pearson (1934), Kendall (1943, 1946), Nair (1947).

* Griffith, A. A. (1920). The phenomena of rupture and flow in solids. *Philosophical Transactions of the Royal Society of London A* **221,** 163-198.

**Summary:** The author reports the results of theoretical and experimental studies of rupture in glass and other solids. The relevant results, based on the "weakest link" or "largest flaw" concept, are the following (pp. 179-180): "The general conclusion may be drawn that the weakness of isotropic solids, as ordinarily met with, is due to the presence of discontinuities, or flaws, as they may be more correctly called, whose ruling dimensions are large compared with molecular distances. The effective strength of technical materials might be increased 10 to 20 times at least if these flaws could be eliminated. . . . Consideration of the consequences of the foregoing general deduction indicated that very small solids of given form, e.g., wires or fibres, might be expected to be stronger than large ones, as there must in such cases be some additional restriction on the size of the flaws. In the limit, in fact, a fibre consisting of a single line of molecules must possess the theoretical molecular tensile strength. In this connection it is, of course, well known that fine wires are stronger than thick ones, but the present view suggests that in sufficiently fine wires the effect should be enormously greater than is observed in ordinary cases. This conclusion has been verified experimentally for the glass used in the previous tests, strengths of the same order as the theoretical tenacity having been observed."

**Citations:** Griffith (1924), Peterson (1930), Reinkober (1931), Weibull (1939a), Tucker (1941), Fisher & Hollomon (1947), Gensamer, Saibel & Lowrie (1947), Gensamer, Saibel & Ransom (1947), Epstein (1948a,b), Epstein & Brooks (1948), McAdam, Geil, Woodard & Jenkins (1948).

* Pearson, Karl (1920). On the probable errors of frequency constants. Part III. *Biometrika* **13,** 113-132.

**Summary:** The author points out that Edgeworth (1893) was in error when he stated (p. 99, footnote), that the displacements of the two quartiles and the median are independent; they are, in fact, considerably correlated. The author determines the standard errors of quantiles of a sample of size N (assumed large) from any known population and the correlations of pairs of such quantiles. He applies these results to the problem of estimating the median (mean) and the standard deviation of a normal

population from pairs of symmetric quantiles of a (large) sample from that population. He gives the best (minimum-variance unbiased) estimators of the median and of the standard deviation from one, two, or three pairs of symmetric quantiles, together with the standard errors of these estimators. Finally, he considers a numerical example dealing with measurements of the intelligence of schoolboys.

**References:** Edgeworth (1893), Sheppard (1899a).

**Citations:** Tippett (1925), Goodrich (1927), Hojo (1931), Pearson & Pearson (1931, 1932), Hojo (1933), Kendall (1943, 1946), Mosteller (1946), Wilks (1948), Benson (1949), Godwin (1949b).

* de Pietri-Tonelli, Alfonso (1920). *Lezioni di Scienza Economica Razionale e Sperimentale.* Industrie Grafiche Italiane, Rovigo.

**Summary:** The author discusses indices of variability (pp. 196-202). He defines three indices of relative variability: (1) the ratio of the sum of absolute deviations from the arithmetic mean to the sum of the data [relative mean deviation from the arithmetic mean]; (2) the ratio of the quadratic mean deviation [standard deviation] to the arithmetic mean (relative quadratic mean deviation) [relative standard deviation or coefficient of variation]; and (3) the ratio of the arithmetic mean of the absolute differences to the arithmetic mean of the data (relative mean difference). He refers to the work of Gini (1912) and Ricci (1916); like the latter, but unlike the former, he does not mention the absolute or relative mean deviation from the median. He computes the values of his three indices for data on the wholesale prices of wheat and of flour at the Rovigo market in 1914. In computing the mean difference, he uses the formula $2\sum_{i=1}^{[(n+1)/2]}(n+1-2i)\,(p_{n-i+1}-p_i)/n(n-1)$, where $p_1 \leq p \leq \ldots \leq p_n$ are the ordered values of the price (n in number).

**References:** Gini (1912), Ricci (1916).

**Citation:** Castellano (1935).

* Stewart, R. M. (1920a). Peirce's criterion. *Popular Astronomy* **28,** 2-3.

**Summary:** The author raises two objections to Peirce's criterion, which states that, "in a series of N observations, the n observations showing the largest residuals should be rejected when the probability of the system of errors obtained by retaining them is less than that of the system of errors obtained by their rejection multiplied by the probability of making n and no more abnormal observations." (p.2). These objections are: (1) "It may be remarked at the outset that the principle as stated is erroneous when n is greater than unity. For let $P_0$ be the probability of the system of errors obtained by retaining all observations, and $P_1, P_2, \ldots, P_n$ the probabilities of the systems obtained by the rejection of 1, 2, . . .,n observations (embodying the probabilities of making 1, 2, . . . , n abnormal observations). Now it is obvious that $P_1, P_2, \ldots, P_n$ may all be greater than $P_0$, while one of them $P_r (r<n)$ may be greater than any of the others. Obviously then the system obtained by rejecting only r observations is the most probable, while the above principle would call for the rejection of n observations. Hence his reasoning must be restricted to the case n = 1." (p.2). (2) "Let y be the probability, supposed unknown, that an observation is of the abnormal character which involves its rejection. . . . [Peirce's argument is based] on the assumption that . . . $y(1-y)^{N-1}$ must have its maximum possible value, or that y = 1/N. Herein lies the fallacy which vitiates the whole reasoning. Of the actual value of y we know nothing; we do know, however, that it is a function of (among other things) the liability to mistakes of the particular observer involved, and will therefore vary with different observers; nor is there any reason why it should so adjust itself as to make . . . [$y(1-y)^{N-1}$] a maximum. The most we can conclude is that $y(1-y)^{N-1}$ cannot be greater than $(N-1)^{N-1}/N^N$, the value attributed to it by Peirce, and that therefore if all residuals are less than the value obtained from Peirce's criterion, no observation should be discarded." (pp. 2-3). The author closes (p.3) with the remark: "It may be worth noting here that the criterion proposed by Chauvenet for the rejection

of a single doubtful observation, which gives results nearly the same as Peirce's criterion, also contains an obvious fallacy, and is of no value.

**References:** Peirce (1852), Chauvenet (1863).

**Citations:** Archibald (1925), Coolidge (1925), Rider (1933).

* Stewart, R. M. (1920b). The treatment of discordant observations. *Popular Astronomy* **28,** 4-6; *Publications of the American Astronomical Society* **4** (1923), 118-120.

**Summary:** Like Stone, Edgeworth, and Newcomb, the author assumes that the precision is not the same for all the observations, but simplifies matters as much as possible by restricting the number of values of the precision constant h to two. If there are $n_1$ observations of precision $h_1$ and $n_2$ observations of precision $h_2$, the law of error is $[n_1/(n_1 + n_2)]\,[h_1/\sqrt{\pi}]\,e^{-h_1^2x^2} + [n_2/(n_1 + n_2)]\,[h_2/\sqrt{\pi}]\,e^{-h_2^2x^2}$, or if we set $h_1 = H$, $h_2 = \alpha H$, $n_2 = \beta n_1$, it becomes $\{H/[(1 + \beta)\sqrt{\pi}]\}\,(e^{-H^2x^2} + \alpha\beta\, e^{-\alpha^2H^2x^2})$. It can be shown that the most probable value of the observed quantity is the weighted mean $(P_1X_1 + P_2X_2 + \ldots + P_nX_n)/(P_1 + P_2 + \ldots + P_n)$, $(n = n_1 + n_2)$, in which the weights $P_i$ are a function of the residuals $x_i$, viz. $P_i = [1 + \alpha^3\beta\, e^{H^2(1-\alpha^2)x_i^2}]/[1 + \alpha\beta\, e^{H^2(1-\alpha^2)x_i^2}]$. The quantities H, $\alpha$, and $\beta$ are determined from the two easily verified equations $\epsilon^2 = \Sigma x^2/(n-1) = \{(\alpha^2 + \beta)/[\alpha^2(1 + \beta)]\}/2H^2$, $\eta = \Sigma|x|/[n(n-1)]^{1/2} = \{(\alpha + \beta)/[\alpha(1 + \beta)]\}/H\sqrt{\pi}$ and from the condition that $\alpha$ shall be a maximum, which requires that $\beta = \alpha$. The resulting value of $P_i$ is given by $P_i = (1 + \alpha^4e^{\gamma_i})/(1 + \alpha^2e^{\gamma_i})$, where $\gamma_i = [(1-\alpha^2)x_i^2]/2\alpha\epsilon^2$. Once the weights have been determined, the weighted mean can easily be found.

**Comments:** Stewart's method is not based on order statistics, but this paper is included for purposes of comparison with other methods of treating discordant observations which are based on order statistics.

**Citations:** Ogrodnikoff (1928), Rider (1933).

* Czuber, Emanuel (1921). *Die Statistischen Forschungsmethoden*. L. W. Seidel & Sohn, Wien; review by E. J. Gumbel, *Zeitschrift für Angewandte Mathematik und Mechanik* **3** (1923), 327. Second edition, 1927; third edition (ed F. Burkhardt), 1938. (JFM 64, 543-544).

**Summary:** In Part II, Chapter 2 (pp. 67-107) the author discusses various averages. In Sections 41 and 42 (pp. 82-85) he considers the median and its relation to the arithmetic mean. He defines the median of a set of observations as a value such that as many observations lie above as below it. He notes that for an odd number (say, $2n+1$) of ordered observations, the median is uniquely defined as the middle $[(n+1)^{th}]$ observation, but that for an even number (say, $2n$) of such observations, the median can be taken as either of the two middle observations $[n^{th}$ or $(n+1)^{th}]$ or any value between them, though for convenience it is often taken as the value midway between them. In Section 46 (pp. 94-96) he discusses relations among the arithmetic mean M, the median C, and the mode D. For distributions skewed to the right, $D<C<M$; for those skewed to the left, $M<C<D$. In either case the median lies approximately one-third of the way from the arithmetic mean to the mode for a broad class of distributions [Pearson (1895)]. In Section 48 (pp. 102-106), the author discusses the logarithmic treatment of data. He points out that the median of the logarithms of a set of observations is equal to the logarithm of their median, a property not enjoyed by other averages. Chapter 4 (pp. 126-157) deals with measures of variability. In Section 64 (pp. 140-142) the author treats the mean (absolute) deviation, noting that the deviations may be taken either from the arithmetic mean or from the median, but that the latter choice has the advantage that it makes the mean deviation a minimum. He discusses quartiles, deciles, percentiles and the quartile deviation in Section 65 (pp. 143-145) and the ratios of the mean deviation and of the quartile deviation to the standard deviation in Section 66 (pp. 145-148). In Part III, Chapter 3 (pp. 278-314) is devoted to the normal frequency curve. Section 130 (pp. 311-314) deals with the precision of estimating the percentiles.

On p. 313 the author gives, for a normal distribution with mean zero, the ratios (5DP) of the median, deciles and quartiles to the standard deviation and the ratios (4DP) of their standard errors to that of the mean.

**Note:** All references to specific chapters and pages are to those in the third edition, the only one the compiler has seen.

**References:** Galton (1889), Pearson (1895), Fechner (1897), Mitscherlich (1903), Charlier (1910) [1920], Yule (1911), {von Mises (1931), Anderson (1935)}.

**Citations:** Goldhizer (1923), Rietz (1924), Jordan (1927), Foster (1934), Rosin & Rammler (1934), Gaede (1942), Kendall (1946).

* Hall, L. Standish (1921). The probable variations in yearly run-off as determined from a study of California streams. *Transactions of the American Society of Civil Engineers* **84,** 191-213; discussion, 214-257.

**Summary:** Author's synopsis (p. 191): "Data on the variations in the quantity of the yearly run-off of rivers are susceptible to treatment by the theory of probability. The purpose of this paper is to study the actual variations in the measured yearly run-off of California streams in order to construct a probability cross-section paper on which, by means of a straight-line relation, the probable minimum and maximum yearly run-offs which are likely to occur once in a period of 100 years, may be determined for any stream from the available run-off data for that stream. In connection with the study, the relation between the arithmetic mean, the median, and the most probable values [modes] of the yearly run-off were also established, at least approximately. The cross-section paper, constructed as a result of this investigation, and which was named by the writer 'Hydraulic Probability Paper,' can be used in estimating the relative wetness and dryness of individual years, in terms of the probable absolute wettest and dryest years. By using these indices of the relative wetness and dryness, the probable values of the yearly run-off can be estimated for streams for which there are few or no run-off records, and for as many years as run-off measurements have been taken on neighboring drainage areas." The Hydraulic Probability Paper mentioned above is so constructed that skewed data typical of the yearly run-offs of California streams plots as a straight line. The plotting positions used are determined by the formula $(2n-1)/2m$ for the $n^{th}$ of m yearly stream flows in order of magnitude from smallest to largest.

**References:** Merriman (1884), Yule (1911), Fuller (1914), Hazen (1914), Tolley (1916).

**Citations:** Foster (1924), Goodrich (1927), Foster (1934, 1936), Lane & Lei (1949).

* Jackson, Dunham (1921). Note on the median of a set of numbers. *Bulletin of the American Mathematical Society* **27,** 160-164. (JFM 48, 291).

**Summary:** The author shows that, for each value of $p>1$, there is a definite number $x = x_p$ which minimizes the sum $S_p(x) = \Sigma_{i=1}^{n} |x-a_i|^p$, where $a_1, a_2, \ldots, a_n$ are a set of real numbers, which may or may not be all distinct. For $p = 2$, $x_p$ is the arithmetic mean of the a's. The limit of $x_p$, as $p \to 1$, is the median, while the limit of $x_p$, as $p \to \infty$, is the midrange. When the sample size is even, say $n = 2k$, and $a_k \neq a_{k+1}$ (where $a_1 \leq a_2 \leq \ldots \leq a_n$), the author regards the median as not being uniquely defined, so that any number x belonging to the interval $a_k \leq x \leq a_{k+1}$ may be taken as the median. He supplements the usual definition by defining the median to be $X = \lim_{p \to 1} x_p$, which in this case is a definite number between $a_k$ and $a_{k+1}$ and which agrees with the usual definition otherwise. The number X is characterized by the equation $(X-a_1) \ldots (X-a_k) = (a_{k+1}-X) \ldots (a_n-X)$. If $n = 2$, $X = (a_1+a_2)/2$; if $n = 4$, $X = (a_4a_3 - a_2a_1)/[(a_4+a_3) - (a_2+a_1)]$. When $n > 4$, the explicit determination of X involves the solution of an algebraic equation of higher degree.

**Citations:** Dodd (1922), Goldhizer (1923), Jackson (1923), Rietz (1924), Whittaker & Robinson (1924), Jordan (1927), Barral Souto (1938), Bruen (1938), Cisbani (1938), Dodd (1938, 1940), Kendall (1943, 1946), Mosteller (1946).

* Jones, D. Caradog (1921). *A First Course in Statistics*. G. Bell and Sons, Ltd., London. (JFM 48, 616—listing only).

**Summary:** Chapters IV and V (pp. 22-41) deal with various averages (arithmetic mean, median, mode, and weighted mean, with brief mention of geometric and harmonic means). The author discusses the computation of the various averages, from grouped and ungrouped data, and their application. On page 39 he states: *"In a reasonably symmetrical distribution of observations, . . .* the values of the mean, the median, and the mode will be found to lie all very close together; and a useful practical rule to remember is that *the median comes in general between the mean and the mode, the difference between the mean and the mode being about three times the difference between the mean and the median."* On page 40 he summarizes the advantages and disadvantages of mean, median and mode. Chapter VI (pp. 42-51) deals with measures of dispersion or variability, including mean deviation (from the mean or the median, preferably the latter), standard deviation, quartile deviation or semi-interquartile range, and coefficient of variation. In addition to the quartiles, the author also mentions the deciles. He notes (page 50) that when the distribution is almost symmetrical, the quartile deviation is approximately 2/3 the standard deviation and the mean deviation (from the mean) is approximately 4/5 the standard deviation. A portion (pp. 61-62) of Chapter VII deals with skewness, of which the author mentions only the Pearsonian measure (mean-mode)/S.D. $\doteq$ 3(mean-median)/S.D. Several examples of the calculation of measures of central tendency, dispersion, and skewness are worked out, and a cumulative frequency curve or ogive is plotted on page 67. In the Appendix, Section 6 (pp. 270-271) is devoted to a proof that the mean deviation is a minimum when measured from the median and Section 7 (pp. 271-273) deals with the method of least squares.

**References:** Bowley (1901), Yule (1911).

**Citations:** Rietz (1924), Yang (1933), Bruen (1938).

* Julin, Armand (1921). *Principes de Statistique Théorique et Appliquée, Tome I: Statistique Théorique*. Marcel Rivière, Paris; Albert Dewit, Bruxelles.

**Summary:** The author gives a detailed discussion of measures of central tendency (Sections 212-275). First, he considers arithmetic, geometric, harmonic and contra-harmonic means and relations among them. In Secs. 256-265 (pp. 405-421), he discusses the median. He quotes Bowley (1901) [1902] and Fechner (1897) on the properties of the median, but states that his definition is nearly the same as that of Yule (1911), which seems to him the clearest. He points out, as does Yule, that if the frequency curve is constructed on a graph, an ordinate erected at the median divides the area under the curve into two equal parts. If the number of observations is odd, he defines the median as the middle value when they have been arranged in order from smallest to largest; he states (incorrectly) that this is the $(n-1)^{st}$ order statistic of a sample of size $2n-1$; actually, it is the $n^{th}$. If the number of observations is even, he takes as the median the value midway between the middle two (or their common value if they are equal, as often happens). He gives examples of the computation of the median both for grouped and for ungrouped data. He shows how chance affects the median (and the arithmetic mean) by computing these measures for the length of the 94 almonds in a half-pound purchased at a store in Brussels and of independent subsamples of sizes 54 and 45. He enumerates the advantages and disadvantages of the median. He also defines the quartiles ($Q_1$ and $Q_3$) and the deciles. Finally, he considers the mode (Mo). Then he turns to measures of dispersion. In Sections 280-283 (pp. 442-446), he discusses the mean deviation (D) [from the median (Me) or from the mean (M)]. He states that Laplace [(1812), Suppl. 2 (1818)] showed that the mean deviation is least when deviations are taken from the median, and he illustrates this point by calculating the mean

deviation of a numerical sample of size 7 from the median and from the mean. In subsequent sections, he discusses the standard deviation ($\sigma$), the quartile deviation [$Q = (Q_3 - Q_1)/2$] (semi-interquartile range), and the following measures of relative variability: $\sigma/M$ (coefficient of variation), D/Me and $(Q_3 - Q_1)/(Q_3 + Q_1)$ [Bowley's measure]. He also discusses measures of skewness, including the quartile measure [$(Q_3 + Q_1 - 2Me)/Q$], Pearson's measure [$(M - Mo)/\sigma$], for which Yule proposed substituting $3(M - Me)/\sigma$, and the standardized third moment $\alpha_3 = \sqrt{\beta_1}$. In Secs. 296-303 (pp. 466-475), he discusses the mean difference ($\Delta$ or $\Delta_R$) (without or with repetition), for which he gives simplified formulas obtained by using the relation $\Sigma_{i=1}^{n} |a_i - a_{n-i+1}| = 2 \Sigma_{i=1}^{n} |a_i - Me|$. He applies these formulas to the previously mentioned data on the lengths of 94 almonds. In Sections 429-431 (pp. 688-690), he gives formulas for the standard errors of the arithmetic mean, the median, and the standard deviation.

**References:** Laplace (1812) [1818], Quetelet (1835 [1869], 1846), Boudin (1865) [1916], Venn (1866), Fechner (1874), Jevons (1874), Galton (1875), Wright (1884) [Wright & Hayford (1906)], Edgeworth (1885), Galton (1889), Venn (1891), Pearson (1895), Galton (1896), Yule (1896a), Fechner (1897), Bowley (1901) [1902], Edgeworth (1905), Yule (1911), Gini (1912), King (1912).

**Citations:** von Bortkiewicz (1931), Gini (1932), Anderson (1935), Castellano (1935), Pizzetti (1941a).

* Keynes, John Maynard (1921). *A Treatise on Probability.* Macmillan and Co., Ltd., London.

**Summary:** The author, like Leibniz but unlike most intervening writers on the subject, considers probability to be a branch of logic rather than of mathematics, but he nevertheless makes heavy use of the latter. Chapter XVII (pp. 186-214) deals with some problems in inverse probability, including averages. The first section is a brief introduction and the second is not relevant to the present study. Sections 3-7 (pp. 194-205) deal with the fundamental theory connecting averages and laws of error, and constitute essentially a reprint of the author's earlier paper [Keynes (1911)] on that subject. Sections 8-11 (pp. 205-213) treat the arithmetic average, the method of least squares, and weighting. He traces the history of the use of the arithmetic mean. He states that it may have been first used scientifically by De Witt in 1671, though Leibniz points out that peasants have used it for a long time. He discusses the efforts of Gauss and Laplace to prove the worth of the arithmetic mean and with it that of the normal law of error and the method of least squares, but points out that these concepts occupy no unique position. In Section 12 (pp. 213-214), he discusses the rejection of discordant observations, or alternatively, assigning them smaller weights. He notes that some have deemed these practices to be in accordance with the dictates of common sense, while others, e.g. Hagen (1837), have denounced them as savoring of forgery.

**References:** N. Bernoulli (1709) [1711], Simpson (1756, 1757), Lagrange (1774), Laplace (1774), D. Bernoulli (1778), Laplace (1781), Trembley (1804), Legendre (1805), Adrain (1808), Gauss (1809), Laplace (1810, 1811a, 1812), Legendre (1814), Gauss (1823), Cauchy (1831), Encke (1832-34) [1888], Hagen (1837), Cournot (1843), Ellis (1844), Quetelet (1846), De Morgan (1847), Herschel (1850), Todhunter (1865), Venn (1866), Whitworth (1867), Glaisher (1872, 1873), Lemoine (1873), Jevons (1874), Lexis (1875), Merriman (1877), Edgeworth (1883a,b), Merriman (1884), Edgeworth (1885, 1886a,b, 1887a,e,f), Galton (1888), Bertrand (1889), Czuber (1891a), Venn (1891), Pizzetti (1892), K. Pearson (1895), Galton (1896), Poincaré (1896), Yule (1896a,b), Fechner (1897), Czuber (1899), Sheppard (1899a), Bowley (1901) [1920], Czuber (1903), Kapteyn (1903), Edgeworth (1905), Kozák (1908-10), Bohlmann (1909), Charlier (1910) [1920], Edgeworth (1911), Keynes (1911), Yule (1911).

**Citations:** Fisher (1922), Rietz (1924), Coolidge (1925), Bowley (1928), Shewhart (1931), Bartlett (1933), Jeffreys (1933), Fisher (1934), Bruen (1938), Jeffreys (1939), Simon (1941), Deming (1943), Kendall (1943), Wilks (1943), Cramér (1946), Kendall (1946).

* Morant, G. M. (1921). On random occurrences in space and time, when followed by a closed interval. *Biometrika* **13,** 309-337.

**Summary:** Neyman and Pearson (1928), p. 226, give the following concise summary of the relevant portion of this paper: "Whitworth [(1867)] . . . has shown that if the intervals of time between events which happen once on the average in time c may be considered as random, then the chance after an event has happened that another will not occur in an interval t is given by $e^{-t/c}$. If $y = f(t)$ represents the frequency distribution of the random intervals between the events, then $e^{-t/c} = \int_t^\infty f(t)dt$, or $f(t) = (1/c)e^{-t/c}$. Morant has extended this idea to the case where there is necessarily a closed interval of length b after an event before another can occur. The frequency distribution of intervals will now be $y = (1/c)e^{-(t-b)/c}$."

**Comments:** The fact that the time intervals between random events, with a closed interval b, follow a two-parameter negative exponential distribution, with location parameter b, is not directly relevant to a study of order statistics. However, like the special case (b = 0) proved by Whitworth (1867), it will prove useful later in studies of the random division of an interval.

**Reference:** Whitworth (1867).

**Citations:** Neyman & Pearson (1928), Sukhatme (1936), Kendall (1943), Greenwood (1946), Kendall (1946), Moran (1947).

* Savorgnan, Franco (1921). La ripartizione regionale della ricchezza privata italiana. *Metron* **1** (3), 100-121.

**Summary:** The author calculates the mean difference [Gini (1912)] for data on the distribution of wealth in Italy, but adds nothing new to the theory.

**Reference:** Gini (1912).

**Citation:** Castellano (1935).

* von Bortkiewicz, L. (1922a). Variationsbreite und mittlerer Fehler. *Sitzungsberichte der Berliner Mathematischer Gesellschaft* **21,** 3-11. (JFM 48, 1277-1278).

**Summary:** The range v is defined as the difference between the largest and smallest values among a given number n of observed values of any quantity. If one takes the true errors of these n values, i.e. their deviations from the true value, and orders them according to their absolute values, from smallest to largest, one obtains a series of positive numbers $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ such that $\epsilon_{i+1} \geq \epsilon_i$, and it follows that $v = \epsilon_n + \epsilon_{n-1}$ (with probability 1/2), $v = \epsilon_n + \epsilon_{n-2}$ (with probability 1/4), . . ., $v = \epsilon_n + \epsilon_1$ (with probability $1/2^{n-1}$), or $v = \epsilon_n - \epsilon_1$ (with probability $1/2^{n-1}$). Hence the mean value of the range is $\bar{v} = (1/2)(\bar{\epsilon}_n + \bar{\epsilon}_{n-1}) + (1/4)(\bar{\epsilon}_n + \bar{\epsilon}_{n-2}) + \ldots + (1/2^{n-1})(\bar{\epsilon}_n + \bar{\epsilon}_1) + (1/2^{n-1})(\bar{\epsilon}_n - \bar{\epsilon}_1) = \bar{\epsilon}_n + (1/2)\bar{\epsilon}_{n-1} + (1/4)\bar{\epsilon}_{n-2} + \ldots (1/2^{n-2})\bar{\epsilon}_2$. Now consider the Gaussian distribution $\Phi(x) = (2h/\sqrt{\pi}) \int_0^x e^{-h^2t^2}\, dt$ with precision h. The author finds expressions (in terms of integrals) for the mean range, its ratio to the standard deviation $\mu = 1/h\sqrt{2}$, and the mathematical expectation of the smallest and the largest (in absolute value) of m errors, and obtains numerical values for special cases. Other values for the mean range (besides $\bar{v}$) are v′, which is two times the most probable [modal] largest (absolute) value, and v″, which is twice the conditional expectation [restricted mean] of the largest value, given that it exceeds a lower limit $u_0 (\to 0)$. Of these $\bar{v}$ is exact [see note on von Bortkiewicz (1922b)], v′ is too low, and v″ is too high. The author shows that $(1/2)(v' + v'')$ is very nearly equal to $\bar{v}$ for $n \geq 8$.

**Reference:** Fechner (1897).

**Citations:** Dodd (1923), von Mises (1923), Neyman (1923), Tippett (1925), Gumbel (1933f, 1934g, 1935a, 1935-36, 1937a, 1939f, 1947), Wilks (1948).

* von Bortkiewicz, L. (1922b). Die Variationsbreite beim Gauss'schen Fehlergesetz. *Nordisk Statistisk Tidskrift* **1,** 11-38, 193-220.

**Summary:** This is essentially an expanded version of the author's earlier paper [von Bortkiewicz (1922a)]. This paper contains fuller discussion, more numerical examples, and references (which are completely absent from the earlier paper) to the work of earlier authors on the range and on the related problem of the largest value. The author gives the same expressions as in the earlier paper for the ratio of the mean range of a sample of size n from a normal population to the standard deviation $\mu$, except for a change in notation. In the earlier paper he calls these expressions $\bar{v}/\mu$, $v'/\mu$, $v''/\mu$ and $(1/2)(v'/\mu + v''/\mu)$, while here he denotes them by $E(v/\mu)$, $\xi(n)$, $\psi(n)$ and $[\xi(n)+\psi(n)]/2$, respectively.

**Note:** An interesting fact is that the method the author used in obtaining $\bar{v}/\mu$ [ or $E(v/\mu)$] is mathematically exact, but that this was not recognized by him or by such later authors as Dodd (1923) and Tippett (1925) because he apparently made numerical errors which gave incorrect results for $n = 48$ and $n = 100$. At the request of the compiler, Donald S. Clemm verified in 1968 on the IBM 7094 computer that the author's method does yield correct results for these as well as for other values of n.

**References:** Fourier (1824a), Quetelet (1846), Lexis (1875), Helmert (1877b), Jordan (1877a,b), Bertrand (1889), Jordan (1890), Czuber (1891a), Fechner (1897), Rodewald (1898), Czuber (1899, 1903), Mitscherlich (1903), Charlier (1910), Yule (1911), Czuber (1918).

**Citations:** Dodd (1923), Neyman (1923), Gumbel (1933f, 1934g, 1935a, 1935-36, 1937a, 1947), Wilks (1948).

* Dodd, Edward L. (1922). Functions of measurements under general laws of error. *Skandinavisk Aktuarietidskrift* **5,** 133-158; related abstracts, *Bulletin of the American Mathematical Society* **25** (1919), 245; **26** (1920), 264; **28** (1922), 9, 158-159. (JFM 48, 617—listing only).

**Summary:** The author states (pp. 134-135): "It will be the purpose of this paper to study certain functions of measurements with reference to their approximation to the so-called true value—in particular, the three functions treated by Laplace, viz., the arithmetic mean, the median, and the average of the two extremes [the midrange]. . . . The average of the two extremes [Daniell (1920)] is the application to one variable [Mansion (1913)] of the method of minimum approximation treated by Laplace [(1799, 1812)], de la Vallée Poussin [(1911)] and Goedseels [(1911)]. . . . Two functions of the measurements may be compared (1) by determining which has the greater probability density at the true value, or (2) by determining for each the mean error or, more generally, the error-risk [Czuber (1903)]. . . . In this paper the first method will be used." Sections 2-4 are not directly relevant. Sections 5 (pp. 148-153) and 6 (pp. 153-157) deal with the median and the midrange, respectively. The conclusions are summarized as follows in Section 7 (p. 158): "The present examination of functions of measurements is not exhaustive. The general conclusion, however, would probably be that in most cases where the law of error is symmetrical (the error function even) the arithmetic mean is better than other functions. The superiority of the arithmetic mean becomes somewhat doubtful: (1) When the number of measurements is small. . . . (2) When the probable error is not small as compared with the arithmetic mean. . . . (3) When the error curve,—as evidenced by the actual distribution of measurements,—falls away from its maximum with considerable rapidity at first, but nevertheless persists at some distance from the maximum. In this case, the median may be better than the arithmetic mean. Section 5. (4) When the error curve is perpendicular to the axis of errors, meeting this axis at equal distances from the origin. In this case, the average of the least and greatest measurements may be better than the average of all the measurements. Section 6."

**References:** Laplace (1799, 1812), Czuber (1891a), Pearson (1895), Poincaré (1896), Czuber (1903), Goedseels (1911), de la Vallée Poussin (1911), Dodd (1913), Mansion (1913), Daniell (1920), Jackson (1921).

**Citations:** Rider (1929), Craig (1932a), Pollard (1934), Olds (1935), Dodd (1940).

* Fisher, R. A. (1922). On the mathematical foundations of theoretical statistics. *Philosophical Transactions of the Royal Society of London A* **222,** 309-368. (JFM 48, 1280). Reprinted 1950 as Paper No. 10 of author's *Contributions to Mathematical Statistics*, John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London. (MR 12, 427).

**Summary:** In this fundamental paper, the author discusses the concepts of sufficiency and intrinsic accuracy. He endeavors to lay a firm mathematical foundation for the method of maximum likelihood. This method is here given that name for the first time, though it had previously been employed, not only by the author [Fisher (1912)], but as we have already seen, by Lambert (1760), D. Bernoulli (1778), and Gauss (1809). More directly relevant to a study of order statistics, but far less important to the subsequent development of mathematical statistics, is the author's discussion (pp. 321-322) of the problem of estimating the central value of the Cauchy distribution. He points out that the distribution of the mean of a sample of any size is the same as that of a single observation, so that the mean is an entirely useless statistic. He advocates the use of the median, whose standard error is proportional to $n^{-1/2}$, where n is the sample size. He also touches upon the treatment of outlying observations, concerning which he states (pp. 322-323): "This example [with regard to the central value of the Cauchy distribution] serves also to illustrate the practical difficulty which observers often find, that a few extreme observations appear to dominate the value of the mean. In these cases the rejection of extreme values is often advocated, and it may often happen that gross errors are thus rejected. As a statistical measure, however, the rejection of observations is too crude to be defended: and unless there are other reasons for rejection than mere divergence from the majority, it would be more philosophical to accept these extreme values, not as gross errors, but as indications that the distribution of errors is not normal. As we shall show, the only Pearsonian curve for which the mean is the best statistic for locating the curve, is the normal or gaussian curve of errors."

**References:** Sheppard (1899a), Fisher (1912, 1920), Keynes (1921).

**Citations:** Neyman & Pearson (1928), Shewhart (1931), Bartlett (1933), Hotelling (1933), Jeffreys (1933), Rider (1933), Jeffreys (1934), Darmois (1936), Pitman (1936), Savur (1937b), Fisher & Yates (1938), Jeffreys (1938), Arley & Buch (1940), Lawley (1940), Craig (1943), Kendall (1943), Wilks (1943), Geary (1944), Carlton (1946), Cramér (1946), Kendall (1946), Huzurbazar (1948), Stevens (1948).

* Crum, W. L. (1923). The use of the median in determining seasonal variation. *Journal of the American Statistical Association* **18,** 607-614; abstract, *Bulletin of the American Mathematical Society* **29,** 8. (JFM 49, 384—listing only).

**Summary:** In determining the indexes of seasonal variation in an economic series, the author takes the median of a series of link-relatives for a particular month as the unadjusted index for that month, as recommended by Persons (1919). To the observed data he fits (1) a normal curve, (2) a Charlier Type A curve, and (3) a composition of two normal curves. He notes that Yule (1911) has shown that if a distribution may be dissected into two normal distributions each of half the original frequency, and if the ratio between the two standard deviations is greater than 2.24, the median has a smaller probable error than the mean. He endeavors to extend this result to the case under consideration, in which the standard deviation of the larger portion, containing about 3/4 of the total number of cases, is 4.8 times that of the smaller portion. He concludes that for this series and for many (but not all) economic series, the median is better than the mean.

**References:** Charlier (1910) [1920], Yule (1911) [1919], Persons (1919).

**Citations:** Wilson (1923), Rietz (1924).

* Dodd, Edward Lewis (1923). The greatest and the least variate under general laws of error. *Transactions of the American Mathematical Society* **25,** 525-539. (JFM 50, 656—listing only); abstract, *Bulletin of the American Society* **29,** 216-217. (JFM 49, 384—listing only).

**Author's summary** (pp. 538-539): "The interval of variation [the range] is the difference between the greatest and the least of n variates in a distribution. Theorems are here given for the greatest variate; corresponding theorems can be stated for the least variate, using $|x|$ in place of x when necessary. In the following table which summarizes these theorems, the letters stand for positive numbers; they are constants except x, n, and G. Moreover, $g<1$; but $b>1$, $c>1$, $\alpha>1$. For each variate $\int_x^\infty \phi(x)\,dx$ is the probability that the variate will be equal to or greater than x. With $\phi(x) = \phi_1(x) \cdot \psi(x)$, the two factors are each described below. As n increases indefinitely, a probability converges to certainty that the greatest variate will take on the stated asymptotic value, with a relative error small at pleasure for the values in Classes I, III, V, and VI, and for $1/\alpha$ and $1/\gamma$ in Classes II and IV. Asymptotic values have a theoretic importance because of the rigidity of the determination. Possibly, they may be used unreservedly in problems where the variates are numerous as atoms; but in most practical problems, their chief value would seem to be in furnishing a rough check upon mean, modal, or median values. The latter can be found by determining G so that $\int_G^\infty \phi(x)dx = 1-2^{-1/n}$."

Asymptotic value of the greatest of n variates
When each variate is subject to $\phi_1(x)\cdot\psi(x)$*

| Class | $\phi_1(x)$ | Conditions for $\psi(x)$ | Greatest variate+ | Applications |
|---|---|---|---|---|
| I | $0,\ x > x_2$ | — | $x_2$‡ | 5 Pearson Types |
| II | $x^{-1-\alpha}$ | $k_1 < \psi < k_2$ | $n^{1/\alpha}$ | 3 Pearson Types |
| III | $g^{x^\alpha}$ | $x^{-\beta} < \psi < x^\beta$ | $(-\log g^n)^{1/\alpha}$ | Gaussian Function<br>Gram's Series (finite) |
| IV | $g^{(\log_c x)^\gamma}$ | $x^{-\beta} < \psi < x^\beta$ | $c^{(-\log_g n)^{1/\gamma}}$ | Jørgensen Logarithmic Function |
| V | $g^{c^x}$ | $b^{-x} < \psi < b^x$ | $\log_c(-\log_g n)$ | Makeham Life Function |
| VI | $x^{-x}$ | $b^{-x} < \psi < b^x$ | G, with $G^G = n$ | Poisson Exponential<br>Charlier B-Series (finite) |

* Or merely subject when x is sufficiently large.
+ Apart from the factor $(1+\epsilon')$ with $|\epsilon'|<\epsilon$, small at pleasure.
‡ Provided $\int_{x_2-\epsilon}^{x_2} \phi_1(x)|dx| \neq 0$.

**References:** Pearson (1895), Dunkel (1909), Charlier (1910) [1920], von Bortkiewicz (1922a,b).

**Citations:** Tippett (1925), Gumbel (1935a,b,c,d, 1935-36, 1937a), Kendall (1946), Huzurbazar (1948), Wilks (1948).

* Edgeworth, F. Y. (1923). On the use of medians for reducing observations relating to several quantities. *Philosophical Magazine* (6) **46,** 1074-1088. (JFM 49, 369).

**Summary:** The author begins with the following introductory remarks (p. 1074): "Herein are submitted some further considerations on a method which the author [Edgeworth (1887c, 1888)] proposed . . . for determining several quantities connected with observations by equations many times more numerous than the sought quantities. The facility and the accuracy of the method are now to be reconsidered. The directions given by Professor Turner [(1887)] in the improved statement of the method which he contributed . . . are very simple; yet in applying them to concrete examples he experienced difficulties. I attempt to lighten those difficulties by restating the rationale of the method and by amplifying the directions for its application. The use of the plural median is an extension of Laplace's method of situation [Laplace (1812), Supp. 2 (1818)]." After the promised restatement and amplification, the author considers several numerical examples, then closes (pp. 1085-1088) with the following conclusions: "The accuracy of the double Median depends on much the same considerations as those which relate to the single Median. . . . The comparison of the (single or double) Median method with that of Least Squares is prejudiced by two misapprehensions exaggerating (a) one the defects of the Median, (b) the other the merits of Least Squares. It is presumed that determination by way of Medians is less exact because it sometimes leaves the segment of a line, or even (in the case of the double Median) a space within which no unique value is distinguished. But the comparative definiteness of the Arithmetic Mean is illusory, considering that the determination is liable to a probable error. . . . The Method of Least Squares enjoys an undue preference in virtue of its connection with the Normal Law of Error. For probably that law is not in general fulfilled by observations so perfectly as to justify the preference given. The preferability varies with the character of the observations. . . . When the curve representing the observations is quite abnormal it is very possible that the Median should have the preference in respect of accuracy. . . . When the extremities of the curve representing the crude observations are abnormally protruberant, the Median is apt to be preferable. . . . In short, the use of the Median (single or double) is often easier, and sometimes more accurate, than the Method of Least Squares. . . . Altogether, we may conclude with Laplace that, in certain cases, the Method of Situation is preferable to the Method of Least Squares."

**References:** Laplace (1799, 1812), Airy (1861), Chauvenet (1863), Merriman (1884), Edgeworth (1887c), Turner (1887), Edgeworth (1888), Yule (1911).

**Citations:** Bowley (1928), Rhodes (1930), Bruen (1938).

* Goldhizer, K. (1923). Bemerkungen über den Medianwert. *Zeitschrift für Angewandte Mathematik und Mechanik* **3,** 321-322. (JFM 49, 384—listing only).

**Summary:** The author summarizes the results of two papers by Jackson (1921, 1923). He points out that although one usually reads that a comparison of the arithmetic mean A and the median M can be used in judging the symmetry of a population, it is really the symmetry of the deviations from an average and not that of the values themselves that is involved. One can specify, in an elementary way, asymmetric series of values for which M=A. Given two series of values $a_1<a_2<a_3<a_4$ and $b_1<b_2<b_3<b_4$, so that $a_1 + b_1<a_2 + b_2<a_3 + b_3<a_4 + b_4$, the author determines, in accordance with the definition of Jackson (1921), the condition which must be satisfied in order that $M(a_i + b_i) = M(a_i) + M(b_i) = X_1 + X_2$, where $X_1$ and $X_2$ are the medians of the a's and the b's, respectively. This condition is given by the equation $X_1\{(b_3 + b_4)-(b_1 + b_2)\} + X_2\{(a_3 + a_4)-(a_1 + a_2)\} + a_1b_2 + a_2b_1-(a_3b_4 + a_4b_3) = 0$.

**Note:** The author apparently confuses symmetry and equal spacing of the values.

**References:** Fechner (1874), Galton et al. (1881), Fechner (1897), Bowley (1901), Czuber (1921), Jackson (1921, 1923).

* Jackson, Dunham (1923). Note on quartiles and allied measures. *Bulletin of the American Mathematical Society* **29,** 17-20. (JFM 49, 376).

**Summary:** The author extends the results of his earlier paper [Jackson (1921)] as follows (pp. 17-18): "Let $a_1, \ldots, a_n$ be a set of real numbers (not necessarily all distinct) arranged in ascending order of magnitude algebraically, and let c be an arbitrary number in the interval $0<c<1$. For $p \geq 1$, let a function $\phi_p(x)$ be defined as follows: $\phi_p(x) = (1-c)x^p$ for $x \geq 0$, $\phi_p(x) = c(-x)^p$ for $x \geq 0$. . . . Let $S_p = \Sigma_{i=1}^{n} \phi_p(x-a_i)$, and let $x_p$ stand for the value of x, or any value of x, which reduces $S_p$ to a minimum. It is . . . shown that $x_p$ is uniquely determined for each value of $p>1$; that as p approaches 1, $x_p$ approaches $x_1$, if $x_1$ has a determinate value; and that in the contrary case, $x_p$ approaches a definite limit belonging to the interval within which $x_1$ is indeterminate." The value $X = \lim_{p \to 1} x_p$ is the c-quantile $a_k$ by the usual definition when cn is not an integer and k is the integer such that $k-1 < cn < k$, or when $cn = k$ (an integer) and $a_k = a_{k+1}$. When $cn = k$ and $a_k \neq a_{k+1}$, the c-quantile is defined to be X, a unique value between $a_k$ and $a_{k+1}$, by an extension of the usual definition. The results given for the median by Jackson (1921) are a special case (for $c = 1/2$) of those given in this paper; the quartiles are found by setting $c = 1/4$ and $c = 3/4$, etc.

**Reference:** Jackson (1921).

**Citations:** Goldhizer (1923), Jordan (1927), Dodd (1938).

* von Mises, R. (1923). Über die Variationsbreite einer Beobachtungsreihe. *Sitzungsberichte der Berliner Mathematischer Gesellschaft* **22,** 3-8. (JFM 49, 376-377).

**Summary:** The author makes two comments on the paper by von Bortkiewicz (1922a). The first is that the expression for the mean range $\bar{v}$ given by von Bortkiewicz can be found by taking the difference between the expected value of the largest observation and that of the smallest observation. Let W(x) be the probability that a random observation does not exceed x, so that $W^n(x)$ is the probability that none of the (independent) observations in a sample of size n exceeds x. Hence the expected value of the largest observation in a sample of size n is $g_n = \int x \, d(W^n(x))$ and similarly that of the smallest value is $k_n = -\int x \, d[(1-W(x)^n]$, so that $\bar{v} = g_n - k_n = n \int w \, [W^{n-1}(x) - (1-W(x))^{n-1}] \, x \, dx$. So, since for the Gaussian distribution $w(x) = (h/\sqrt{\pi})e^{-h^2x^2}$, $W(x) = (h/\sqrt{\pi}) \int_{-\infty}^{x} e^{-h^2x^2} \, dx = (1/2)[1 + \Phi(hx)]$, if one sets $hx = z$ and writes $\Phi_z$ for $\Phi(z)$, $\bar{v} = (n/h\sqrt{\pi}) \int z \, e^{-z^2} \{[(1 + \Phi_z)/2]^{n-1} - [(1-\Phi_z)/2]^{n-1}\} dz$. This is the result given by von Bortkiewicz in his formulas (5)-(7). The second comment deals with a series approximation to $\bar{v}$, which is valid asymptotically but is not a good approximation for small or moderate values of n.

**References:** Frank (1918), von Bortkiewicz (1922a).

**Citations:** Gumbel (1933a,f), de Finetti (1934), Gumbel (1934d,g, 1935a, 1935-36, 1936c), von Mises (1936), Gumbel (1937a, 1939f, 1945a).

* Sloane, Charles S. (1923). *Fourteenth Census of the United States, 1920: Center of Population and Median Lines and Centers of Area, Agriculture, Manufactures and Cotton.* U. S. Government Printing Office, Washington.

**Summary:** The following statements (p. 5) quoted by Eells [(1930), pp. 33-34] are relevant: "A somewhat technical significance, different from that frequently given to it, attaches to the term 'Center of population' as used in census publications. The center is often understood to be the point of intersection of a north and south line, which divides the population equally, with an east and west line which likewise divides it equally. This point of intersection is, in a certain sense, a center of population; it is here, however, designated the median point to distinguish it from the point technically defined as the center. There are points different in character which may be termed 'center of population', but at each census the term 'center of population' has been applied to the point which may be considered as the center of gravity

of the United States; in other words, the point upon which the United States would balance, if it were a rigid plane without weight and the population distributed thereon, each individual being assumed to have equal weight and to exert an influence upon the central point proportional to his distance from the point. The pivotal point, therefore, would be its center of gravity and is the point referred to by the term used in the census as 'center of population'. This is sometimes confused with another point, which we term the 'median point'. Continuing the above analogy it may be noted that the median point, which may be described as the numerical center of population, is in no sense a center of gravity. In determining the median point, distance is not taken into account and the location of the units of population is considered only in relation to the intersecting median lines—as being north or south of the median parallel and east or west of the median meridian. It is evident that extensive changes in the geographical distribution of the population may take place without affecting the position of the median point. In this respect the median point differs essentially from the center of population, which responds to the slightest population change in any section of the country. . . . If all the people in the United States were to be assembled at one place, the center of population would be the point which they could reach with the minimum aggregate travel, assuming that they all travelled in direct lines from their residence to the meeting place. No such statement holds of the median point."

**Comments:** Eells (1930) has pointed out that the statement in the next to last sentence of the above quotation is incorrect.

**Citations:** Eells (1930), Ross (1930), Galvani (1933), Gini, Boldrini, Galvani & Venere (1933), Scates (1933).

* Splawa-Neyman, J. [Neyman, J.] (1923). Sur les valeurs théoriques de la plus grande de n erreurs. (Polish summary). *Prace Matematyczno-Fizyczne* (Warsaw) **33,** 49-56. (JFM 52, 515).

**Author's summary** (p. 54, compiler's translation): "(a) The theoretical value of the largest of n errors $\alpha(n)$ exceeds the mathematical expectation of this error whatever be the probability law $\phi(x)$: $\alpha(n)>\beta(n)$. (b) If this law is that of Gauss, $\alpha(n)$ exceeds also the most probable values $\delta(n)$ and $\gamma(n)$: $\alpha(n)>\delta(n)>\gamma(n)$. [$\alpha(n) = n \int_{x_1}^{x} \phi(x)\, x\, dx$ is the mathematical expectation of the error exceeding $x_1$, where $x_1$ is the number such that $n \int_{x_1}^{x} \phi(x)dx = 1$; $\beta(n) = n\int_{-x}^{x} F^{n-1}(x)\phi(x)\, x\, dx$ is the mathematical expectation of the largest of n errors; $\gamma(n)$ is the most probable value of the largest (in absolute value) of $m = n/2$ errors; $\delta(n)$ is the most probable value of the largest of n errors.] If we adopt with von Bortkiewicz the principle that the theoretical value of the range of n errors subject to the law of Gauss is equal to twice that of the largest of the errors, we verify that the results obtained seem to be in harmony with the empirical observation of von Bortkiewicz that $\alpha(n)$ is a little too large a theoretical value of the largest of n errors. (c) If we consider the results of the calculations of von Bortkiewicz [(1922b)], we observe that the theoretical values of the range corresponding to the values of $\gamma(n)$ are constantly a little less than those given by the statistic. In view of the inequalities [$\alpha(n)>\delta(n)$, $\delta(n)>\gamma(n)$] one can assume that $\delta(n)$ is a better theoretical value than $\gamma(n)$ of the largest of n errors."

**References:** von Bortkiewicz (1922a,b).

**Citations:** Tippett (1925), Gumbel (1935-36, 1936-37).

* Wilson, Edwin Bidwell (1923). First and second laws of error. *Journal of the American Statistical Association* **18,** 841-851.

**Summary:** The author re-examines the data of Crum (1923) and reaches the following conclusions: "With the exception of the extreme positive deviations which have not been well fitted by any of Professor Crum's three suggestions, these data give internal evidence of following Laplace's first law of error instead of his second law and should be fitted to that law. By simple graphical means using arith-log [semi-log] paper an extremely good fit [to Laplace's first law] may be had in a very few minutes'

work (all that is necessary is to plot the grouped data, draw a straight line, and read the graph). . . . Professor Crum's plea for the use of the median in certain types of statistics is much reinforced by the behavior of these data when discussed in relation to the first law of error." (pp. 850-851).

**References:** Laplace (1774, 1781), Crum (1923).

**Citation:** Gumbel (1937c).

* Foster, H. Alden (1924). Theoretical frequency curves and their application to engineering problems. *Transactions of the American Society of Civil Engineers* **87,** 142-173; discussion, 174-203.

**Summary:** "The solution of certain types of engineering problems may be aided by the use of probability methods. The ordinary theory of probability, as developed in Least Squares, is not usually suitable for the analysis of natural phenomena. In general, an unsymmetrical probability formula is better adapted to engineering problems. The object of this paper is to develop such a formula and show its application to practical work. Frequency and duration [cumulative frequency] curves are first defined, and their chief characteristics are described. A theoretical frequency curve [Pearson Type III] is then developed, the mathematical steps being given only in outline. The method of evaluating the frequency formula is explained in greater detail. A table . . . is given, from which may be obtained the ordinates of the duration curve, for any values of the coefficient of variation and coefficient of skew. . . . Finally, the practical application of the theoretical curves is discussed, and several examples [from the field of hydrology] are presented to illustrate the usefulness of the method." (Author's synopsis, p. 142). A section is included (pp. 163-164) on the use of probability paper. In the discussion it is brought out that the Pearson Type III curve, whose use is proposed by the author, is not applicable in many cases. The author himself states in his conclusions (p. 172): "The main limitation to be placed on the use of these curves is the fact that the coefficient of skew [the skewness, as defined in statistical literature] must be greater than twice the coefficient of variation." Some of the discussants advocate the use of the Pearson Type I curve when this condition is not satisfied. In his closure (pp. 192-203), the author agrees with this suggestion and gives an analysis of the Type I curve, together with a table of percentage points for various coefficients of skew.

**References:** Merriman (1884), Fuller (1914), Hazen (1914), Hall (1921).

**Citations:** Goodrich (1927), Hazen (1930), Foster (1934), Slade (1934), Foster (1936), Saville (1936), Kimball (1938), Beard (1942), Lane & Lei (1949).

* Griffith, A. A. (1924). The theory of rupture. *Proceedings of the First International Congress for Applied Mechanics* (Delft), pp. 55-63.

**Summary:** The author describes an experiment whose results tend to verify the "weakest link" or "largest flaw" theory advanced by the author in an earlier paper [Griffith (1920)] to explain the much larger observed tensile strength (per unit cross-sectional area) of thin wires or fibers as compared with those of greater diameter. He reports (pp. 60-61): "A fairly hard glass containing much potash and alumina was chosen. By the usual mechanical tests, its Young's modulus was found to be 6,430 kg./sq. mm., and its tensile strength 18.3 kg./sq. mm. . . . The glass was heated to a temperature somewhat below the temperature at which boiling commenced and was then drawn down to thin rods. Immediately after preparation, these rods were found to possess strengths as high as 630 kg./sq. mm. or about half the probable value of the theoretical strength. This strength diminished spontaneously with time, however, until after the lapse of a few hours a steady value was reached which depended on the diameter of the rod. Rods of 1 mm. dia. or over had a strength equal to the strength of the glass as ordinarily determined, while the permanent strength of the finest fibres which could be made (about 1/300th. mm. dia.) was about 350 kg./sq. mm., or more than a quarter of the theoretical strength. By extrapolation, it was inferred that the strength of fibres of infinitesimal diameter would be about 1,130 kg./sq. mm. . . .

**Reference:** Griffith (1920).

**Citations:** Gensamer, Saibel & Ransom (1947), Epstein (1948a,b), Epstein & Brooks (1948).

* Haag, J. (1924). Sur la combinaison des résultats d'observation. *Comptes Rendus de l' Académie des Sciences (Paris)* **179,** 1388-1390. (JFM 50, 653).

**Summary:** For a sample of size n from $N(\mu,1)$, the mean is normally distributed with standard deviation $1/\sqrt{n}$. If n is odd, the median is asymptotically normal with standard deviation $\sqrt{\pi/2n}$, so that its relative precision is $\sqrt{2/\pi} \doteq 4/5$. If n is even, the median is defined as $(1/2)(x_m + x_{m+1})$, where $2m = n$. In this case the relative precision of the median is more difficult to obtain, at least when n is finite. More generally, consider $X = (1/2)(x_p + x_{n-p+1})$ {a quasi-midrange, which is equal to the midrange for $p = 1$ and equal to the median for $p = [(n+1)/2]$}. If n is very large and $n-2p = q$ small, X is asymptotically normal with standard deviation $1/\sqrt{kn}$, where $k = \{2e^{-a^2}/\pi[1-\Theta(a)]\}\{e^{-a^2}/[1-\Theta(a)] - a\sqrt{\pi}\}$, where $2a = x_{n-p+1} - x_p$ and $\Theta(a) = (2/\sqrt{\pi})\int_0^a e^{-x^2}dx = \text{erf}(a)$. When a increases from 0 to $+\infty$, k increases from $2/\pi$ to 1. But the probable value of a is very small. Its asymptotic value is $\sqrt{\pi}(q+1)/2n$. If n is very large and p small, X is again asymptotically normal with standard deviation $1/\sqrt{hn}$, where $h = 4ae^{-a^2}/\sqrt{\pi}\Theta(a)$. When a increases from 0 to $+\infty$, the relative precision decreases from $\sqrt{2/\pi}$ to 0. The most probable value of a is asymptotically $\sqrt{\ln n}$. The standard deviation is asymptotically $(1/2)(\pi/\ln n)^{1/4}$, which decreases much more slowly than $1/\sqrt{n}$. So the rule which consists of adopting the midrange is by far the worst.

**Citations:** Kolmogoroff (1931), Eyraud (1934b), Gumbel (1934e, 1935a, 1937c).

* Horton, Robert E. (1924). Determining the mean precipitation on a drainage basin. *New England Water Works Association Journal* **38** (1), 1-43; discussion, 44-47.

**Summary:** The relevant portion is the section entitled "Use of the Range Ratio", in which the author proposes the use of the ratio of the difference between the maximum and minimum annual rainfall amounts (theoretical or observed) to the mean in testing the validity of a rainfall record. The author states (pp. 21-25): "If a series of annual rainfall records is plotted with the years arranged in order of magnitude [of rainfall], a curve drawn through the plotted points may be used as an integral frequency curve. Thus if there are 100 years of record, the 10th highest value will have been equalled or exceeded on an average once in 10 years, the 20th highest once in 5 years, and so on. . . . Assuming the truth of the proposition, which probably everyone will accept, that there is some, though perhaps unknown, limit to the amount of rainfall, either maximum or minimum, which nature can produce in one year at a given place as long as the general climatic conditions remain as they are, then it is evident that as the exceedance interval increases, the exceedance curve . . . should approach a horizontal or asymptotic line representing this maximum value. . . . The average interval at which an event may be expected to occur having a value less than some assigned value is designated for convenience as a subceedance interval. . . . The annual rainfall ratios also approach an asymptotic line as the subceedance interval increases. . . . An expression, the graph of which fits the better determined portion of the rainfall exceedance curve and meets the requirement of approaching a horizontal asymptote as a limit, as the exceedance interval increases, is $R = B + (A-B)[1-e^{-k(t-1)^n}]$ and for subceedance curves, $R = B + (A-B)e^{-k(t-1)^n}$ where R is the ratio of the rainfall having an exceedance or subceedance interval to the mean rainfall. A, B, k and n are constants to be evaluated from the better determined portion of the curve. . . . The application of these results to the matter of testing the validity of any rainfall record may now be considered. For this purpose it is best to use the maximum range ratio or ratio of the difference between the limiting maximum and minimum annual rainfall amounts to the mean. The observed ranges and the calculated and limiting ranges for . . . eleven [New England] stations are given. . . . In no case has the actual range ratio been as great as unity. The calculated maximum range ratio obtained by taking the difference between the constants A and B is usually near unity and sometimes slightly exceeds it. It is safe to assume that if anywhere in New England a rainfall record is found in which the range ratio much exceeds unity, the

record is open to suspicion. This criterion is better than that of variability. It is simple, direct and applies to a short as well as to a long record. Unfortunately, limiting values of the range ratios are not available outside of New England."

**Citations:** Jarvis (1926), Horton (1936), Kimball (1938).

* Jackson, Dunham (1924). On the method of least $m^{th}$ powers for a set of simultaneous equations. *Annals of Mathematics* (2) **25,** 185-192; abstract, *Bulletin of the American Mathematical Society* **27** (1921), 304. (JFM 48, 299—listing only).

**Summary:** "If there is given a set of p simultaneous equations in n unknown quantities, $p>n$, the question may be raised of determining values for the unknowns so that these equations shall be approximately solved, in the sense that the sum of the $m^{th}$ powers of the absolute values of the errors is a minimum. For $m = 2$, this is the classical problem of least squares. Professor Jackson treats the general problem by methods analogous to those used in recent papers on the approximate representation of a function of a continuous variable x, the independent variable now being represented by an index ranging from 1 to p. . . ." (Abstract, p. 304). The author shows that the problem has at least one solution for every $m>0$, and a unique solution for $m>1$. The limiting case as $m\to\infty$ is equivalent to the problem of minimizing the maximum error.

**Reference:** de la Vallée Poussin (1919).

**Citation:** Bruen (1938).

* Rietz, H. L. (editor) (1924). *Handbook of Mathematical Statistics.* Houghton Mifflin Company, Boston-New York-Chicago-San Francisco; The Riverside Press, Cambridge. (*JFM 50, 355—listing only*).

**Summary:** Chapter I (pp. 1-19), written by E. V. Huntington, is entitled "Mathematical Memoranda." It contains a section (pp. 4-7) on definitions of various kinds of means or averages, including the arithmetic mean, the geometric mean, the harmonic mean, the contra-harmonic mean $[\Sigma(x_i^2)/\Sigma(x_i)]$, the root-mean-square [quadratic mean], the median, the mode, and weighted means. On page 7 we find the following statement: "A general mathematical formula including the arithmetic mean, the median, and the mode is due to D. Jackson and R. M. Foster: Let X be the value of x which minimizes $\Sigma|x_i-x|^p$. Then if $p = 2$, $X =$ *the arithmetic mean*; if $p\to 1$, lim X = *the median*; if $p\to 0$, lim X = the mode. We note also that if $p\to\infty$, lim $X = (1/2)(x_1 + x_n)$ [the midrange], where $x_1$ is the smallest and $x_n$ the largest of the given quantities." In Chapter II (pp. 20-33), Rietz gives a further discussion of averages, mainly for grouped data, and discusses the appropriate average to use. He also discusses measures of dispersion or variability [including the standard deviation, the mean deviation (which may be taken from the arithmetic mean or from the median, but is least when taken from the median), and the quartile deviation or semi-interquartile range] and measures of skewness [(arithmetic mean-mode)/ standard deviation and (upper quartile + lower quartile $-2\cdot$median)/(upper quartile−lower quartile)]. In Chapter IV (pp. 62-70), E.V. Huntington discusses curve fitting by the method of least squares and the method of moments. He mentions the method of averages, but does not mention any of the alternatives based on order statistics. There is also an extensive bibliography (pp. 195-208), prepared by W. L. Crum.

**References:** Laplace (1812), Quetelet (1846), Chauvenet (1863), Todhunter (1865), Venn (1866), Glaisher (1872), Fechner (1874), Jevons (1874), Galton (1875), Edgeworth (1883b), Merriman (1884), Edgeworth (1885, 1887-90), Galton (1888, 1889), Venn (1891), K. Pearson (1895), Yule (1896a,b), Fechner (1897), Sheppard (1899a), Bowley (1901), Czuber (1903), Kapteyn (1903), Edgeworth (1905), Charlier (1910), Edgeworth (1911), Keynes (1911), Yule (1911), King (1912), Trachtenberg (1915), Brunt (1917), Secrist (1917), Czuber (1921), Jackson (1921), Jones (1921), Keynes (1921), Crum (1923).

**Citations:** Foster (1934), Slade (1934), Snedecor (1937), Barral Souto (1938), Bruen (1938), Nair & Shrivastava (1942), Kendall (1946).

* Whittaker, E. T.; Robinson, G. (1924). *The Calculus of Observations.* Blackie & Son, Ltd., London-Glasgow. (JFM 50, 348-349). Second edition, 1926; third edition, 1940; fourth edition, 1944.

**Summary:** Chapter VIII (Secs. 81-105, pp. 164-208; see note below) deals with the normal frequency distribution. Section 102 (pp. 197-199) deals with the probable error of the median, which is found to be $\sqrt{\pi/2} \doteq 1.253$ times that of the mean, assuming that the samples are drawn from a normal population. The extended definition of the median due to Jackson (1921) is given. Section 103 (pp. 199-204) deals with the probable errors of various measures of precision. The results of Gauss (1816) for the probable errors determined from the $m^{th}$ powers of the absolute errors ($m=1,2,3,4,5,6$) and the median of the absolute deviations (from the true value) are given. Chapter IX (Secs. 106-131, pp. 209-259) deals with the method of least squares, with an account of the contributions of Legendre (1805), Gauss (1809), Laplace (1812) and others. Of particular interest is Sec. 131 (pp. 258-259), which is concerned with alternatives to the method of least squares, of which the authors mention three: (1) the method of Tobias Mayer (1750); (2) the method of minimum approximation [Laplace (1799), Goedseels (1909), de la Vallée Poussin (1911)]; and (3) the method of Edgeworth (1887c, 1888).

**Note:** All specific numbers of chapters, sections, and pages are those in the fourth edition.

**References:** Mayer (1750), Simpson (1756), Laplace (1799), Legendre (1805), Gauss (1809), Laplace (1812), Gauss (1816, 1823), Encke (1832-34), Gerling (1843), Edgeworth (1887c, 1888), Galton (1889), Pizzetti (1892), Poincaré (1896), Goedseels (1909), de la Vallée Poussin (1911), Fisher (1920), Jackson (1921).

**Citations:** Peirce (1926), Lidstone (1930), Shewhart (1931), Dodge (1933), Hotelling (1933), Bond (1935), Hotelling (1936a), Aitken (1937), Savur (1937b), Bruen (1938), Jeffreys (1938), Hulme & Symms (1939), Arley & Buch (1940), Simon (1941), Deming (1943), Wilks (1943), Kendall (1946), Duncan (1947).

* Winkler, Wilhelm (1924). Einkommen. *Handwörterbuch der Staatswissenschaften, 4. Auflage* **3,** 367-401.

**Summary:** This article is divided into three sections, dealing respectively with the idea of income, income statistics, and income politics. In the second section (pp. 380-398), the author discusses the Pareto law and other laws for the distribution of income. He mentions several measures of central tendency that may be applied to income distributions, including the arithmetic mean, the midrange, the mode, and the median [with respect to the amount of income (see Note) or with respect to the number of taxpayers]. As measures of dispersion, he mentions the simple mean (absolute) deviation, the quadratic mean deviation (standard deviation), and the quartile deviation, among others. He points out that, when the Pareto formula is known to be applicable, its parameter a [$\alpha$] is a much more easily and quickly found measure than the simple mean deviation.

**Note:** The median with respect to the amount of income is the value which later came to be called the medial [see March (1930)], while the median with respect to the number of taxpayers (income earners) is the usual median.

**References:** Pareto (1897), Persons (1909).

**Citations:** Gumbel (1927), von Bortkiewicz (1931), Bresciani-Turroni (1939).

* Amoroso, Luigi (1925). Ricerche intorno alla curva dei redditi. *Annali di Matematica Pura ed Applicata* (Bologna) **4** (2), 123-155. (JFM 51, 405-406).

**Summary:** The author discusses Pareto's law for the distribution of income [see Pareto (1897)] and the Lorenz *curve of concentration* [see Lorenz (1905)]. He proposes a generalization of Pareto's distribution of income given by $y = c\, e^{-\gamma(x-h)^{1/s}}(x-h)^{(p-s)/s}$ ($h<x<\infty$; c, h, $\gamma$, p essentially positive constants), which he rewrites in the form $\log y = \alpha + \beta \log(x-h) - \gamma(x-h)^{\epsilon}$, where $\alpha = \log c$, $\beta = (p-s)/s$ and $\epsilon = 1/s$. For $s = 1$, this distribution reduces to the Pearson type III, and closely resembles Pareto's second approximation; for $s = -1$, it reduces to the Pearson type V. The number of incomes n and their arithmetic mean m are given by $n = \int_h^x y\ dx = c \cdot |s| \cdot \Gamma(p)/\gamma^p$ and $m = h + \Gamma(p+s)/\gamma^s\Gamma(p)$. The modal value of x is $\nu = h + [(p-s)/\gamma]^s$ if $p-s>0$; otherwise, the frequency curve is monotone decreasing. The degree of skewness (schiacciamento) increases with increase in $(\nu-h)/(m-h) = (p-s)^s\Gamma(p)/\Gamma(p+s)$ or in $\Delta=(p-s)\,[\Gamma(p)/\Gamma(p-s)]^{1/s}$. When $p-s>0$, $|\Delta|<1$, but when $p-s<0$, $\Delta$ can take any negative value, however large. The author therefore calculates $q = 1 - 2^{\Delta}$, which has sign opposite to that of $\Delta$ and decreases as $\Delta$ increases, varying from +1 to −1. He calls q the determinant of the curve. The Lorenz curve is contained in the triangle OAB with vertices O (0,0), A (1,0) and B (1,1). The line OB is called the *line of equidistribution*. The area contained between the Lorenz curve and the line of equidistribution is called the *area of concentration* and the ratio r of that area to the area (1/2) of triangle OAB is called the *ratio of concentration* [Gini (1914a)]. Let S be the area of the portion of triangle OAB outside the area of concentration. Then $r = 1-2S$, so that, as S varies from 0 to 1/2, r varies from 1 to 0, and either r or S can be used as a measure of concentration of income. For Amoroso's distribution, $r = [(m-h)/m]\{1-2H(p,s)\}$, where $H(p,s)=[1/\Gamma(p)\Gamma(p+s)]\int_0^{\infty} e^{-x}x^{p-1}dx \int_0^{x} e^{-t}t^{p+s-1}dt$. The author proves the recurrence formula $H(p,s)=H(p,s-1) - \Gamma(2p+s-1)/2^{2p+s-1}\Gamma(p)\Gamma(p+s)$, where $H(p,0) = 1/2$. He uses this result to find H(p,s) and, from it, r for the special cases $s = +1$ (Pearson type III) and $s = -1$ (Pearson type V).

**Reference:** Gini (1914a).

**Citations:** Gini (1926), Gumbel (1928), d'Addario (1930, 1931, 1932), Gini (1932), Pietra (1932a), Castellano (1933a, 1935), Pietra (1935a), d'Addario (1936), Gini (1936), Bresciani-Turroni (1939), Gini (1939), Davis (1941).

* Archibald, R. C. (1925). Benjamin Peirce. V. Biographical sketch. *American Mathematical Monthly* **32,** 8-30.

**Summary:** On page 13 the author writes: "In order to bring out more clearly the place Peirce occupies in the development of American mathematics it seems desirable to comment . . . on four of the subjects which he discussed in a notable manner: 1. criterion for the rejection of doubtful observations; . . . [the other three are not relevant to the present study]. 1. 'Peirce's criterion,' as the term has appeared in scientific literature, had as its object the solution of a delicate and practically important problem of probability; this problem is: 'Being given certain observations of which the greater part is to be regarded as normal and subject to the ordinary law of error adopted in the method of least squares, while a smaller unknown portion is abnormal, and subject to some obscure source of error, to ascertain the most probable hypothesis as to the partition of the observations into normal and abnormal.' This criterion has been regarded as one of Peirce's best contributions to science. . . . The excessive brevity of Peirce's statement concerning the criterion, when it appeared in 1852, resulted in frequent misunderstandings. The tables which Gould published three years later facilitated its application. But it was not until 1878 that Peirce somewhat remedied his original statement by giving fuller explanations. In 1920, however, R. M. Stewart proved the statement fallacious."

**References:** Peirce (1852), Gould (1855), Airy (1856), Winlock (1856), Chauvenet (1863), Stone (1868), Glaisher (1873), Stone (1873b), Glaisher (1874), Stone (1874), Peirce (1878), Stewart (1920a), Coolidge (1925).

**Citation:** Rider (1933).

* Bottema, O. (1925). Over de toevallige verdeeling van een lijnsegment. *Nieuw Archief voor Wiskunde* (2) **15,** 22-27. (JFM 51, 400).

**Summary:** A line segment of unit length is divided into n parts by $(n-1)$ points chosen at random. Let $y_i$ be the length of the $i^{th}$ longest of the n parts $(i=1, 2, \ldots, n)$. Then the mean value of $y_i$ is $\bar{y}_i = (1/n)[1/n+1/(n-1) + \ldots + 1/i]$, an expression given by Mantel (1915). The mean value of the $m^{th}$ power of $y_i$ is given by $\overline{y_i^m} = [m!(n-1)!/(m+n-1)!]\,\Sigma\{1/[n^{e_1}(n-1)^{e_2}\ldots i^{e_{n-i+1}}]\}$, where the summation is over all sets of values of the $e_k$'s $(k=1,2,\ldots,n-i+1)$ such that $e_1 + e_2 + \ldots + e_{n-i+1} = m$. As a special case, we have $\overline{y_i^2} = [1/n(n+1)]\,\{1/n^2 + 1/(n-1)^2 + \ldots + 1/i^2 + [1/n + 1/(n-1) + \ldots + 1/i]^2\}$. It follows that the variance of $y_i$ is given by $\operatorname{var} y_i = \overline{y_i^2} - \bar{y}_i^2 = [1/n(n+1)]\,\{1/n^2 + 1/(n-1)^2 + \ldots + 1/i^2 - (1/n)[1/n + 1/(n-1) + \ldots + 1/i]^2\}$. Similarly, the mean value of the product of $y_i$ and $y_j$ (where $i<j$, so that $y_i \geq y_j$) is given by $\overline{y_i y_j} = [1/n(n+1)]\,\{1/n^2 + 1/(n-1)^2 + \ldots + 1/j^2 + [1/n + 1/(n-1) + \ldots + 1/i]\,[1/n+1/(n-1) + \ldots + 1/j]\}$. It follows that the correlation between $y_i$ and $y_j$ is given by $r_{ij} = [\overline{y_i y_j} - \bar{y}_i \cdot \bar{y}_j]/\sqrt{\operatorname{var} y_i \cdot \operatorname{var} y_j} = \{1/n^2 + 1/(n-1)^2 + \ldots + 1/j^2 - (1/n)[1/n + 1/(n-1) + \ldots + 1/i]\,[1/n + 1/(n-1) + \ldots + 1/j]\}/\{1/n^2 + 1/(n-1)^2 + \ldots + 1/i^2 - (1/n)[1/n + 1/(n-1) + \ldots + 1/i]^2\}^{1/2}\,\{1/n^2 + 1/(n-1)^2 + \ldots + 1/j^2 - (1/n)\,[1/n + 1/(n-1) + \ldots + 1/j]^2\}^{1/2}$.

**Reference:** Mantel (1915).

* Coolidge, Julian Lowell (1925). *An Introduction to Mathematical Probability.* Oxford University Press, Oxford-New York. (JFM 51, 380). Reprinted 1962 by Dover Publications, Inc., New York. German edition, B. G. Teubner, Leipzig, 1927. (JFM 53, 491-492).

**Summary:** Portions of Chapter VII (pp. 101-129) on "Errors of Observation" are relevant to our subject. Section 1 (pp. 101-113) of this chapter deals with determination of the "best value". On pp. 112-113 the author defines the median as "the middle term in order of magnitude of an odd number of terms and the average of the middle terms of an even number of terms." He states the following theorem, which he attributes to Fechner (1874): "The sum of the numerical values of the divergences of a number from a given series of numbers will be a minimum if the number in question be the median." Section 2 (pp. 113-124) deals with the Gaussian law of error, for which the author gives essentially Gauss' first deduction [Gauss (1809)], with passing mention of that of Hagen (1837) based on the composition of elementary errors. Section 3 (pp. 125-129) is devoted to doubtful observations. The author reviews the contributions of a number of authors, including Bertrand, Peirce, Airy, Winlock, Stewart, Chauvenet, Stone, and Glaisher (see references below). Apparently [see Rider (1933), p. 10] the author would agree with the criticism of Stone's criterion offered by Edgeworth (1887a), whom he does not mention.

**References:** Gauss (1809), Hagen (1837), Peirce (1852), Airy (1856), Winlock (1856), Chauvenet (1863), Stone (1868), Glaisher (1873), Stone (1873b), Fechner (1874), Glaisher (1874), Stone (1874), Bertrand (1889), Poincaré (1896), Czuber (1899, 1903), Stewart (1920a), Keynes (1921).

**Citations:** Archibald (1925), Shewhart (1931), Rider (1933), Eyraud (1935), Gumbel (1935-36), Bruen (1938), Simon (1941).

* Goedseels, E. (1925). *Exposé Rigoreux de la Méthode des Moindres Carrés.* Ceuterick, Louvain. Review by L. Alliaume, *Revue des Questions Scientifiques* (4) **8,** 527-528. (JFM 51, 397). Abstract, *Comptes Rendus de l'Association Française pour l'Avancement des Sciences* (Grenoble, 1925), 146-147. Supplement, *Annales de la Société Scientifique de Bruxelles* **45** (1926), (1), 242-245. (JFM 52, 524).

**Summary:** This brochure was published in lieu of a fifth edition of the author's book [Goedseels (1909)]. It contains two parts, a new exposition of the method of least squares (Chapter I-III) and an adaptation of this method (Chapter IV) to the problem forming the object of Cauchy's method. In the first part, the author considers n equations in three unknowns x, y, z: $a_i x + b_i y + c_i z = m_i + r_i$ $(i = 1, 2, \ldots, n)$.

The quantities $r_i$ are the errors affecting the equations, which evidently have finite upper bounds. The author denotes by M any upper bound whatever of the errors $r_i$, remarking that in practice one tries to give M as small a value as possible. He denotes by X, Y and Z the solutions of the least-squares normal equations, and by $R_i$ the corresponding values of $r_i$. Finally, he denotes by R the root-mean-square error. Two new notions, the quadratic approximation (see Note) and the indicating hyperbola, allow him to treat the method of least squares rigorously. He distinguishes three cases: $R = M$, $R < M$ and $R > M$. In the case $R = M$, which he considers the most interesting, he shows that the least-squares solutions X, Y, Z are the exact values of the unknowns. In the case $R < M$, which is the usual case, he shows that the least-squares solutions are the most approximative (quadratically) of the unknowns and have approximations of the form $Q\sqrt{M^2 - R^2}$ where Q is a value given in the supplement. Finally, he shows that the case $R > M$ is absurd, occurring when observers attribute to their observations a degree of precision which they do not have. He states that one usually thinks that conditional equations are better the closer R is to zero, but that his work establishes that they are better the closer R is, not to zero, but to M. In the supplement, the author proves the following theorem: When one introduces one or more new items (unknowns) into the conditional equations, and lets all the old terms remain, the new root-mean-square error $R_2$ is less than $R_1$, the old root-mean-square error. He gives further discussion of the indicating hyperbola, of which the ordinates of the upper branch are the upper bounds of x (or y or z) and the ordinates of the lower branch are the lower bounds. In the case $R = M$, the indicating hyperbola has an asymptote parallel to the axis with ordinate equal to X. In the case $R < M$, it has two branches with ordinates $X + Q\sqrt{M^2 - R^2}$ and $X - Q\sqrt{M^2 - R^2}$. The most approximative value (quadratically) is X and the quadratic approximation is $Q\sqrt{M^2 - R^2}$. In the absurd case $R > M$, the ordinates of the two branches of the indicating hyperbola vary from $-\infty$ to $+\infty$.

**Note:** According to the review by Alliaume, the author developed the quadratic approximation because practitioners found the calculations required by the most approximative method (or the method of minimum approximation, as the case may be) too laborious.

**Reference:** Goedseels (1909) [1914].

**Citation:** Alliaume (1927a).

* Irwin, J. O. (1925a). The further theory of Francis Galton's individual difference problem. *Biometrika* **17,** 100-128. (JFM 51, 391-392).

**Summary:** Assuming a normal distribution of the underlying population, the author finds the mean difference between the $p^{th}$ and $q^{th}$ individuals in order of magnitude (from largest to smallest) of a sample of size n to be $\text{Mean}(x_p - x_q) = \text{Mean } x_p - \text{Mean } x_q = \int_{-\infty}^{\infty} \{I_\alpha(n-q+1,q) - I_\alpha(n-p+1,p)\}\,dx$, where $I_x(r,s)$ is the incomplete Beta-function ratio. The $r^{th}$ moment about the origin of the frequency distribution of differences between the $p^{th}$ and $(p+1)^{th}$ individuals, where $\alpha = \int_{-\infty}^{x} \phi(u)\,du$, $\alpha' = \int_{-\infty}^{y} \phi(u)du$, $\phi(u) = e^{-u^2/2}/\sqrt{2\pi}$, is found to be $\mu_r' = \{[r(r-1)n!]/[(n-p)!\,p!]\} \int_{-\infty}^{\infty} \int_{-\infty}^{x} (1-\alpha)^p (1-\alpha')^{n-p} (x-y)^{r-2}\,dx\,dy$. A complicated expression for the actual frequency distribution of the difference is obtained. A useful approximation is obtained for the frequency distributions of differences between the first and second and between the second and third individuals in order of magnitude, it being shown that these distributions are closely represented by J-shaped curves of the form $y = y_0 \exp\{-\frac{1}{2}[(x+h)^2 - h^2]/s^2\}$. Values of $y_0$, h and s ($\Sigma$ in the author's notation) are tabulated for p = 1,2 and n = 3,10,50,100,500,1000; also p = 1, n = 2 and p = 2, n = 1200, use being made of the incomplete normal moment function $\int_{h'}^{\infty} (1/\sqrt{2\pi})e^{-x^2/2} x^r\,dx$, where $h' = h/s$.

**References:** Galton (1902), Pearson (1902).

**Citations:** Irwin (1925b), Pearson (1925), Tippett (1925), Pearson (1926), Rider (1933), Sastry (1935), Thompson (1935), Cramér (1946), Kendall (1946), Wilks (1948), Godwin (1949a,b).

* Irwin, J. O. (1925b). On a criterion for the rejection of outlying observations. *Biometrika* **17,** 238-250. (JFM 51, 398).

**Summary:** The author proposes a criterion based on the frequency distribution of the differences between the $p^{th}$ and $(p+1)^{th}$ individuals in order of magnitude in a random sample of size n from a normal population. Using the approximation he developed in an earlier paper [Irwin (1925a)], he calculates the probabilities that the differences between the first and second and between the second and third individuals should be greater than $\lambda$ times the standard deviation of the sampled population. If these probabilities become too small, he advocates rejecting the first (the last) or the first two (the last two) individuals as not belonging to the same homogeneous group as the remainder, so that a table of this probability for varying values of $\lambda$ provides a criterion for the rejection of outlying observations. If $\sigma$ is the standard deviation of the sampled population and $P(\lambda)$ is the probability that a difference will be greater than $\lambda\sigma$, the author shows that $P(\lambda) = \int_{h'+\lambda'}^{\infty} e^{-t^2/2}\,dt / \int_{h'}^{\infty} e^{-t^2/2}\,dt$, where $h = h's$, $\lambda\sigma = \lambda's$ and h and s are tabulated in the earlier paper [Irwin (1925a)] for various values of n and p = 1,2. He tabulates $P_p(\lambda)$, the probability that the difference between the $p^{th}$ and $(p+1)^{th}$ individuals in order of magnitude exceeds $\lambda\sigma$, for p = 1,2; n = 2 [for p = 1 only], 3,10(10)100(100)1000; $\lambda$ = .1(.1)5.0. On page 240 he states: "It is a matter of opinion, which the mathematician cannot settle, how small $P(\lambda)$ must be before it is justifiable to reject the observation, but about .05 seems reasonable while .01 is perhaps on the side of caution." In deciding whether to reject one observation one uses Table II [$P_1(\lambda)$], and if there are two outlying observations rather close together and then a large gap, one uses Table III [$P_2(\lambda)$] in deciding whether to reject both. The author works out three numerical examples in which he compares the performance of his criterion with that of the criteria of Peirce (1852) and Chauvenet (1863).

**References:** Airy (1856), Chauvenet (1863), Pearson (1902), Irwin (1925a).

**Citations:** Student (1927), Pearson (1931), Rider (1933), McKay (1935), Thompson (1935), Kendall (1946), Nair (1948b).

* Lévy, Paul (1925). *Calcul des Probabilités.* Gauthier-Villars, Paris. (JFM 51, 380).

**Summary:** Part 2, beginning on page 135, deals with the mathematical theory of probability. Chapter VII (pp. 278-294) treats the theory of errors. Of particular interest is Section 64 (pp. 285-288), which is concerned with the determination of parameters of precision. On page 286, the author states: "The parameter of precision is a quantity which it is useful to know approximately, and simple and nearly correct formulas are the best. The simplest procedure, given n approximate values $x_1, x_2, \ldots, x_n$ of the same quantity X, consists of discarding the 25% of the values which are the largest, and the smallest in the same proportion. The length of the interval containing the values retained should be approximately equal to a [$a = \sigma\sqrt{2}$, where $\sigma$ is the standard deviation], or more exactly to 0.95a, twice the probable error of artillerymen, for the law of Gauss. One then deduces a. If for example n = 12, one discards on the one hand the smallest values $x_1, x_2, x_3$, on the other hand the largest $x_{10}, x_{11}, x_{12}$. The interval containing the values retained is $x_9 - x_4$. It is evidently preferable to take as extremities of this interval $(x_3 + x_4)/2$ and $(x_9 + x_{10}/2$, and to define a by the formula $(x_9 + x_{10})/2 - (x_3 + x_4)/2 = 0.95a$. If n = 10, one is led similarly to write $x_8 - x_3 = 0.95a$. One can also discard the extreme values in a different proportion." (Compiler's translation.) Section 66 (pp. 291-293) deals with the method of least squares; the author does not present any alternative methods.

**Comments:** The reader will note that the author gives two methods of determining the quartiles (and from them the interquartile range and the semi-interquartile range or quartile deviation). Later [see comments on the paper by Hojo (1931)] we shall present reasons for preferring a third method.

**Citations:** Fréchet (1927), Cramér (1928), Lévy (1929), d'Addario (1931), de Finetti (1932), Cantelli (1933), de Finetti (1933), Neyman (1937b), Kendall (1943), Cramér (1946), Kendall (1946).

* Pearson, Karl (1925). On the multiple correlation of brothers, being a note on Mr. J. O. Irwin's memoir, and on my statement of the application of Galton's difference problem to the determination of the degree of relationship of brothers, made in August 1902. *Biometrika* **17,** 129-141. (JFM 51, 412).

**Summary:** The author corrects a misinterpretation by Irwin (1925a) of a statement made by the author in an earlier paper [Pearson (1902)] concerning the application of the difference problem of Galton (1902) to the determination of the degree of relationship of brothers. He makes use of the theory developed by Irwin (1925a) in the course of working out numerical examples illustrating what he meant by his earlier statement and expanding upon it.

**References:** Galton (1902), Pearson (1902), Irwin (1925a).

* Pietra, G. (1925). The theory of statistical relations with special reference to cyclical series. *Metron* **4** (3-4), 383-557. (JFM 51, 381).

**Summary:** In the introduction the author summarizes several measures of connection, concordance, and dissimilarity, proposed by Gini in a series of publications during the period 1912-18, for seriations (successions of quantities which measure the intensity of a character classified according to the intensities of another character) or series (successions of quantities which measure the intensity of a character classified according to the qualities of another character to the successive modalities of which a succession of numbers may correspond). In Part I the author deals with dissimilarity between cyclical series and in Part II with connection and concordance between cyclical series, thus extending Gini's results to the case of cyclical series. Given two groups of n ordered quantities $a_1 \geq a_2 \geq \ldots \geq a_r \geq \ldots \geq a_{n-r+1} \geq \ldots \geq a_n$ and $b_1 \geq b_2 \geq \ldots \geq b_r \geq \ldots b_{n-r+1} \geq \ldots b_n$, the quantities $a_r$ and $b_r$ are said to be *cograduate*, while the quantities $a_r$ and $b_{n-r+1}$ (or $a_{n-r+1}$ and $b_r$) are said to be *contragraduate*. Then sums of absolute values or of squares of n differences of quantities, one from each group (chosen without repetition) are least when cograduate quantities are paired and greatest when contragraduate quantities are paired. That is, $\Sigma_{r=1}^{n} |a_r - b_r|$ is a minimum and $\Sigma_{r=1}^{n} |a_r - b_{n-r+1}|$ is a maximum. Likewise $\Sigma_{r=1}^{n} (a_r - b_r)^2$ is a minimum and $\Sigma_{r=1}^{n} (a_r - b_{n-r+1})^2$ is a maximum. The *simple index of dissimilarity* is $D = (1/n) \Sigma_{i=1}^{n} |a_i - b_i|$ and the *quadratic index of dissimilarity* is $^2D = \sqrt{(1/n)\Sigma_{i=1}^{n} (a_i - b_i)^2}$. The author gives analogous indices of dissimilarity between deviations from the arithmetic mean, the quadratic mean or the median. As an index of variability, the author suggests simple or quadratic mean difference with or without repetition. Measures of the connection or the concordance between two characters arranged in a bivariate frequency table are based on the indices of dissimilarity between the cell frequencies in a column (row) and the corresponding row (column) totals. None of these indices as such is applicable to cyclical series, but the author finds analogues which are applicable in this case. He gives several numerical examples involving relations between wind directions and months or seasons of the year or hours of the day and between months of birth of mothers and sons.

**References:** Gini (1912, 1914a,b, 1915a,b, 1916a,b, 1917, 1918).

**Citations:** Gini (1926), Gini & Galvani (1929), de Finetti & Paciello (1930), de Gleria (1930), de Finetti (1931a), Castellano (1935), Dodd (1940), Pizzetti (1940, 1941a), Kendall (1946).

* Reilly, Joseph; Rae, William Norman; Wheeler, Thomas Sherlock (1925). *Physico-Chemical Methods.* D. van Nostrand, New York.

**Summary:** Chapter II (pp. 3-19), which deals with experimental errors, is the only part of this book which is relevant to our subject. The following statement (p. 10) is of special interest: "A number of rules have been suggested for the rejection of observations. Perhaps the best of these is Wright and Hayford's [second edition (1906) of the book by Wright (1884)] rule: 'Reject each observation for which the residual exceeds 5 times the P. E. for a single determination. Examine carefully each observation for which the

residual exceeds 3.5 times the P. E. and reject it if any of the accompanying conditions are such as to produce lack of confidence.'" On pp. 12-13, the authors explain the determination of empirical constants by the method of least squares; they do not present any alternative methods.

**Reference:** Wright (1884) [Wright & Hayford (1906)].

* Tippett, L. H. C. (1925). On the extreme individuals and the range of samples taken from a normal population. *Biometrika* **17,** 364-387. (JFM 51, 392).

**Summary:** The probability integral of the largest of n individuals from a normal population is given as $\int_{-\infty}^{x_p} f(x)dx = \alpha_p^n$ where $\alpha_p = \int_{-\infty}^{x_p} \phi(x)dx$, and $\phi(x) = e^{-x^2/2}/\sqrt{2\pi}$. The mean (5DP), standard deviation (4DP), and $\beta_1$ and $\beta_2$ (3DP) are given for the distribution of the largest of n individuals in samples from a normal population for n = 2,5,10,20,60,100,200,500,1000. Values of the extreme variate (4DP) occurring with limiting probabilities of 5% and 1% in samples of size n are given for n = 10,20, 50,100,200,500,1000. The mean range is given as $\overline{w} = \int_{-\infty}^{\infty} \{1-(1-\alpha)^n-\alpha^n\}dx$. The $m^{th}$ moment (about the mean) of the range is given as $\mu_m = \Sigma_{s=0}^{n-2}(-1)^s \{(n-1)!/[(s+1)!\,(n-2-s)!]\}(-w)^m + (-\overline{w})^{m-1}\Sigma_{s=0}^{n-2}(-1)^s \{n!/[(s+1)!\,(n-s-1)!]\}m \int_{-\infty}^{\infty} (1-\alpha_1^{n-s-1})\,\alpha_1^{s+1}\,dx_2 + m(m-1)\,\Sigma_{s=0}^{n-2}(-1)^s \{n!/[(s+1)!\,(n-s-1)!]\} \int_{-\infty}^{\infty} \int_{-\infty}^{x_1} (1-\alpha_1^{n-s-1})\,\alpha_n^{s+1}\,(x_1-x_n-\overline{w})^{m-2}\,dx_1\,dx_n$. For even values of n, this reduces to $\mu_m = m(m-1) \int_{-\infty}^{\infty} \int_{-\infty}^{x_1} \{1-\alpha_1^n-(1-\alpha_n)^n+(\alpha_1-\alpha_n)^n\}(x_1-x_n-\overline{w})^{m-2}\,dx_1\,dx_n-(m-1)(-\overline{w})^m$. The probability integral (7DP) of the distribution of the largest individual in samples of size n from a normal population is tabulated for n = 3,5,10: x = −2.6(0.2)5.8; n = 20,30,50: x = −0.1(0.1)6.0; n = 100(100)1000: x = 1.0(0.1)6.5. The mean range (5DP) is tabulated for samples of size n = 2(1)1000 and $\sigma$ (3DP), $\beta_1$ (3DP), and $\beta_2$ (3DP) for the distribution of range are given for samples of size n = 2,10,20,60,100,200,500,1000. The author remarks in a footnote that little reliance can be placed on the last figure of the values for $\beta_1$ and $\beta_2$ in the latter table. It turns out [see Pearson (1926)] that in some cases the next-to-last figure is inaccurate.

**References:** Pearson (1902, 1920), von Bortkiewicz (1922a), Dodd (1923), Splawa-Neyman (1923), Irwin (1925a).

**Citations:** Pearson (1926), Peirce (1926), Student (1927), Fisher & Tippett (1928), Neyman & Pearson (1928), Pearson & Adyanthāya (1928), Sophister (1928), Pearson & Adyanthāya (1929), Rider (1929), Hojo (1931), Pearson (1931), Pearson & Pearson (1931), Shewhart (1931), Pearson (1932), Dodge (1933), Rider (1933), Davies & Pearson (1934), Gumbel (1934g, 1935a, 1935-36), Pearson & Haines (1935), Sastry (1935), Snedecor (1937), Newman (1939), Daniels (1941), Simon (1941), Tucker (1941), Beard (1942), Hartley (1942), Pearson & Hartley (1942), Kendall (1943), Knudsen (1943), Gumbel (1944, 1945a), Gurney (1945), Tucker (1945a,b), Cramér (1946), Daly (1946), Kendall (1946), Mosteller (1946), Egudin (1947), Eisenhart, Hastay & Wallis (1947), Elfving (1947), Grubbs & Weaver (1947), Gumbel (1947), Gurney (1947), Lord (1947), Cox (1948), Epstein (1948a), Frankel (1948), Nair (1948b), Wilks (1948), Godwin (1949b), Howell (1949).

* Chadwick, W. B. (1926). Concerning the probability curves of n points taken at random on a straight line segment of constant length. *American Mathematical Monthly* **33,** 444-449. (JFM 52, 527).

**Summary:** Consider n points uniformly and independently distributed on a straight line segment of length a, which we may, without loss of generality, take to be the segment from the origin to the point (a,0). Let $P_k$ be the $k^{th}$ from the left of the n points. The author proves the following results: (1) The probability curve of $P_k$ has a zero of the (k−1)th order at the origin and a zero of the (n−k)th order at the point (a,0). (2) The probability curve of $P_k$ (k = 1,2, . . . ,n) is of degree (n−1). (3) The probability curve of $P_k$ is given by $y = [(n/a^n)(n-1)(n-2)\ldots(n-k+1)/(k-1)!](a-x)^{n-k}x^{k-1} = (n/a^n)$ multiplied by the $k^{th}$ term in the binomial expansion $[(a-x)+x]^{n-1}$. (4) The maximum ordinate of this curve is the one corresponding to the abscissa x = a(k−1)/(n−1) [this modal value of x is found by setting dy/dx = 0 and solving for x]. Thus the abscissas of the n points divide the x-axis from the origin to (a,0) into (n−1) equal parts [with

one point at the origin and one at (a,0)]. (5) The corresponding ordinates are given by the equation $y = n(n-1)(n-2)\ldots(n-k+1)(n-k)^{n-k}(k-1)^{k-1}/a(k-1)!(n-1)^{n-1}$. (6) The n maximum points lie on the curve $y = (a-x)^{(n-1)(a-x)/a}\, x^{(n-1)x/a} / \int_0^a (a-z)^{(n-1)(a-x)/a}\, z^{(n-1)x/a} dz$, which is continuous for $-a/(n-1) < x < na/(n-1)$, and they are symmetric with respect to the line $x = a/2$. (7) Each one of the n probability curves intersects every other one once and only once in the region under consideration, except for intersections at the origin and at (a,0). The author works out a numerical example (n=5) for which he determines the probability curves, their maximum points, the curve on which the maximum points lie, and the points of intersection of the probability curves.

* Estienne, J. E. (1926-27). Introduction à une théorie rationnelle des erreurs d'observation. *Revue d'Artillerie* **97** (1926), 421-441; **98** (1926), 542-562; **100** (1927), 471-487.

**Summary:** The author proposes replacing the classical theory of errors of observation based on the arithmetic mean and the method of least squares by what he calls a rational theory based on the median and the method of least (absolute) first powers, which he derives from the notion that, for a conscientious observer, every measurement has the same subjective probability 1/2 of being too large as of being too small. He states (p. 439 of the first installment): "When one wishes to characterize by a single number a quantity G, constant or variable, following discordant measurements, of which each one inspires a priori the same confidence, the choice of the median m is imperative. If G is constant, the choice is imperative because m is the most probable value of G. If G has varied during the course of the operations, the same choice is again imperative for the reasons . . . : (1) The number m is probably closer than any other one of the measurements taken at random; (2) The sum S of the absolute deviations of the measurements from a number a is a minimum when a = m; (3) The median value always corresponds to the same state of the quantity, whatever be the characteristic x or $\phi(x)$ adopted to estimate this quantity." (Compiler's translation). Interestingly enough, the author does not take the next logical step and propose replacing the Gaussian law of error by Laplace's first law of error. Instead, he insists that his so-called rational theory is consistent with any one of a whole family of error laws, including the Gaussian law, which may be used to approximate the unknown true law. He continues (p. 439): "If one wishes to deduce approximately the subjective probability c to attribute to the statement that the median m of n measurements of a quantity presumed constant differs by less than $\epsilon$ from the median M which results from a large number of observations made under similar conditions, one can generally apply the classical formula $c = \Theta(h\epsilon)$, h being given by the formula $1/h\sqrt{\pi} = S/n$." He also proposes to treat the case of systematic errors, which Gauss (1823) specifically excluded from his treatment, by taking the median of the medians of repeated measurements under each of several conditions. In the second installment, he proposes a method of combining m linear equations in n unknowns ($2n > m > n$) based on his rational theory. Even when the restriction $m < 2n$ is satisfied, his method does not appear to be practical. In the third installment, he insists that the importance of an error, at least in many cases, is proportional to its absolute value and not to its square as postulated by Gauss (1823); that, however numerous be a set of measurements taken to the nearest unit, they cannot provide well founded reasons to adopt a value stated to a fraction of that unit; that the accuracy of a set of measurements cannot be judged by their consistency, since there may be systematic errors; and, finally, that time and money are better spent in perfecting measuring instruments so as to obtain more accurate observations than in increasing the number of observations.

**References:** Gauss (1809, 1823), Estienne (1890).

**Citation:** Estienne (1927).

* Gini, Corrado (1926). Contributions of Italy to modern statistical methods. *Journal of the Royal Statistical Society* **89,** 703-724. (JFM 52, 39).

**Summary:** Among the contributions mentioned are those of the author [Gini (1912, 1914a)] on the mean difference and its relation to the mean deviation from the arithmetic mean or from the median,

concerning which he writes (p. 711): "Statisticians had already taken from the physical and astronomical sciences the concept of deviations from an average in order to measure the variability of the characters. The method answers well when the average represents a really existing objective value, from which the measurements furnished by observation diverge owing to accidental errors, or at least when it represents a typical value, as in the Gaussian distribution. But in many other cases, as in the series of the prices of a commodity at different times or in different places, the average from which the deviations are taken has not such a meaning, and the problem of the measurement of variability does not consist therefore in determining how much the different measurements differ from an average, but how much the different magnitudes differ among themselves. Therefore, the notion of the *mean of all possible differences* was introduced and has already been used by several authors in the field of economic and demographic phenomena." The author also reviews the work of Italian statisticians on indices of concentration [Gini (1909, 1910, 1912, 1914a), Savorgnan (1915), Ricci (1916), Amoroso (1925)], indices of dissimilarity [Gini (1914b), Pietra (1925)], relations between cograduate quantities [Gini (1914b, 1917)], and measures of concordance and discordance [Gini (1915a,b, 1916a,b)].

**References:** Pareto (1897), Gini (1909, 1910, 1912, 1914a,b, 1915a,b), Pietra (1915), Savorgnan (1915), Gini (1916a,b), Ricci (1916), Gini (1917), Amoroso (1925), Pietra (1925).

**Citations:** Gini & Galvani (1929), von Bortkiewicz (1931), Castellano (1935), Gini (1936), Davis (1941).

* Jarvis, C. S. (1926). Flood flow characteristics. *Transactions of the American Society of Civil Engineers* **89,** 985-1032; discussion, 1033-1104.

**Summary:** The paper itself is not directly relevant to a study of order statistics, but parts of the discussion are relevant, especially the comments by E. W. Lane, Weston E. Fuller, and Robert E. Horton. Lane points out that the many factors which combine to cause floods may be divided into two groups: (1) Those depending on the physiography of the basin in question; and (2) those depending on the weather. Jarvis considers only the former, which are constant for a given point on a given stream, but Lane insists that one must also take account, as Fuller (1914) has done, of the latter, which are variable, but follow some probability law. Lane proposes an equation of the form $Q = K(\log I + B)$, where Q is the discharge, in second-feet per square mile, which would be equalled or exceeded on the average once in I years, K is a constant for the station in question; and B is a constant for all the streams in a given area. If one denotes by $Q_1$ the magnitude of a flood with an average interval of one year (the "one-year flood"), then $Q_1 = KB$ or $K = Q_1/B$, so that $Q = (Q_1/B)(\log I + B) = Q_1[1 + (1/B) \log I]$, which is of the same form as the Fuller formula: $Q = CA^{0.8}(1 + 0.8 \log T)$. Lane states that his method is almost identical with that used by Horton in his contribution to the discussion of the paper by Fuller (1914), but differs from it in that Lane takes into account not only the largest flood in each year, but all floods on the stream, which he considers to be an improvement. Lane holds that the average interval I between floods of a given magnitude (return period) is more meaningful that the interval I/2 (used by Fuller) for which the chances of such a flood occurring or not occurring are equal, but Fuller disagrees. This disagreement affects also the choice of plotting positions on probability paper. Horton insists that there is a finite upper limit on the size of a flood, so that one should use a formula such as the one he developed in an earlier paper [Horton (1924)], which reflects this fact, rather than the Fuller or Lane formulas, which do not.

**References:** Fuller (1914), Horton (1924).

**Citations:** Foster (1936), Saville (1936), Gumbel (1941a).

* Pearson, Egon S. (1926). A further note on the distribution of range in samples taken from a normal population. *Biometrika* **18,** 173-194. (JFM 52, 531).

**Summary:** The author applies geometric methods to extend the work of Tippett (1925). For n = 2(1)6, he tabulates the following momental constants of the distribution of the largest value u and the smallest value v in a random sample of size n from a standard normal population [N(0,1)]: Mean, $\sigma$, $\mu_2$, $\mu_3$, $\mu_4$ (6DP) and $\beta_1$, $\beta_2$ (4DP) for the marginal distribution of u; Cross-products $p_{11}$, $p_{12}$, $p_{13}$, $p_{22}$ and correlation coefficient r (all to 6DP). For the range w = u − v, he tabulates the mean (5DP), standard deviation, $\beta_1$ and $\beta_2$ (all to 4DP) for n = 2(1)6 and the mean (5DP), standard deviation and $\beta_1$ (3DP), and $\beta_2$ (2DP) for n = 10,20,60,100,200,500,1000. Also included are graphs from which values of these moment constants can be read approximately for other values of n and a note on the relation between $\overline{w}_n$ = mean range and $\chi_{n,p}$ = mean difference between $p^{th}$ and $(p+1)^{th}$ order statistics of a sample of size n.

**References:** Irwin (1925a), Tippett (1925).

**Citations:** Student (1927), Neyman & Pearson (1928), Pearson & Adyanthāya (1928), Sophister (1928), Hojo (1931), K. Pearson (1931), E. S. Pearson (1932), Jeffreys (1938), Newman (1939), Simon (1941), Hartley (1942), Pearson & Hartley (1942), Kendall (1943, 1946), Egudin (1947), Elfving (1947), Gumbel (1947), Lord (1947), Nair (1947), Cox (1948, 1949), Godwin (1949b), Gumbel (1949).

* Peirce, Frederick Thomas (1926). Tensile strength for cotton yarns v.— "The weakest link"—theorems on the strength of long and of composite specimens. *Journal of the Textile Institute* **17,** T355-T368.

**Summary:** Epstein (1948a), pp. 140-141, has given the following excellent summary: "As far as is known to the present author, the first worker to realize the close connection between the strength of a specimen and the distribution of smallest values was F. T. Peirce [(1926)] of the British Cotton Industry Research Association. To use his own words: 'It is a truism, of which the mathematical implications are of no little interest, that the strength of a chain is that of its weakest link.' This sentence states in compact form the essence of the assumed model. Peirce was led to this problem in connection with some researches on the construction of reliable procedures for the testing of yarn. In order to accomplish this primary aim it was essential to study the effect of certain variables which one might expect on *a priori* grounds to be capable of greatly influencing the results unless properly controlled. Among the factors considered was the length of the specimen. It was observed, for example, that if specimens of some fixed length $\ell$ broke under normally distributed loads, then specimens of length $n\ell$, where n is an integrer greater than one, will break under loads whose distribution is negatively skewed the larger the value of n. [The author determines the decrease in the mode and the median of the tensile strength corresponding to an n-fold increase in length, and estimates the corresponding decrease in the mean by making use of the fact that the difference between the mean and the mode is approximately three times that between the mean and the median.] Peirce wished to account for this phenomenon, which is incidentally a characteristic of breaking strength and thus was led to the formulation of his 'chain' model and to the equivalent statistical problem. It is interesting that the paper by Peirce was published at the time when the first basic statistical researches on the distribution of extreme values were carried out by the British statistician, Tippett [(1925)]. Peirce mentions explicitly in his paper that the theoretical work of Tippett is of importance in obtaining quantitative results concerning the distribution of breaking strengths."

**Note:** Apparently neither Peirce nor Epstein was aware of the work of Chaplin (1880, 1882).

**References:** Whittaker & Robinson (1924), Tippett (1925).

**Citations:** Daniels (1945), Gurney (1945, 1947), Gurney & Pearson (1947), Epstein (1948a,b), Epstein & Brooks (1948), Higuchi, Leeper & Davis (1948), Peterson (1949b).

* Saibante, Mario (1926). I profitti delle società per azioni e la concentrazione dei capitali industriali. *Metron* **6** (1), 165-201.

**Summary:** The author defines an index of cograduation $C_{ab} = [\Sigma_{i=1}^{n} |p_{ai} - p_{\beta i}| - \Sigma_{i=1}^{n} |p_{ai} - p_{bi}|] / (n^2/2)$ [*or* $(n^2+1)/2$ for n odd], where $p_{ai}$ is the $i^{th}$ smallest of n values of character A and $p_{\beta i}$ and $p_{bi}$ are respectively the $i^{th}$ largest and $i^{th}$ smallest of the n corresponding values of character B. The numerator of the expression for $C_{ab}$ is thus the sum of n absolute differences of contragraduate quantities minus the sum of n corresponding absolute differences of cograduate quantities [see Gini (1914b, 1915a)]. He also discusses the ratio of concentration R [see Gini (1914a)]. He gives two measures of the difference between two ratios of concentration, $R_{a_1}$ and $R_{a_2}$ (expressed as percentages): $D = (R_{a_2} - R_{a_1}) / R_{a_1}(100 - R_{a_1})$ and $D' = (R_{a_2} - R_{a_1}) / \sqrt{R_{a_1}(100 - R_{a_1})}$, proposed by Vinci (1918) and by Gini, respectively.

**References:** Gini (1914a), Vinci (1918).

**Citations:** Saibante (1928), Gini (1930, 1936, 1939), Davis (1941).

* Alliaume, M. (1927a). La notion d'approximation a posteriori de plusieurs mesures d'une même grandeur et les conséquences de cette notion pour la théorie des erreurs. *Annales de la Société Scientifique de Bruxelles* (A) **47** (1), 5-14. (JFM 53, 505).

**Summary:** A quantity has been measured n times with the results $m_1, m_2, \ldots m_n$ (arranged in ascending order). All results are a priori equally precise. Both the most approximative method and the method of minimum approximation [see Goedseels (1909)] yield the midrange $(m_1 + m_n)/2$ as the best value of the measured quantity that one can obtain from the n measurements. Practitioners are reluctant to accept this result, protesting that it is based on precisely those values which one has the greatest reason to believe inexact. The author introduces an a posteriori approximation in an attempt to interpret these objections mathematically and to satisfy them. Let $\mu$ be that value adopted in a first calculation, whatever the reason for adopting it. Let P be a positive number which is called the a posteriori approximation factor. The a posteriori approximation of a measurement $m_i$ is taken to be a linear function of the absolute value of the difference $(\mu - m_i)$, consisting of a constant term denoted by M and a first degree term with coefficient P. If $m_i$ is less than $\mu$, the possible values m of the measured quantity, if one takes into account only the $i^{th}$ observation, satisfy the double inequality $-M - (\mu - m_i)P \leq m - m_i \leq M + (\mu - m_i)P$; on the contrary, if $m_i$ is greater than $\mu$, it is necessary to write $-M - (m_i - \mu)P \leq m - m_i \leq M + (m_i - \mu)P$. The author shows that it is when the coefficient P is greater than one that the a priori approximation is sufficiently obliterated by the a posteriori approximation so that the most advantageous final value is no longer the midrange. When practitioners reject this final value, it is necessary to admit that they assign to P a value greater than one, which leads to the following result: When one wishes to interpose the a posteriori approximation in employing several measurements of the same quantity with the same a priori approximation, taking account of the manner in which the measurements are distributed in the series of numbers, it is necessary to adopt, as the most advantageous final value, half the sum of the two closest values; when several pairs of consecutive measurements define equal intervals, one applies this rule to those of which the mean is closest to half the sum of the extreme values. The author states that this conclusion is generally capable of setting in accord the practitioners who allow themselves to reject measurements of which they esteem the results too divergent and those whom the arbitrary character of this estimation prevents from rejecting any observation whatever carried out under satisfactory conditions.

**References:** Boudin (1865) [1916], Goedseels (1909), Mansion (1913), Goedseels (1925).

* Alliaume, M. (1927b). Sur la méthode la plus approximative de la théorie des erreurs. *Annales de la Société Scientifique de Bruxelles* (A) **47** (1), 60-68. (JFM 53, 505).

**Summary:** After n indirect measurements affecting simultaneously two unknowns, x and y, the equations of observation, complete with residues, are of the linear form (or reduce to the linear form) $a_i x + b_i y - m_i = k_i r_i$ $(i = 1,2, \ldots, n)$, with the conditions $-R \leq r_i \leq +R$, for the approximation factor R. The

numerical value of the factor R forms part of the data of the problem posed by the practitioner. It is limited only by the condition of being at least equal to a quantity $R_0$, the minimum approximation factor, calculable from the numbers $a_i$, $b_i$, $m_i$, $k_i$. The substitution $R = R_0$ defines the method of minimum approximation. It can happen, on the contrary, that the value of R proposed by the practitioner surpasses $R_0$ sufficiently so that certain simplifications become possible in the application of the most approximative method. The author defines two numbers $R_1$ and $R_2$ such that it suffices that the approximation factor adopted is at least equal to the smaller of the two. He does this first for the special case in which, the final unknowns being u and v, the equations of observation are $u + iv - m_i = 0$ $(i = 1, 2, \ldots, n)$. Then $R_1$ is the smallest value of the approximation factor for which there exist values of u which render decreasing (or stationary) the lower and upper bounds of v obtained by solving the inequalities $-M_i \leqslant u + iv - m_i \leqslant M_i$, where $M_i = (\mu + \nu i)R$ and $\mu$ and $\nu$ are positive coefficients for which it is natural that the practitioner should adopt the absolute values of the roots of the equations of observation. Similarly, $R_2$ is the smallest value of the approximation factor for which there exist values of u which render those bounds increasing (or stationary). The calculation of $R_0$ is long and tedious. If, in order to avoid this calculation, one wishes to admit a larger approximation, this can be done by stopping at either $R_1$ or $R_2$, to which correspond expressions for the final values of the unknowns which are very simple, and which are found, once for all, in literal form. Lastly, the author shows how to proceed from the special case to the general case.

* Burgess, Robert Wilbur (1927). *Introduction to the Mathematics of Statistics.* Houghton Mifflin Company, Boston-New York-Chicago-Dallas-San Francisco; The Riverside Press, Cambridge. (JFM 53, 510—listing only). Review, *Bulletin of the American Mathematical Society* **34** (1928), 372-373.

**Summary:** Chapter V (pp. 80-107) deals with averages and their use. The author defines the arithmetic mean (with equal or unequal weights), the median, the mode, the harmonic mean, and the geometric mean and shows how to compute each of them from grouped and ungrouped data. He notes that, for a certain type of frequency curves [the Pearson system, especially the Pearson Type III], it is very nearly true that 3(Mean − Median) = Mean − Mode. In comparing the various averages, he considers the ease of computation and manipulation and the effect of extreme observations. He notes that the median is usually the easiest to compute, but that the arithmetic mean is much better than the median or the mode as a basis for algebraic discussion. On page 101 he states:"If a few unusually large or small items are included, as often happens in wage or salary distributions, the median may be desirable as a substitute for or in addition to the arithmetic mean." Chapter VII (pp. 135-154) deals with measures of dispersion. The author defines the range R, the quartile deviation Q, the mean deviation (or average deviation) A, the standard deviation $\sigma$, and the coefficient of variation and shows how to compute them for grouped and ungrouped data. On page 135 he states: "One of the commonest of these measures is the range, the interval from the smallest to the largest item in the distribution. This measure is in common use, though not favored scientifically." The quartile deviation of N observations is defined as half the difference between the first quartile and the third quartile, which are defined, by analogy with the median [the $(N+1)/2$ ordered value], as the $(N+1)/4$ and $3(N+1)/4$ ordered values. If $N+1$ is not an integral multiple of 4, interpolation between adjacent ordered values is required in computing the quartiles. Other quantiles (tertiles, deciles, and percentiles) are defined analogously. On pages 138-139 the author states: "An important property of the mean deviation is that *it is a minimum about the median*; this will be proved in a later section [page 143]. In spite of this property, the mean deviation is more often taken about the mean than about the median." On page 147, he gives a table of the ratios $A/\sigma$, $Q/\sigma$ and $Q/A$ for various theoretical and observed distributions, from which he concludes (page 152): "As a general thing, the mean deviation is about four fifths, and the quartile deviation about two thirds, of the standard deviation."

**Citations:** Eells (1930), Shewhart (1931), Yang (1933).

* Estienne, J. E. (1927). Exposé d'une théorie rationnelle des erreurs d'observation. *Bulletin de la Société Mathématique de France* **55** (2), 24-25 (Comptes Rendus des Séances); discussion, 25-26. (JFM 53, 508).

**Summary:** The following summary of the author's communication is given on pp. 24-25: "General Estienne criticizes first the principles on which the theory of errors rests. He denies the value of the notion of the law of error on which this theory rests; according to him, the probability of such and such value of the error is not something well defined. He denies the value of the arithmetic mean of Gauss; he denies even that one can deduce by any method from a series of measurements a value more precise than that which can result from a single measurement. He proposes a theory based on the notion of the conscientious observer who has no more reason to be mistaken in one direction than the other. When a series of measurements has been accomplished, if they are not equally precise, he takes account of only the most precise. If they are equally precise, he chooses the median value, such that, among the values found, there will be the same number of values larger than the median as of values smaller. He indicates several simple properties of the median value which appear to him to justify the role which it plays in this notion; he insists especially that the value chosen does not change if one changes the independent variable, contrary to what happens in the classical theory. But he does not think, however, that in any case one can gain a decimal by adopting the median value. A complete exposition of the ideas of General Estienne has been published [Estienne (1926-27)]. . . ." (Compiler's translation). Paul Lévy, one of five persons contributing to the discussion, declares that it is incontestable that one can increase the precision by the repetition of measurements provided that the error is not systematic. Although he does not agree exactly with the set of ideas expounded by Gen. Estienne, he does not believe that his conclusions are devoid of practical value. He states that in numerous cases, theory and experience confirm the law of Gauss, in which case absolutely rigorous reasoning shows that the arithmetic mean of equally precise measurements is the best value to take, the standard error of the median being 25% greater than that of the mean. When the law of Gauss does not apply, the median may be more precise than the mean. But it is better to discard a definite proportion of the extreme measurements and to take the mean of those retained. This method includes both the classical method and that of Gen. Estienne as special cases.

**Reference:** Estienne (1926-27).

* Fréchet, Maurice (1927). Sur la loi de probabilité de l'écart maximum. *Rocznik Polskie Towarzystwo Matematiczne* [*Annales de la Société Polonaise de Mathématique* (Cracow)] **6,** 93-116. (JFM 54, 562).

**Summary:** The author gives a new exposition of the theory of errors based on the probability law of the maximum deviation instead of that of the sum of a number of errors as in the exposition of Lévy (1925). In the author's theory, the role of the integral of Laplace, $(1/\sqrt{\pi}) \int e^{-x^2} dx$, is taken, in the reduced form of the cumulative distribution function, by the function $2^{-x^{-\alpha}} (\alpha > 0)$. The expression for the probability density function equal in the first case to $(1/\sqrt{\pi}) e^{-x^2} dX$ is here $\alpha X^{-\alpha-1} 2^{-x^{-\alpha}} \log 2 \cdot dX = \alpha (\log 2)\, dX / X^{\alpha+1} 2^{1/X^{\alpha}}$. One notices that this probability is zero for $X = 0$ and maximum for a value of $X \neq 0$. The author remarks that his exposition distinguishes very nicely the respective roles of two conditions studied by Lévy. The condition of a normal family has the effect of creating a class of cumulative distribution functions equally near at infinity to that of Gauss and containing the law of probability of error of the sum of a finite or infinite number of partial errors. It is the second condition: $M_n/B_n$ small [where $M_n = \max (A_1, A_2 \ldots, A_n)$ and $B_n^{\alpha} = A_1^{\alpha} + A_2^{\alpha} + \ldots + A_n^{\alpha}$, $A_i$ $(i = 1, 2, \ldots, n)$ being independent elementary errors] because of which the resultant error of n errors follows approximately the law of Gauss, not only at infinity, but everywhere.

**Comments:** The following quotation from Gumbel [(1958a), p. 3] attests to the importance of this paper: "The first paper based on the concept of a type of initial distributions different from the normal one is due to M. Fréchet (1927). He was also the first to obtain an asymptotic distribution of the largest value. More important, he showed that largest values taken from different initial distributions sharing a common property may have a common asymptotic distribution. He introduced the stability postulate according to which the distribution of the largest value should be equal to the initial one, except for a linear transformation. Fréchet's paper, published in a remote journal, never gained the recognition it merited.

This was due to the fact that the type of initial distributions considered is not very frequent."

**Reference:** Lévy (1925).

**Citations:** Lévy (1929), Gumbel (1935a,b, 1935-36, 1939f), Gnedenko (1941), Gumbel (1941a), Gnedenko (1943), Gumbel (1946b), Wilks (1948), Juncosa (1949).

* Friedrich, Konrad (1927). Ueber Punktgenauigkeit. *Zeitschrift für Vermessungswesen* **56,** 33-41, 65-79.

**Summary:** The author recalls that, as is known, the theorem of Schreiber (1882) holds not only for a single function, e.g. the side of a triangle, but also for the mean error of a point, which depends on two functions. He states the following extension of Schreiber's theorem: For a given position of a new point in a network of triangles, one obtains the most favorable weight distribution for the estimate of the point by making the error ellipse a circle by using the smallest number of excess observed quantities (angles, lengths), or, in case this is not possible, without such excess measurements. He gives several applications of the extended Schreiber theorem.

**Citation:** Friedrich (1937).

* Galvani, Luigi (1927). Dei limiti a cui tendono alcune medie. *Bollettino della Unione Matematica Italiana* **6** (4), 173-179.

**Summary:** The author recalls that Dunkel (1909) generalized the geometric mean. Let $y_n(x) = [(a_1^x + a_2^x + \ldots + a_n^x)/n]^{1/x}$, where $a_i > 0$ $(i = 1, 2, \ldots, n)$ and x is real. For $x = -1$, $x = 1$ and $x = 2$, $y_n(x)$ is the harmonic mean, the arithmetic mean, and the quadratic mean, respectively. Dunkel showed that $\lim_{x \to 0} y_n(x) = \sqrt[n]{a_1 a_2 \ldots a_n}$ (the geometric mean of the $a_i$). He also showed that $\lim_{x \to -\infty} y_n(x) = \min(a_1, a_2, \ldots, a_n)$ and $\lim_{n \to \infty} y_n(x) = \max(a_1, a_2, \ldots, a_n)$. Thus, if $a_1 \leq a_2 \leq \ldots \leq a_n$ is an ordered sample from some distribution (known or unknown), $\lim_{x \to -\infty} y_n(x) = a_1$ (the first order statistic) and $\lim_{x \to \infty} y_n(x) = a_n$ (the $n^{th}$ order statistic). The author gives further generalizations, but they are not relevant to a study of order statistics.

**Reference:** Dunkel (1909).

**Citations:** Galvani (1931), Cisbani (1938), Dodd (1940), Zappa (1940), Boldrini (1942).

* Goodrich, R. D. (1927). Straight line plotting of skew frequency data. *Transactions of the American Society of Civil Engineers* **91,** 1-43; discussion, 44-118.

**Summary:** In his synopsis (pp.1-2), the author states: " . . . This paper deals with the application of new equations for both the 'Integral' or 'Duration' Curve and the 'Frequency' or 'Differential' Curve; and for expressing the variations in the occurrence of phenomena which follow 'Skew Frequency Laws' as to their distribution. These equations are particularly applicable to the plotting of hydraulic data (rainfall and run-off) having a limited number of observations in the series. . . . Numerous examples of the use of these equations in the plotting of curves on a new type of skew frequency paper are shown. Ordinary and logarithmic coordinate paper may also be used. These curves cover a greater variety of conditions than have heretofore been presented. Curves are plotted with equal facility when either or both the coefficient of variation and the coefficient of skew are negative as well as when the mode is greater than the mean. The same plotting paper is used when either one or both of the limits are finite. The equations can also be applied to the representation of certain population curves." On pp. 10-11 the author discusses factors for classifying frequency curves, including the quartile deviation $Q = (Q_3 - Q_1)/2$ and the quartile ratio $(Q_3 + Q_1 - 2Q_2)/Q$ [where $Q_1$, $Q_2$, and $Q_3$ are the quartiles ($Q_2$ = median)], both of which were discussed by Yule (1911) [1917, p. 147]. On page 14, he discusses plotting positions on probability paper. For the $i^{th}$ order statistic of a sample of size n, he proposes $(2i - 1)/2n$; in a footnote, he mentions that if one uses $i/n$, it is impossible to plot the last point. The discussion includes contributions by several well-known

hydrologists and statisticians, including H. Alden Foster, W. A. Shewhart, L. Standish Hall, Arne Fisher, Allen Hazen, and the author. Shewhart (p. 53) discusses the precision of the median and the quartile deviation relative to that of the mean and the standard deviation, respectively, for a normal distribution. There is frequent mention, both in the paper itself and in the discussion, of the normal curve, with references to the work of Adrain (1808), Gauss (1809), and Hagen (1837), and of skew frequency curves, with references to the work of Pearson (1895) and Kapteyn (1903) [Kapteyn & van Uven (1916)].

**References:** Adrain (1808), Gauss (1809), Hagen (1837), Pearson (1895), Kapteyn (1903) [Kapteyn & van Uven (1916)], Yule (1911) [1917], Fuller (1914), Hazen (1914), Brunt (1917), Pearson (1920), Hall (1921), Foster (1924).

**Citations:** Hazen (1930), Foster (1934), Slade (1934), Foster (1936), Saville (1936), Beard (1942), Lane & Lei (1949).

* Gumbel, E. J. (1927). Ein Mass der Konzentration bei pekuniären Verteilungen. *Archiv für Sozialwissenschaft und Sozialpolitik* **58,** 113-139.

**Summary:** The author defines a measure of concentration for pecuniary distributions (e.g., income and wealth). The independent variable is called x and $\phi(x)dx$ is the relative frequency in the interval x to x + dx, $c_1$ and $c_2$ being the lower and upper limits of the distribution of x. If the variable of integration is denoted by z, the area of the distribution is $\mu_0 = \int_{c_1}^{c_2} \phi(z)dz = 1$ (1). The first moment about the origin, in practice denoted as the arithmetic mean, is defined by $\mu_1 = \bar{x} = \int_{c_1}^{c_2} z\phi(z)dz$. The relative portion of the variable up to x, also denoted as the incomplete first moment, amounts to $u(x) = \int_{c_1}^{x} z\phi(z)dz/\bar{x}$ (2). We have then $0 \leq u \leq 1$, if $c_1 \leq x \leq c_2$. The proportion of the population up to the value x of the variable amounts to $v(x) = \int_{c_1}^{x} \phi(z)dz$ (3), where again $0 \leq v \leq 1$, if $c_1 \leq x \leq c_2$. Now in order to determine how great a part of the population corresponds to a given part of the first moment, we eliminate x from (2) and (3). The equation so obtained, $v = f(u)$ (4), is denoted as the concentration equation and its graphical representation as the concentration curve. The area under the concentration curve is $I = \int_0^1 v\,du$. The area between the concentration curve and the line v = u, the optimum concentration curve corresponding to equal income (or wealth), is I − 1/2. The author defines the measure of concentration $\zeta = (I - 1/2)/(1/2) = 2I - 1$, so that $\zeta = 0$ corresponds to the optimum and $\zeta = 1$ to the infinitely concentrated distribution. He finds the value of $\zeta$ for the special and general Pareto distributions, and applies the results to distributions of income in Austria, Prussia and Japan. He also applies his measure of concentration to the distribution of energy of gas molecules. In a footnote, he remarks that the investigations of Amoroso, Gini, Lorenz and Watkins first became known to him during the printing of this paper.

**Reference:** Winkler (1924).

**Citations:** Gumbel (1928), von Bortkiewicz (1931), Gini (1932), Pietra (1932a), Castellano (1933a).

* Jordan, Charles (1927). *Statistique Mathématique.* Gauthier-Villars, Paris. (JFM 57, 509—listing only). Review, *Bulletin of the American Mathematical Society* **34** (1928), 375-377.

**Summary:** Chapter VI (pp. 150-179) deals with various means, including the arithmetic, geometric and harmonic means (A, G and H, respectively), the median M (and other quantiles—specifically the quartiles, deciles, and centiles) and the mode $M_0$. There is also a discussion of measures of dispersion (standard deviation $\sigma$, mean deviation $\chi$, and quartile deviation $\rho$), of asymmetry [the quartile measure $S = (Q_1 + Q_3 - 2M)/(Q_3 - Q_1)$, Pearson's measure $S_1 = (A - M_0)/\sigma \doteq 3(A - M)/\sigma$ and $\beta_1 = \alpha_3^2$, where $\alpha_3$ is the standard third moment] and of kurtosis ($\beta_2 = \alpha_4$, the standard fourth moment). The author defines the median of an even number of values as any value at all between the two middle ones when arranged in order of magnitude, but notes that, to give it a definite value, it is often taken to be the value midway

between them. He also gives the new definition proposed by Jackson (1921). He states that the median, whose introduction he attributes (incorrectly) to Cournot (1843), is one of the most important constants of a series of observations, being especially useful when the individual values are difficult to measure, but easy to arrange in order of magnitude. He notes that the "mean man" of Quetelet is actually the median man, that the median can be calculated for grouped data with open-ended intervals at the extremes, and that it is not influenced by extreme observations, as is the arithmetic mean. If the median is large, one knows that at least half the observations are large, but a large mean may be caused solely by one or a few very large observations. The median, however, has the defect that if one knows the medians of two sets of observations, one cannot find from them the median of the combined sets. The author gives two numerical examples of the use of the median. He shows that the sum of the absolute deviations of a set of observations is less when taken from the median than from any other value. The author's definitions of other quantiles are the standard ones; he mentions also the definitions of Jackson (1923) analogous to his definition of the median. He notes the empirical relation $A - M_0 \doteq 3(A - M)$ among mean, median and mode first observed by Karl Pearson (1895). He defines the mean deviation $\chi$ in terms of deviations from the median, thus making it a minimum, and the standard deviation $\sigma$ and the quartile deviation $\rho$ in the usual manner. As measures of relative dispersion, he mentions $2\rho/(Q_1+Q_3)$ and $\chi/M$, as well as the usual coefficient of variation $\sigma/A$.

**References:** Quetelet (1835) [1869], Cournot (1843), Bowley (1901) [1920], Czuber (1921), Jackson (1921, 1923).

**Citations:** Gibrat (1930), d'Addario (1931), Shewhart (1931), Barral Souto (1938), Kendall (1943, 1946).

* "Student" [W. S. Gosset] (1927). Errors of routine analysis. *Biometrika* **19,** 151-164. Reprinted 1942 in *Student's Collected Papers* (ed. E. S. Pearson and John Wishart), pp. 135-149. Cambridge University Press, Cambridge, England.

**Summary:** Using the results of Tippett (1925) and E. S. Pearson (1926), the author finds the equations of curves fitted to the distributions of ranges in samples of size n=2(1)6,10,20,60 from a standard normal population. As an application he proposes (pp. 161-162) the following rule for the treatment of discordant observations, a use suggested by Irwin (1925b): "Let $W_n$ be the limit at which with a sample of size n, the chance of obtaining a greater range than $W_n$ is p (say .05), then if $w_n$ the actual range of a sample be greater than $W_n$ repetition should be made. Let $w_{n+1}$ be the range of the new sample including the repetition, then if $w_{n+1} < W_{n+1}$ the mean of the n+1 results should be accepted. If, on the other hand, $w_{n+1} > W_{n+1}$ the most outlying observation should be rejected, and if then the resulting $w_n < W_n$ the mean of these n should be accepted, but if not, a further repetition should be made and the whole n+2 observations examined afresh, and so on until a sample of at least n is obtained lying within the required limits." An example of the use of this rule is given.

**References:** Irwin (1925b), Tippett (1925), E. S. Pearson (1926).

**Citations:** Pearson & Adyanthāya (1928), Sophister (1928), K. Pearson (1931), E. S. Pearson (1932), Pearson & Haines (1935), Pearson & Chandra Sekar (1936), Newman (1939), Pearson (1939), Simon (1941), Knudsen (1943), Pearson & Hartley (1943), Kendall (1946), Gumbel (1947), Nair (1948b), Cox (1949).

* Tucker, John, Jr. (1927). The compressive strength dispersion of materials, with applications. *Journal of the Franklin Institute* **204,** 751-781.

**Summary:** In connection with a study of the dispersion of the compressive strength of materials (columns of Portland cement mortar and concrete), the author considers various measures of dispersion (absolute and relative), including the mean difference, which he attributes to [von] Andrae [(1869,

1872)]. He also includes a section on the variation in strength with length of columns. He presents the "weakest link" theory, according to which the strength of a column n units long is the strength of the weakest of the n individual units, so that the average strength decreases with the length. He points out, however, that this theory applies only when the unit length is chosen so that failure of a specimen of unit length will occur in exactly the same manner as the failure of the column. Since failure of a short concrete column normally occurs by skew in a plane inclined at approximately 35° to the column axis, the theory is invalidated if unit length/diameter $< k \doteq 1.5$.

**References:** [von Andrae (1869, 1872)].

**Citations:** Tucker (1941, 1945a,b), Frankel (1948).

* von Verschuer, Otmar (1927). Die vererbungsbiologische Zwillingsforschung. *Ergebnisse der inneren Medizin und Kinderheilkunde* **31,** 35-120.

**Summary:** In a comparative study of differences in various measures between identical and between fraternal twins, the author makes routine use (pp. 93-95) of various measures of variability, including the mean difference.

**Citation:** Lenz & von Verschuer (1928).

* Bartels, J. (1928). Die Schwankungsweite als Funktion der Beobachtungsdauer. *Meteorologische Zeitschrift* **45,** 489-490.

**Summary:** The author notes that Köppen (1928) has considered the question of how the difference between the highest and lowest yearly totals of precipitation at a station (the range) depends on the length of the period of observation, and states that Köppen's conjecture that the desired functional relation must result from general statistical considerations is correct, as can be shown by extension of an observation of Helmert (1872) [1907, p. 364] concerning greatest errors. Let n be the number of observations and $\mu$ the standard deviation of a series of observations obeying the Gaussian law of error. The most probable number of observations that deviate in absolute value from the mean by more than b is, with $\Theta(t)=(2/\sqrt{\pi})\int_0^t e^{-t^2}\,dt$, given by $n[1-\Theta(b/\mu\sqrt{2})]$. The lower limit a of the interval in which the largest absolute deviation is to be expected to lie will thus, with $\alpha=a/\mu$, be given by $1-\Theta(\alpha/\sqrt{2})=1/n$. This equation can be solved for $\alpha$ with the aid of tables for $\Theta(t)$. The expected value of the greatest deviation is $g=(2/\sqrt{\pi})\int_{\alpha/\sqrt{2}}^{\infty} t\mu\sqrt{2}\,e^{-t^2}\,dt/[1-\Theta(\alpha/\sqrt{2})]=\mu\sqrt{2/\pi}\,e^{-\alpha^2/2}/(1/n)$, so that $\gamma=g/\mu=n\sqrt{2/\pi}\,e^{-\alpha^2/2}$. The range S will be defined as the difference of the highest and lowest observations. Out of n observations with standard deviation $\mu$, $n(n-1)/2$ differences can be formed, of which the mean square is equal to $2\mu^2$. Thus one obtains $\sigma=S/\mu$ by multiplying by $\sqrt{2}$ the value of $\gamma$ corresponding to $n(n-1)/2$. An estimate $\sigma^*$ for $\sigma$ is constructed from the following plausible but not rigorous consideration. By symmetry the two absolute largest errors are equally large, one positive and the other negative; $\sigma^*$ for n observations is therefore double the value of $\gamma$ for n/2 observations. Naturally always $\sigma<2\gamma$; for large n, $1/n=1-\Theta(\alpha/\sqrt{2})=\sqrt{2/\pi}\,e^{-\alpha^2/2}$ holds asymptotically, so $\gamma/\alpha\to 1$. The $i^{th}$ largest observation is greater than $\beta_i\,\mu$, if $1-\Theta(\beta_i/\sqrt{2})=i/n$; $\beta_i$ is thus equal to the $\alpha$ corresponding to n/i. Since the proportional increase of $\alpha$ in passing from n/i to n, with fixed i, tends to 0, it follows that asymptotically $(\alpha+\beta_i)/\alpha=2$; consequently, since $\sigma=\alpha+\beta_i$ and $\alpha\to\gamma$, we have also asymptotically $\sigma=2\gamma$, as is already discernible, from a table given by the author, for large values of n. The author compares his theoretical values with Köppen's empirical ones.

**References:** Jordan (1869), Helmert (1872) [1907], Köppen (1928).

**Citation:** Birkeland & Frogner (1935).

* Birkeland, B. J. (1928). Ältere meteorologische Beobachtungen in Bergen: Luftdruck und Temperatur seit 100 Jahren. *Geofysiske Publikasjoner* (Oslo) **5** (8), 1-56.

**Summary:** This publication does not appear to contain anything of theoretical interest. It does contain extensive records of pressure and temperature in Bergen, Norway over a period of more than 100 years, including monthly maximum and minimum temperatures and pressures (pp. 30-31, 39-47) and maximum and minimum monthly averages of temperature and pressure for each calendar month over a period of years (pp. 34, 36); also mean maxima and minima, absolute maxima and minima, etc.

**Citation:** Barricelli (1943).

* Bowley, A. L. (1928). *F. Y. Edgeworth's Contributions to Mathematical Statistics*. Royal Statistical Society, London. (JFM 54, 573).

**Summary:** This work contains a summary and an annotated bibliography (74 items) of Edgeworth's contributions, including important work on the law of error and the choice of means, with special emphasis on the median. In the latter connection, the author writes (pp. 101-102): "The median has the disadvantages that its standard deviation of error (1 divided by $2\sqrt{n}$ times the greatest ordinate where the area of the frequency curve is unity) is greater by 25% than that of the arithmetic mean in the normal curve. This excess is not, however, serious in the rough measurements of credibility with which we are generally concerned in statistics, and in some non-normal curves the median is more accurate than the arithmetic mean. . . . It has the well-known advantage that it can be computed when the measurements away from the centre are known only roughly, and generally in graded observations interpolation is only needed in the central parts . . . . If no graduation is necessary the median is evidently the easiest mean to compute, and if the maximum ordinate is known its probable error can at once be written down. As is well known, the median is that position which makes the sum of the deviations from it (all taken as positive) a minimum, the test of least detriment suggested by Laplace. . . . Finally, if a mean is required of discordant observations, where discordance signifies that the observations are taken from facility curves with different moduli, there is 'a peculiar propriety in the use of the Median' . . . . For these reasons Edgeworth attached great importance to the median in a considerable number of problems. His advocacy extends to its use for two or more unknowns . . . ." The last statement is exemplified by seven pages (pp. 103-109) on the method of situation of Laplace (1812) [1818] and its extension to two or more unknowns by Edgeworth (1887b,c, 1888, 1923).

**References:** Laplace (1812), Venn (1866), Edgeworth (1883a,b, 1885, 1886a,b, 1887a,b,c,d,e,f, 1887-90, 1888, 1893, 1898, 1905, 1911), Yule (1911), Edgeworth (1913), Keynes (1921), Edgeworth (1923).

**Citations:** Rhodes (1930), Bruen (1938), Kendall (1943, 1946).

* Cramér, Harald (1928). On the composition of elementary errors. *Skandinavisk Aktuarietidskrift* **1-1**, 13-74, 141-180. (JFM 54, 557).

**Summary:** This paper is divided into two parts, the first on mathematical deductions and the second on statistical applications. In the second part the author introduces what is now called the Cramér-von Mises statistic. He writes (pp. 142-145): "It has been proved in I that, if our variable can be regarded as the sum of a large number of mutually independent components, we have under certain further conditions an asymptotic expansion (1) $F(x) = \Phi(x) + P_1(\Phi) + P_2(\Phi) + \ldots$, where $\Phi(x) = (1/\sqrt{2\pi}) \int_{-\infty}^{x} e^{-t^2}$ dt while . . . the $P_i$ are linear expressions in the derivatives of $\Phi(x)$: (2) $P_1(\Phi) = (1/3)S\Phi^{(3)}(x)$, $P_2(\Phi) = (1/3)$ $E\,\Phi^{(4)}(x) + (1/18)\,S_2\,\Phi^{(6)}(x)$, $P_3(\Phi) = T\Phi^{(5)}(x) + (1/9)E\,S\Phi^{(7)}(x) + (1/162)S^3\Phi^{(9)}(x), \ldots$ [where $S = -\mu_3/2\sigma^3$, $E = (1/8)\,(\mu_4/\sigma^4 - 3)$, and $T = -(1/120)\,(\mu_5/\sigma^5 - 10\mu_3/\sigma^3)$, $\mu_i$ being the $i^{th}$ central moment and $\sigma$ the standard

deviation]. . . . Let us first suppose that we know the 'true values' of m [the mean], $\sigma, \mu_3, \mu_4, \ldots$ and that, in a series of n mutually independent observations, we have found the values $x_1, x_2, \ldots x_n$ of the variable X. Let us put $\nu$ = the number of those $x_i$ which are $< m + \sigma x$, where x is a given quantity such that no $x_i$ is exactly equal to $m + \sigma x$. . . . Hence if n is large and if we mark, for a set of values of x, the corresponding points $y = \nu/n$ in a diagram, these points will give us an idea of the shape of the curve $y = F(x)$, which rises from 0 to 1 as x increases from $-\infty$ to $+\infty$. In the case of a statistical variable generated by elementary components, this curve will more or less approach to the normal curve $y = \Phi(x)$ and we are particularly interested in any systematic deviations shown from this form. These deviations are often very small compared with the whole range of the ordinates of the curves, and so an inconveniently large figure is required if we want to make a detailed examination of the deviations. In order to simplify this examination, we form the differences $\triangle_1(x) = \nu/n - \Phi(x)$ and mark the corresponding points $y = \triangle_1(x)$ in our diagram where, now, the scale of the ordinates may be taken much larger without causing any inconvenient dimensions of the figure. If the expansion (1) holds, the curve $y = P_1(\Phi)$ ought to show a tolerably good fit to the points $y = \triangle_1(x)$. A further test on the expansion is supplied by the differences $\triangle_2(x) = \triangle_1(x) - P_1(\Phi) = \nu/n - \Phi(x) - P_1(\Phi)$ and the corresponding points $y = \triangle_2(x)$ which ought to agree with the curve $y = P_2(\Phi)$. —If there are still systematic deviations, we may obviously continue in the same way and form the differences $\triangle_3(x) = \triangle_2(x) - P_2(\Phi) = \nu/n - \Phi(x) - P_1(\Phi) - P_2(\Phi)$ that may be compared with the curve $y = P_3(\Phi)$, and so on. If the expansion (1) holds, the fit to the observed data must be improved as we pass from the normal curve to the curves $y = \Phi + P_1$, $y = \Phi + P_1 + P_2, \ldots$, and so the differences $\triangle_1(x), \triangle_2(x), \ldots$, will, on the average, form a decreasing sequence. As a standard for judging the values of the differences found in an actual case we shall choose the absolutely convergent integrals $J_i = \int_{-\infty}^{\infty} [\triangle_i(x)]^2 \, dx$ that can be approximately calculated if the values of x for which $\triangle_i(x)$ is known lie sufficiently close together. . . ." Expressions are given on page 147 for the mathematical expectations of $J_i$ ($i = 1,2,3$).

**References:** Kapteyn (1903) [Kapteyn & van Uven (1916)], Edgeworth (1905), Charlier (1906, 1910 [1920]), Lévy (1925).

**Citations:** Lévy (1929), Steffensen (1930), Wold (1935), Neyman (1937a), Gumbel (1942a), Kendall (1943), Cramér (1946), Kendall (1946), von Mises (1947).

* Dufton, A. F. (1928). Correlation. *Nature* **121,** 866.

**Summary:** The author states: "For the determination of a linear function of X approximating to Y for a range of corresponding values (X,Y), a graphic method is desirable. If the plotted points are divided into two classes by the median of X, in each case, the same number of points should lie on each side of the required line, which is thus found by inspection. The method is much more expeditious than that of least squares. It appears, moreover, to be free from undue bias by outlying values. For the values plotted in Fig. 1 [the points (1,7), (2,6), (3,7), (3,9), (5,4), (5,5), (6,6), (7,2), (8,3), (8,5), (9,3), (10,2), (10,4) and (11,1)], the straight line is $Y = (40 - 3X)/5$, with a mean deviation of 1.00. The method of least squares gives $Y = (1940 - 137X)/236$, with a mean deviation of 1.05."

**Comments:** The compiler of this bibliography has recently shown by counterexamples that the condition proposed by Dufton (that, on each side of the median of X, there be an equal number of points on each side of the line) is neither necessary nor sufficient to minimize the mean deviation, as inferred by Bruen (1938), who calls Dufton's method "the method of . . . least deviations sums."

**Citation:** Bruen (1938).

* Fisher, R. A.; Tippett, L. H. C. (1928). Limiting forms of the frequency distribution of the largest or smallest member of a sample. *Proceedings of the Cambridge Philosophical Society* **24,** 180-190. (JFM 54, 560).

**Summary:** "The limiting distribution, when n is large, of the greatest or least of a sample of n, must satisfy a functional equation which limits its form to one of two main types. Of these one has, apart from size and position, a single parameter h, while the other is the limit to which it tends when h tends to zero. The appropriate limiting distribution in any case may be found from the manner in which the probability of exceeding any value x tends to zero as x is increased. For the normal distribution the limiting distribution has h=0. From the normal distribution the limiting distribution is approached with extreme slowness; the final series of forms passed through as the ultimate form is approached may be represented by the series of limiting distributions in which h tends to zero in a definite manner as n increases to infinity. Numerical values are given for the comparison of the actual with the penultimate distributions for samples of 60 to 1000, and of the penultimate with the ultimate distributions for larger samples." (Authors' summary, p. 190).

**Reference:** Tippett (1925).

**Citations:** Gumbel (1933c), de Finetti (1934), Gumbel (1934b,g, 1935a,d, 1935-36), von Mises (1936), Gumbel (1937a,b, 1939f)), Cochran (1941), Daniels (1941), Gnedenko (1941), Gumbel (1941a, 1942c), Kimball (1942), Gnedenko (1943), Gumbel (1943a,c), Kendall (1943), Gumbel (1944, 1945a), Cramér (1946), Gumbel (1946b), Kendall (1946), Kimball (1946a), Mosteller (1946), Elfving (1947), Gumbel (1947), Gurney & Pearson (1947), Cox (1948), Epstein (1948a,b), Epstein & Brooks (1948), Wilks (1948), Gumbel (1949), Juncosa (1949), Langbein (1949), Press (1949).

* Gumbel, E. J. (1928). Das Konzentrationsmass. *Allgemeines Statistisches Archiv* **18,** 279-300.

**Summary:** In Section 1 of this paper, the author summarizes and extends the results on the concentration curve and the measure of concentration $\zeta$ [Gini's concentration ratio R] developed in his earlier paper [Gumbel (1927)] and much earlier, in slightly different form, by Lorenz (1905) and by Gini (1914a). In Section 2, he shows that $\zeta$ is equal to the mean difference divided by twice the mean ($\bar{x}$). For a monotone decreasing distribution (p.d.f.) between the limits $c_1$ and $c_2$, he shows that $\zeta > 1/3 - c_1/3\bar{x}$; for distributions beginning at $c_1 = 0$, this inequality reduces to $\zeta > 1/3$. For a monotone increasing distribution (p.d.f.) starting with the value 0 at $c_1 = 0$, he shows that $\zeta > (c_2 - \bar{x})/3\bar{x}$. Following Gini, he also develops an addition theorem for incompletely known distributions (for which either the beginning or the end is not known). In Section 3, he computes $\zeta$ for the linear distribution with p.d.f. $\phi(x) = a - bx$, which is trapezoidal except in special cases, in which it may be right triangular ($c_2 = a/b$ or $a = 0, b < 0, c_1 = 0$) or even uniform ($b = 0$). In Section 4, he computes $\zeta$ for the exponential, half-normal and Maxwell distributions, and then proceeds to compute the mean of the absolute differences of the velocities of gas molecules, assuming that the velocity follows a Maxwell distribution. In Section 5, he summarizes his results, noting that when $c_1 = 0$, $\zeta$ is 1/3 for the uniform distribution, 0.2 for the linear increasing distribution ($a = 0, b < 0$), 0.4 for the linear decreasing distribution ($c_2 = a/b$), $\sqrt{2} - 1 = 0.41421$ for the half-normal distribution, and 0.5 for the exponential distribution.

**References:** Lorenz (1905), Gini (1912, 1914a), Amoroso (1925), Gumbel (1927).

**Citations:** von Bortkiewicz (1931), Gini (1932), Castellano (1933a), Gini (1936), Pizzetti (1941a).

* Köppen, W. (1928). Die Schwankungen der jährlichen Regenmenge. *Meteorologische Zeitschrift* **45,** 281-291.

**Summary:** The author investigates the influence of the length of the observation period (in years) and the average annual rainfall on the range of a series of observations on annual rainfall, and obtains empirical relations based on extensive data. As an example, he finds that the range of the annual rainfall in Great Britain increases by about 10% of the average annual rainfall whenever the period of observation is doubled. He indicates the need for a theoretic study of the relation between the range and the number of observations.

**Citations:** Bartels (1928), Birkeland & Frogner (1935).

* Lenz, F.; von Verschuer, O. (1928). Zur Bestimmung des Anteils von Erbanlage und Umwelt an der Variabilität. *Archiv für Rassen- und Gesellschafts-Biologie* **20** (4), 425-428.

**Summary:** The authors give the following formula for the mean difference d for grouped data: $d = 2x[p_1(p_2 + 2p_3 + 3p_4 + \ldots + [k-1]p_k) + p_2(p_3 + 2p_4 + \ldots + [k-2]p_k) + \ldots + p_{k-1} \cdot p_k] / n \cdot (n-1)$, where $p_1$ = number of observations in $i^{th}$ class ($i = 1, 2, \ldots, k$), k = number of classes, x = difference between successive class marks, and n = total number of observations $[\Sigma_{i=1}^{k} p_i]$. They state that for large samples from a binomial distribution, the relation between d and the mean deviation e (from the arithmetic mean) is given by $e/d = 0.7$.

**Reference:** von Verschuer (1927).

**Citations:** Gottschick (1939), Pizzetti (1941a).

* Neyman, J.; Pearson, E. S. (1928). On the use and interpretation of certain test criteria for purposes of statistical inference. Parts I, II. *Biometrika* **20A,** 175-240, 263-294. (JFM 54, 565-566).

**Summary:** Test criteria for samples from normal, rectangular, and exponential populations are considered in Part I. The greatest attention is given to the normal case (pp. 178-208), which is the one of least interest in our study. The rectangular case (pp. 208-211) is of more interest, since it is here that test criteria based on the range can be used to advantage. Consider a sample, with midrange G and range W, assumed to have come from some rectangular population. We wish to test the hypothesis that the midrange is g and the range is w, against the alternative hypothesis that the range and midrange are w' and g', respectively. If the criterion of likelihood is used, the authors state (p. 209): ". . . we need consider only the range of the sample, and our confidence in the hypothesis will increase as $W \rightarrow w$. If $W > w$ or any of the observations in the sample fall outside . . . [the interval $(g - w/2, g + w/2)$], then of course the hypothesis becomes impossible." The probability integral of the sample range W is given by $P_w = \{nw-(n-1)W\ W^{n-1}/w^n$, while that of $z' = 2(G-g)/W$ is given by $P_{z'} = (1+z')^{-n+1}$. The mean, standard deviation, $\beta_1$, and $\beta_2$ for $W/\sigma$ ($\sigma$ = standard deviation) are tabulated for samples of size n = 3,4,6,10,20 from rectangular and normal distributions. A comparison is made of tests based on rectangular and normal assumptions. Formulas are given on p. 217 for the moment constants of the range for samples from a rectangular distribution. The only part of the treatment of the exponential case (pp. 221-230) which is relevant to our study is the section on tests for random intervals. Whitworth (1867) has shown that the intervals of time between random events which happen on the average once in time c follow the one-parameter negative exponential distribution with pdf $f(t) = (1/c)e^{-t/c}$. Morant (1921) has extended this idea to the case in which there is a closed period of length b after an event before another can occur. In this case the intervals between events follow the two-parameter negative exponential distribution with pdf $f(t) = (1/c)\, e^{-(t-b)/c}$. Let $t_1, t_2, \ldots, t_v$ be the times of occurrence of $v = n+1$ successive observations of an event, and let $i_1, i_2, \ldots, i_n$ be the n intervals between occurrences. Let $i_s$ be the shortest of these intervals. On pp. 226-227 the authors show how to test the hypothesis that the intervals are random with closed period b, using the test statistic $z'' = x_1/\ell$, where $x_1 = i_s - b$ and $\ell = (i_1 + i_2 + \ldots + i_n)/n - i_s = (t_v - t_1)/n - i_s$, and its probability integral $P_{z''} = (1+z'')^{-n+1}$. Part II does not contain anything relevant to our study.

**References:** Whitworth (1867), Morant (1921), Fisher (1922), Tippett (1925), Pearson (1926).

**Citations:** Pearson & Adyanthāya (1928), Sophister (1928), Pearson & Adyanthāya (1929), Rider (1929), Pearson & Pearson (1931), Shewhart (1931), Craig (1932a), Pearson & Pearson (1932), Craig (1933), McKay & Pearson (1933), Olds (1935), Sastry (1935), Nair (1936), Pearson & Chandra Sekar (1936), Sukhatme (1936), Gumbel (1942a), Rajalakshman (1943), Wilks (1943), Geary (1944), Cramér (1946), Kendall (1946), Bickerstaff (1947), Lehmann (1947), Nair (1948a), Rao (1948).

* Ogrodnikoff, Kyrill (1928). On the occurrence of discordant observations and a new method of treating them. *Monthly Notices of the Royal Astronomical Society* **88,** 523-532. (JFM 52, 520; 55, 918).

**Summary:** The author divides methods for the treatment of discordant observations into two groups: (1) those in which it is assumed that each observer, no matter how skilled, occasionally makes observations which are affected by blunders and not merely by errors following a certain law, the aim of these methods being to detect and reject the observations resulting from blunders; (2) those in which no observation is rejected, but smaller weights are ascribed to observations which differ greatly from the mean value. The first group includes the methods of Peirce (1852), Chauvenet (1863), and Stone (1868). Objections to these methods have been raised by mathematicians [see, e.g., Bertrand (1889)] on theoretical grounds and by astronomers on practical grounds. The second group includes the methods of De Morgan (1847), Glaisher (1873), Newcomb (1886), Stewart (1920b) [1923], and the author, who summarizes his own method as follows (p. 524): "In the present paper I show that if the mean error changes during the observations, the resultant frequency curve of residuals must show an excess of large residuals as compared with what is predicted by a Gaussian law, and I introduce the notion of 'mean weights' of observations differing by the same amount from the mean value of the unknown quantity. For the determination of these weights I give a simple formula which permits their calculation without solving the integral equation which appears when we assume the observed frequency curve to be composed of Gaussian ones. The weights prove to be diminishing with increasing deviations." The author notes that Czuber (1891a) gives "a very instructive history of the question."

**References:** De Morgan (1847), Peirce (1852), Chauvenet (1863), Stone (1868), Glaisher (1873), Newcomb (1886), Bertrand (1889), Czuber (1891a), Stewart (1920b) [1923].

* Pearson, Egon S.; Adyanthāya, N. K. (1928). The distribution of frequency constants in small samples from symmetrical populations (preliminary notice). *Biometrika* **20A,** 356-360. (JFM 54, 566).

**Summary:** For samples of size n from a rectangular population with range w (standard deviation $\sigma = w/2\sqrt{3}$), formulas for the frequency constants (mean, standard deviation, $\beta_1$, and $\beta_2$) of the sample range have been given by Neyman and Pearson (1928), p. 217. The standard errors of the center (midrange) G and of the median M are given by $\sigma_G = \sigma\sqrt{6/(n+1)(n+2)}$, $\sigma_M = \sigma\sqrt{3/(n+2)}$ for n odd, and $\sigma_M = \sigma\sqrt{3n/(n+1)(n+2)}$ for n even. For samples from a normal population, the frequency constants of the range have been tabulated by Pearson (1926), p. 192, while the standard errors of the median [for samples of size n=2(1)5 only] and of the midrange are calculated by the authors from integrals given by Pearson (1926), p. 178, and from the formula $\sigma_G = \sigma_u\sqrt{(1+r_{uv})/2}$, where $\sigma_u$ and $r_{uv}$ have been given by Tippett (1925). For the rectangular and normal populations ($\beta_2$ = 1.8 and 3.0, respectively), the theoretical values of the mean and standard deviation of the range (in terms of the population standard deviation $\sigma$), of $\beta_1$ and $\beta_2$ for the range, and of the standard errors of the midrange and the median (as multiples of the standard error of the mean, $\sigma/\sqrt{n}$) are tabulated on p. 359 for n = 2,3,4,5,10,20, with the exception noted above in the case of the median. Values of the same constants, obtained by experimental sampling, for n = 2,5,10,20 are given in the same table for three other symmetric populations: Pearson Type II ($\beta_2$ = 2.500), Type VII ($\beta_2$ = 4.122), and Type VII ($\beta_2$ = 7.069). [For n=2, values of $\beta_1$ and $\beta_2$ for the range are missing, and theoretical values (all equal to 1.000) are given for the standard errors of the midrange and the median.] On p. 360, the authors tabulate (to 3DP) the probability (theoretical for $\beta_2$ = 1.80, 3.00 and empirical for $\beta_2$ = 2.50, 4.12, 7.07) that the range of a sample of size n=2,5,10,20 exceeds $k\sigma$, k = 1(1)11.

**References:** Tippett (1925), Pearson (1926), Student (1927), Neyman & Pearson (1928).

**Citations:** Pearson & Adyanthāya (1929), Rider (1929), Hojo (1931), Pearson (1931), Craig (1932b), Davies & Pearson (1934), Kendall (1943), Cramér (1946), Kendall (1946), Bickerstaff (1947), Plackett (1947).

* Saibante, Mario (1928). La concentrazione della popolazione. *Metron* **7** (2), 53-99.

**Summary:** The author gives a theoretical discussion of the Pareto distribution and the parameter $\alpha$ of that distribution, which is often used as an index of concentration, along with Gini's index of concentration $\delta$ and his ratio of concentration R. He proposes the use of $\alpha$, $\delta$ and R to measure the concentration of population in large cities, and calculates their values for several countries. As expected, the values of $\delta$ and R are highly correlated (positively) and both are highly correlated (negatively) with $\alpha$. Increasing values of $\delta$ and R and decreasing values of $\alpha$ are associated with increasing concentration of the population in large cities. The author also examines the relation of these measures to population density (negatively correlated with $\delta$ and R and positively correlated with $\alpha$) and average wealth (positively correlated with $\delta$ and R and negatively correlated with $\alpha$). As a measure of correlation, the author uses Gini's index of cograduation instead of the Pearsonian product-moment correlation coefficient.

**References:** Pareto (1897), Gini (1910) [1922], 1914a), Saibante (1926).

**Citations:** Castellano (1935), Gini (1936), Davis (1941).

* "Sophister" (1928). Discussion of small samples drawn from an infinite skew population. *Biometrika* **20A,** 389-423. (JFM 54, 567).

**Summary:** The author reports the results of drawing 1000 random samples of each of two sizes (5 and 20) from a Pearson Type III distribution, $y=y_0e^{-x}x^7$. The parts of this paper which are relevant to our study are Section 3 (pp. 394-395) and the accompanying Table 2 (p. 396) and Diagrams III and IV (pp. 397-398), which deal with distributions of ranges of the above mentioned samples. For each sample size, the author fits a normal curve and the best fitting Pearson curve (Type VI for $n=20$ and Type I for $n=5$) to the distribution of ranges. The fit of the normal curve is satisfactory for $n=5$, but not for $n=20$.

**References:** K. Pearson (1895), Tippett (1925), E. S. Pearson (1926), "Student" (1927), Neyman & Pearson (1928).

**Citations:** Pearson & Adyanthāya (1929), Rider (1929), K. Pearson (1931), E. S. Pearson (1939), Kendall (1946).

* Fisher, R. A. (1929). Tests of significance in harmonic analysis. *Proceedings of the Royal Society of London* (A) **125,** 54-59. (JFM 55, 950—listing only). Reprinted 1950 as Paper No. 16 of R. A. Fisher's *Contributions to Mathematical Statistics*, John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd. London.

**Summary:** The following portions are relevant to our study: "The practice of picking out the larger values of x, not in advance, but by reason of their exceptional magnitude, requires as Sir Gilbert Walker [(1914)] has shown, an important modification of the test of significance. For, if we wish to test the significance of the largest observed value of x, we must compare the value observed with the sampling distribution of the largest of n independent values, and not with that of any one value chosen in advance. If P stand for the probability which we adopt, as sufficiently small to be used as a criterion of significance, the corresponding value of x will be given by $e^{-x/c} = P$ [where c is the mean value of x], for any particular term, but if x is chosen to be the largest of n independent values, it is necessary that the probability should be $1-P$ that all the n values shall be less than x, so that $(1-e^{-x/c})^n = 1-P$ is the equation which determines the least value of x to be judged significant. This is the criterion derived by Walker." (p. 55)." . . . The probability that the largest of the n terms [as a fraction of the total] should exceed g is, so long as this probability is small, naturally not far from n times the values given by [$P=(1-g)^{n-1}$, the probability that any one harmonic term as a fraction of the total exceeds g], and has been shown to be exactly $P=n(1-g)^{n-1} - \ldots + (-)^{k-1}\{n!/k!(n-k)!\}(1-kg)^{n-1}$, where k is the largest integer less than $1/g$." (p. 58). An author's note and a table of 5% and 1% critical values of g for $n=5(1)50$ have been added to the reprint.

**Reference:** Walker (1914).

**Citations:** Fisher & Yates (1938), Fisher (1939, 1940), Garwood (1940), Chandra Sekar & Francis (1941), Cochran (1941), Davis (1941), Finney (1941), Greenwood (1946), Kendall (1946), Hartley (1949).

* Gini, Corrado; Galvani, Luigi (1929). Di talune estensioni dei concetti di media ai caratteri qualitativi. *Metron* **8** (1-2), 3-209. (JFM 56, 1084). Partial English translation by A. J. Lotka, *Journal of the American Statistical Association* **25** (1930), 448-450.

**Summary:** The following statements (pp. 136-138), translated by A. J. Lotka and published by F. A. Ross [(1930), pp. 448-450], are relevant: "The customary definition of the median in a linear series, implying the existence of a greatest and of a least variate, cannot have any meaning for a two-dimensional series, because the dyads of variates $(X_i,Y_i)$ of such a series cannot be arranged 'in order of magnitude.' If, on the other hand, we designate as *median* that pair of variates X and Y represented by the point $(\bar{x},\bar{y})$ for which the following relation holds $\theta(x,y) = \Sigma_{i=1}^{s} \sqrt{(x_i-x)^2+(y_i-y)^2}\, T_i = \min$. the radical being taken with a positive sign, then this term has a definite meaning. [$T_i$ apparently represents a frequency.] . . . It is to be observed that in contradistinction from what is the case for the arithmetic mean, the definition of the median of a series in two dimensions cannot be made to depend on that of the median of an ordinal series. In fact, if we consider respectively the medians $\bar{\bar{X}}$ and $\bar{\bar{Y}}$ of . . . two series . . . we cannot conclude that the dyad of variates $(\bar{\bar{X}},\bar{\bar{Y}})$ constitutes the median of the given two-dimensional series in the sense set forth above. . . . The 'median center' . . ., as it is ordinarily defined with reference to the distribution of population in a territory, is the point of intersection of the meridian and the parallel so chosen that each separately divides this distribution into two parts containing equal numbers. Evidently the 'median center' so defined not only lacks the characteristic property of a median, namely of making a minimum the sum of the absolute displacements, but furthermore, the median thus defined is not invariant with regard to the system of coördinates to which the points of the plane are referred; that is to say, if these points were referred to coördinates other than latitude and longitude, the median center thus obtained would be different from that defined in terms of latitude and longitude. Our definition of the median, on the contrary, has a purely objective meaning with regard to the distribution under consideration. . . . The point in which a population can be concentrated with the minimum aggregate travel from the present location is precisely that which constitutes the median in the sense of the definition given by us [and not, as stated in various publications of the U. S. Bureau of the Census, the center of population (center of gravity), for which it is not the sum of the displacements but the sum of their squares which is a minimum, or the 'median point', as defined in those publications]". The above quotation is but one example of the authors' use of what they call the "principle of maintenance of formal laws" to extend the customary definition of the median. In Sections 27-30 (pp. 83-96), they use this principle to define the median of a series of values $X_i (i=1,2,\ldots,s)$ with corresponding frequencies $Y_i$, where the $X_i$ are not ordered in the usual sense but are cyclically ordered, so that the usual definition is meaningless, as the value X which minimizes $\Sigma_{i=1}^{s} |X-X_i|Y_i$.

**References:** Galton (1875), Gini (1912, 1914b, 1915a,b, 1918), Pietra (1925), Gini (1926).

**Citations:** Gini (1930), Ross (1930), Galvani (1933), Gini, Boldrini, Galvani & Venere (1933), Griffin (1933), Scates (1933), Cisbani (1938), Dodd (1940), Kendall (1943, 1946).

* de Gleria, Amadio (1929). Una abbreviazione nel calcolo della differenza media. *Rivista Italiana di Statistica, Economia e Finanziaria* **1** (4), 383-384.

**Summary:** Let $a_1 \leq a_2 \leq \ldots \leq a_n$ be an ordered series [e.g. the order statistics of a sample of size n from some population (specified or unspecified)]. Let $s_1 = a_1, s_2 = a_1 + a_2, \ldots, s_n = a_1 + a_2 + \ldots + a_n; S = s_1 + s_2 + \ldots + s_n; s'_1 = a_n, s'_2 = a_n + a_{n-1}, \ldots, s'_n = a_n + a_{n-1} + \ldots a_1; S' = s'_1 + s'_2 + \ldots + s'_n$. Czuber (1914) has proposed the following formulas for calculating the mean difference $\Delta$ and the mean difference with

repetition $\triangle_R$: $\triangle = 2(S'-S)/n(n-1)$ and $\triangle_R = 2(S'-S)/n^2$. The author makes use of the obvious relation $S + S' = (n+1)\sum_{i=1}^{n} a_i$ or $S' = (n+1)\sum_{i=1}^{n} a_i - S$ to reduce Czuber's formulas to $\triangle = 2\{(n+1)\sum_{i=1}^{n} a_i - 2S\}/n(n-1)$ and $\triangle_R = 2\{(n+1)\sum_{i=1}^{n} a_i - 2S\}/n^2$.

**References:** Gini (1912), Czuber (1914).

**Citations:** de Finetti & Paciello (1930), de Finetti (1931a), Pietra (1932a), Pizzetti (1941a,c).

* Lévy, Paul (1929). Sur quelques travaux relatifs à la théorie des erreurs. *Bulletin des Sciences Mathématiques* (2) **53,** 11-32. (JFM 55, 917).

**Summary:** The second law of large numbers can be stated roughly as follows: The sum of a large number of variables (errors) which are independent and very small obeys the law of Gauss. One of the necessary hypotheses is that each of the errors has a finite standard deviation which, by neglecting an additive constant, one can represent by $m_i x_i$, $m_i$ denoting a positive constant and $x_i$ a reduced variable (i.e., one with zero mean and unit variance). The author shows that the maximum deviation considered by Fréchet (1927) is almost surely finite if the series $m_1^{\alpha}+m_2^{\alpha}+\ldots+m_n^{\alpha}+\ldots$ $(0<\alpha<2)$ is convergent and almost surely infinite if that series is divergent. In the latter case, the method of least squares cannot be rigorously justified, and the arithmetic mean is not necessarily the best average. In that case, the author states that it is natural to discard, in a definite proportion, the largest and smallest values and take the mean of the values retained, a method which is good in every case and gives a standard error asymptotically equal to $a/\sqrt{n}$, a depending on the law of error and the ratio of the number of values retained to the total number of measurements. He goes on to say that one may wish to determine that ratio so as to minimize a, but that it is not necessary to do this exactly, since a varies slowly in the vicinity of the minimum. He also mentions General Estienne's proposal that only one value, the median, be retained. This method, he says, is less precise than the preceding one, but has the advantage of being simple to use in practice, e.g. in artillery fire.

**References:** Lévy (1925), Frechét (1927), Cramér (1928).

* Pearson, Egon S.; Adyanthāya, N. K. (1929). The distribution of frequency constants in small samples from non-normal symmetrical and skew populations. *Biometrika* **21,** 259-286. (JFM 55, 922).

**Summary:** The only part of this paper relevant to our study is Section 6 (pp. 280-286). On pp. 280-281 the authors state: "In a recent paper [Neyman & Pearson (1928), p. 212] it was shown that in sampling from a rectangular population the appropriate criterion to use in testing a hypothesis regarding the position of the mean, a, was not z but the ratio $z' = (G-a)/(R/2)$, where u and v are the highest and lowest values of the variable in the sample, G is the sample 'centre', $= (u+v)/2$, R is the sample range, $= u-v$. The theoretical distribution of $z'$ in samples of n from this population was obtained, and it was suggested that perhaps it might be of wider application, just as 'Student's' z-distribution has been found to be adequate for populations differing considerably from the normal. Further analysis, however, soon showed that the 'rectangular theory' $z'$-distribution would not be appropriate for samples from the populations of common statistical experience." Approximate distributions of $z'$ in samples of 5 and 10 from a normal population are found and curves are fitted to them to obtain 'normal theory' $z'$-curves. The sensitivities of z and $z'$ tests are compared, not only for the normal population but for three neighboring non-normal populations with $\beta_1 = 0.0$, $\beta_2 = 2.5$; $\beta_1 = 0.0$, $\beta_2 = 4.1$ and $\beta_1 = 0.2$, $\beta_2 = 3.3$. As might be expected, "for samples of 5 the z-test is not very much more sensitive than the $z'$-test, but . . . when n has increased to 10 the former has a very marked advantage. The difference is least for the platykurtic population." (p. 284).

**References:** Tippett (1925), Neyman & Pearson (1928), Pearson & Adyanthāya (1928), "Sophister" (1928), Rider (1929).

**Citations:** Kendall (1946), Tukey (1948b, 1949b), Walsh (1949b).

* Rider, Paul R. (1929). On the distribution of the ratio of mean to standard deviation in small samples from non-normal universes. *Biometrika* **21,** 124-143. (JFM 55, 304, 921). Abstract, *Bulletin of the American Mathematical Society* **35,** 440.

**Summary:** A substantial part of this paper (the part described in the title) does not involve order statistics. The relevant portions are those comparing the distributions of various statistics in small samples from discrete rectangular populations with varying number of classes with the corresponding distributions, previously worked out by other authors, for samples from continuous populations. The author's results for discrete populations are compared with those of Pearson & Adyanthāya (1928) for the median, of Neyman & Pearson (1928) for the range, and of Dodd (1922) for the extreme average (midrange). Results are also given for the distributions of greatest and least variates in samples from discrete rectangular populations. On p. 139 the author states the following general conclusions: "A number of statistical parameters (mean, median, range, extreme average, greatest variate and least variate) in samples from a rectangular universe are distributed in polynomials, which are apparently, except in the case of greatest variate and least variate, of degree one less than the number in the sample. There appears to be little difference in the distributions of these variates when the universe is continuous and when it is discrete."

**References:** Dodd (1922), Tippett (1925), Neyman & Pearson (1928), Pearson & Adyanthāya (1928), Sophister (1928).

**Citations:** Pearson & Adyanthāya (1929), Shewhart (1931), Craig (1932a, 1933), Olds (1935), Kendall (1946), Bickerstaff (1947).

* Smith, David Eugene (1929). *A Source Book in Mathematics.* McGraw-Hill Book Company, Inc., New York-London. (JFM 55, 583-585).

**Summary:** The section on the field of probability (pp. 546-604) contains translations of all or portions of five classical works, including those of Legendre (1805), pp. 72-75, on the method of least squares and of Laplace (1812), pp. 304-327, on the probability of errors in the mean results of a great number of observations, etc. The former was translated by Henry A. Ruger and Helen M. Walker and the latter by Julian L. C. A. Gÿs.

**References:** Legendre (1805), Gauss (1809), Laplace (1812).

* Wilson, Edwin B.; Hilferty, Margaret M. (1929). Note on C. S. Peirce's experimental discussion of the laws of error. *Proceedings of the National Academy of Sciences* (USA) **15** (2), 120-125. (JFM 55, 303).

**Summary:** The authors re-examine an extensive series of observations for which C. S. Peirce (1873) concluded that the normal law was verified, and reach the opposite conclusion. Peirce's data consist of about 500 observations each day for 24 different days. The authors give the standard deviations of the median and of the mean for each day, and proceed to compare them, reaching the following conclusions (p. 124): "The ordinary statement based on the normal law is that the determination of the median is 25% worse than that of the mean. A comparison of the standard deviations of the median and mean . . . shows that for these observations the median is better determined than the mean on 13 days, worse determined on 9 days, and equally well determined on 2 days. Roughly speaking this means that mean and median are on the whole equally well determined."

**Comments:** The authors use their results as evidence that the data have not come from a normal distribution; the results also tend to show, as confirmed by theory [see, e.g., Gumbel (1937c)], that for some distributions the median is more precise than the mean.

**References:** Peirce (1873), Poincaré (1896) [1912].

**Citations:** Luyten (1932), Fréchet (1935), Gumbel (1937c), Fréchet (1941).

* d'Addario, R. (1930). Un'indagine sulla dinamica distributiva. *Rivista Italiana* **2** (3), 269-304.

**Summary:** The author points out that Pareto's equation ($N_x = A/x^{\alpha}$) is that of a [cumulative] distribution curve, while Gini's [$N_x/N = (R_x/R)^{\delta}$] is that of a curve of concentration, and he therefore asks why one should be confused with the other. The first gives the number of individuals having an income greater than a specified value x, while the second gives the fraction of the total number of individuals having a specified fraction of the total income. He also discusses Gini's ratio of concentration R and its relation to the ratio S of half the mean deviation (from the arithmetic mean) to the arithmetic mean. He states that it is easy to demonstrate, for a given distribution, that $R > S$ [see Pietra (1915), Ricci (1916)]. He calculates R and S and plots the curve of concentration for industrial and commercial income and for professional income in the city of Taranto in the years 1908, 1914, 1922 and 1928. Lastly, he discusses the relation between cograduate quantities and corresponding quantities [Gini (1917)] and applies it to incomes of individuals in a population at widely separated times.

**References:** Pareto (1897), Lorenz (1905), Gini (1909, 1910), Mortara (1911), Gini (1914a), Pietra (1915), Ricci (1916), Gini (1917), Amoroso (1925).

**Citations:** d'Addario (1932), Castellano (1935), Bresciani-Turroni (1939).

* Eells, Walter Crosby (1930). A mistaken conception of the center of population. *Journal of the American Statistical Association* **25,** 33-40.

**Summary:** The author (pp. 33-34) quotes at length from p. 5 of the Census Bureau bulletin by Sloane (1923), italicizing the following sentences: "If all the people in the United States were to be assembled at one place, the center of population would be the point which they could reach with the minimum aggregate travel, assuming that they all travelled in direct lines from their residence to the meeting place. No such statement holds of the median point." He notes that the same statement is found in various other Census Bureau reports on the Fourteenth Census (1920) and the Thirteenth Census (1910). He remarks (p. 35): "It cannot be denied that the form of statement used in the quotations in italics from the census gives a much more picturesque, vivid, and popular definition of the center of population than does the more abstract and mathematical definition of it as the centroid or center of gravity. But unfortunately it lacks one rather essential attribute—*truthfulness*. Before demonstrating its falsity, however, it may be of importance to consider a little further the history of the interesting point known as the center of population and of the closely related 'median point'." He points out that the erroneous statement concerning minimum aggregate travel is not found in the 1900 census or in any earlier ones, although treatment of the center of population goes as far back as an 1874 Census Bureau publication on the ninth (1870) census, in which it is defined correctly as the center of gravity of the population. He states that the "median point" was first computed for the 1900 census, at which time it was also computed for 1880 and 1890 for comparison. He considers three examples of small populations, and finds that for two of them the aggregate distance from the median point is less than that from the center of gravity, while the opposite is true for the other one. Thus neither of these points has in general the minimum distance property. The author considers the problem of what is the true point of minimum aggregate travel, but is able to solve it only for the very special case of three points forming an isoscles triangle.

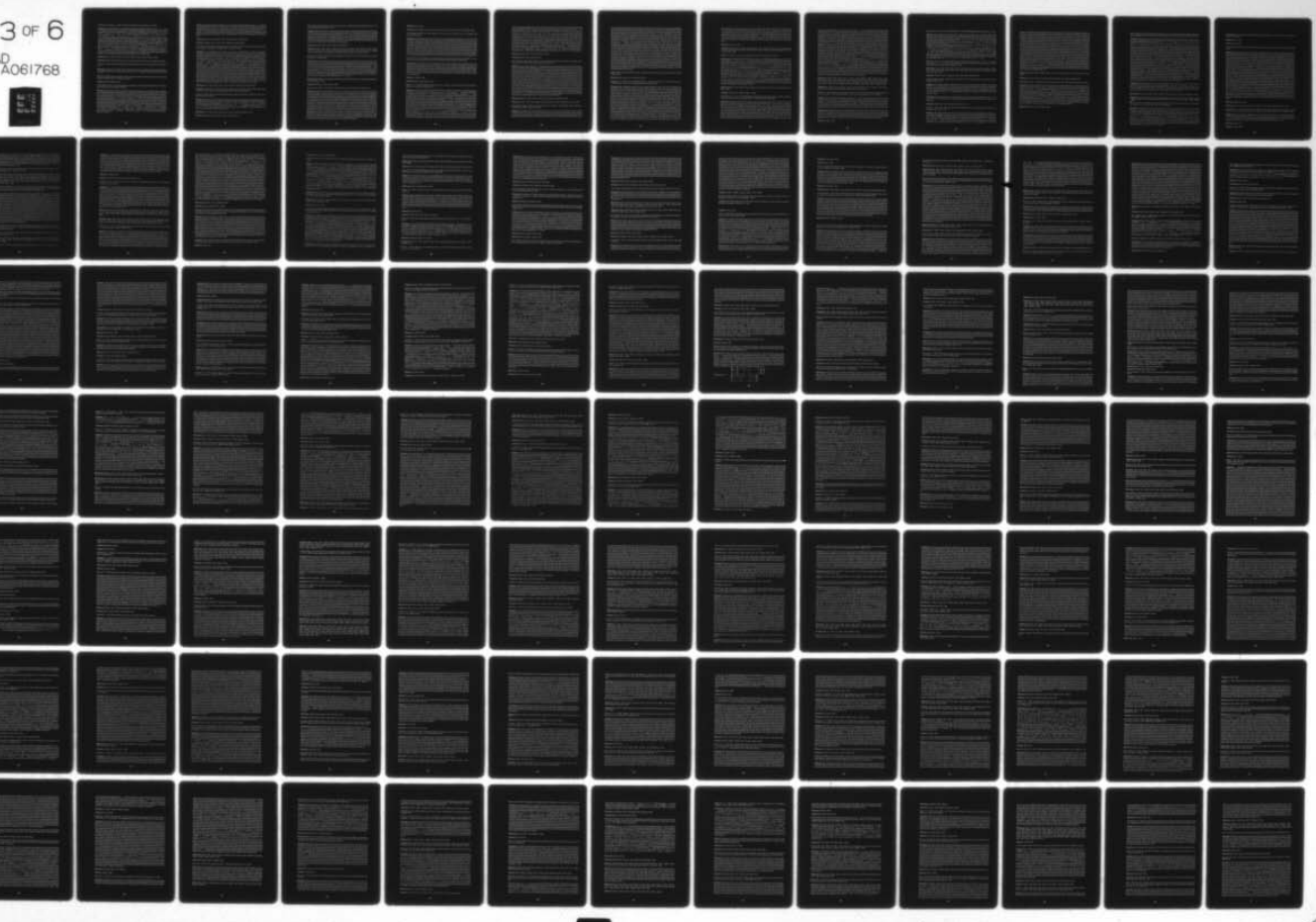
**References:** Bowley (1901) [1902], King (1912), Sloane (1923), Burgess (1927).

**Citations:** Ross (1930), Gini, Boldrini, Galvani & Venere (1933), Scates (1933).

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* de Finetti, B.; Paciello, U. (1930). Calcolo della differenze media. *Metron* **8** (3), 89-94.

**Summary:** The authors discuss various methods of computing the mean difference. Let $a_1 \leq a_2 \leq \ldots \leq a_n$ be a sample of size n from some (specified or unspecified) population. Gini (1912) gave the following formula for calculating the mean difference $\Delta$ and/or the mean difference with repetition $\Delta_R$: $\Delta_R = [(n-1)/n]\Delta = (1/n^2)\sum_{i=1}^{n}(n+1-2i)|a_i - a_{n-i+1}|$. Czuber (1914) gave the formula $\Delta_R = [(n-1)/n]\Delta = (2/n^2)(S' - S)$, where $S = s_1 + s_2 + \ldots + s_n$, $s_i = a_1 + a_2 + \ldots + a_i$, $S' = s'_1 + s'_2 + \ldots + s'_n$, $s'_i = a_n + a_{n-1} + \ldots + a_{n-i+1}$. A further simplification, which the authors mention but do not specify, was given by de Gleria (1929). The authors note that Gini (1914a) showed that $\Delta_R = [(n-1)/n]\,\Delta = 2AR$, where A is the arithmetic mean of the $a_i$'s and R is the ratio of concentration, so another way to find $\Delta_R$ and/or $\Delta$ is first to calculate A and R and then use the above relation. The authors go back to the basic definitions, $\Delta_R = \sum_{i,j=1}^{n}|a_i - a_j|/n^2$ and $\Delta = \sum_{i,j=1}^{n}|a_i - a_j|/n(n-1)$. They use the relation $\sum_{i,j=1}^{n}|a_i - a_j| = 2\sum'_{i,j=1}^{n}(a_i - a_j)$, where $\Sigma'$ extends over only those values of i and j such that $i > j$. Then they show that $\sum'_{i,j=1}^{n}(a_i - a_j) = \sum_{h=1}^{n-1} h(n-h)(a_{h+1} - a_h)$. Substituting these results back into the defining formulas gives $\Delta_R = 2\sum_{h=1}^{n-1} h(n-h)(a_{h+1} - a_h)/n^2$ and $\Delta = 2\sum_{h=1}^{n-1} h(n-h)(a_{h+1} - a_h)/n(n-1)$. The authors also adapt their formulas to grouped data, and use them to calculate the mean difference in the birth rates for the 16 Compartments of Italy in 1921, weighted according to the population of each Compartment.

**References:** Gini (1912), [Czuber (1914)], Gini (1914a), Pietra (1925), de Gleria (1929).

**Citations:** de Gleria (1930), de Finetti (1931a), Yntema (1933), Pizzetti (1941a,c), Kendall (1943, 1946).

* Fogelson, S. (1930). Sur la détermination de la médiane. *Revue Trimestrielle de Statistique* (Warsaw) **7** (2), 1- .

**Summary:** Gumbel (1937c) states that the author shows that, for any distribution having finite mean and standard deviation, the median of a sample of size N is asymptotically normally distributed (as $N \to \infty$).

**Note:** The compiler has been unable to obtain a copy of this paper, and hence has had to rely upon the paper by Gumbel (1937c) for information about it.

**Citations:** Gumbel (1935a, 1937c).

* Gibrat, Robert (1930). Une loi des répartitions économiques: l'effet proportionnel. *Bulletin de la Statistique Générale de la France* **19** (4) 469-513.

**Summary:** The author reviews the Pearson system of frequency curves, Kapteyn's skew frequency curves, and other theoretical distribution curves. He proposes that the lognormal law (which he calls the law of proportional effect) be used instead of Pareto's law to approximate distributions of salaries, incomes, inheritances, etc., as well as the concentration of enterprises. The equation that he gives for the law of proportional effect, $y = (1/\sqrt{\pi})\exp\{-[a\log(x-x_0) + b]^2\}$, is not a proper density function [see Note below]. He discusses three methods of estimating the parameters of this and other distributions: the method of least squares, the method of Cauchy, and Pearson's method of moments. He states that the method of moments exaggerates the effect of extreme observations. He proposes use of the parameter a of the lognormal distribution to define an index of concentration $C = 100/a$. He shows that, for the lognormal distribution, $\log(M - x_0) = -b/a$, $\log(A - x_0) = 1/4\,a^2\log e - b/a$ and $\log(M_0 - x_0) = -1/2a^2\log e - b/a$, where A = arithmetic mean, M = median and $M_0$ = mode. He states in a footnote (p. 505) that he has proved the formulas $C = 45.5\,D$ and $C = 105\log q_3/q_1$, where $D = (q_3 - q_1)/M$, $q_3$ and $q_1$ being the upper and lower quartiles.

**Note:** It has been pointed out by d'Addario (1931) that Gibrat omitted the factor a log e/$(x-x_0)$ needed on the right-hand side of his equation to make it a proper density function. This omission did not affect his results for log $(M-x_0)$, log $(A-x_0)$ and log $M_0-x_0$, which are consistent with d'Addario's results, as Gibrat (1931) has pointed out. It is of interest that log $(M-x_0)$ lies one-third of the way from log $(A-x_0)$ to log $(M_0-x_0)$, which is reminiscent of the rule of thumb [see, e.g. Pearson (1895) and Yule (1896a)] that the distance between the mean and the median is approximately one-third of that between the mean and the mode.

**References:** Pareto (1897), Bowley (1901), Kapteyn (1903), Jordan (1927).

**Citations:** d'Addario (1931), Gibrat (1931, 1932a,b), Castellano (1935).

* Gini, Corrado (1930). Sul massimo degli indici di variabilità assoluta e sulle sue applicazione agli indici di variabilità e al rapporto di concentrazione. *Metron* **8** (3), 3-65.

**Summary:** The author defines the *maximum of an index of variability* [for a series of nonnegative quantities, e.g. sample observations] as the value which this index assumes in the case of maximum inequality of the quantities in the series. For a series of n quantities, the maximum inequality occurs when one quantity is T = nA (where A = arithmetic mean) and the rest are equal to zero. The maxima of six common indices are given by $\text{Max}\,{}^1S_A = 2[(n-1)/n]\,A$, $\text{Max}\,{}^1S_M = A$, $\text{Max}\,{}^2S_A = \sqrt{n-1}\,A$, $\text{Max}\,{}^2S_M = \sqrt{n}\,A$, $\text{Max}\,\Delta = 2A$, $\text{Max}\,\Delta_R = 2[(n-1)/n]A$, where ${}^1S_A$ = simple mean deviation from the arithmetic mean, ${}^1S_M$ = simple mean deviation from the median, ${}^2S_A$ = quadratic mean deviation from the arithmetic mean, ${}^2S_M$ = quadratic mean deviation from the median, $\Delta$ = mean difference (without repetition) and $\Delta_R$ = mean difference with repetition. If the upper limit (maximum sample value) is L (< nA), the author gives different (more complicated) formulas for the maxima. He also discusses the curve of concentration, the area of concentration and the ratio of concentration (R) in relation to the indices of variability. In particular, he gives maximum values of the six indices of variability considered above when R is specified (in relation to A or M). He also discusses unweighted and weighted arithmetic means and medians, and calculates their values for birth, marriage and death rates of the 16 Compartments of Italy during the period 1920-1923. The weighted (according to population) arithmetic mean is equal to the value for the whole nation, and the weighted median is much closer to this value than either of the unweighted measures. The author also applies the formulas for the case L <nA to obtain maximum values, for the same demographic data, of the six indices of variability discussed earlier.

**References:** Gini (1912, 1914a), Vinci (1918), Saibante (1926), Gini & Galvani (1929).

**Citations:** Galvani (1931, 1932), Gini (1932), Castellano (1933a), Yntema (1933), Pietra (1935a), Wold (1935), Gini (1936, 1939), de Vergottini (1940), Davis (1941), Kendall (1946).

* de Gleria, Amadio (1930). Sul calcolo della differenza media. *Rivista Italiana di Statistica, Economia e Finanziaria* **2** (2), 155-161.

**Summary:** The author points out that the simplification of Czuber's formula for calculating the mean difference which he gave in his earlier note [de Gleria (1929)] is not applicable to grouped data. He gives an analogous formula for grouped data: $\Delta = 2\{(\Sigma_{i=1}^{n} f_i + 1)(\Sigma_{i=1}^{n} f_i a_i) - 2S\}/(\Sigma_{i=1}^{n} f_i)(\Sigma_{i=1}^{n} f_i - 1)$, where $a_i$ is the class mark and $f_i$ is the frequency of the $i^{th}$ class ($i = 1,2,\ldots,n$ = number of classes) and $2S = \Sigma_{i=1}^{n} f_i a_i(f_i + 1 + 2\Sigma_{k=i+1}^{n+1} f_k)$. He applies this formula to calculate the mean difference of the age differences of newly married couples in Italy in the year 1914.

**References:** Gini (1912, 1914a), Pietra (1915, 1925), de Finetti & Paciello (1930).

**Citations:** de Finetti (1931a), Pietra (1932a), Pizzetti (1941a,c).

* Hazen, Allen (1930). *Flood Flows. A Study of Frequencies and Magnitudes.* John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London.

**Summary:** Material relevant to our study is found in Chapters III (Method of Plotting Annual Floods), VII (The Coefficient of Skew), and VIII (Drawing a Smooth Curve). In Chapter III (pp. 13-18), the author gives a discussion of probability paper and proposes $P = (2m-1)/2n$ as a plotting position for the $m^{th}$ order statistic of a sample of size n. In Chapter VII (pp. 42-47), he summarizes the work of Foster (1924) and Goodrich (1927) on skew curves. In Chapter VIII (pp. 48-59), which is based largely on the work of Foster, he gives instructions for plotting logarithmic skew curves and Pearson Type I and Type III skew curves, making use of Tables II-IV (pp. 188-189) in the Appendix.

**References:** Hazen (1914), Foster (1924), Goodrich (1927).

**Citations:** Foster (1934), Slade (1934), Foster (1936), Grassberger (1936), Saville (1936), Gumbel (1937-38), Kimball (1938), Creager (1939), Gumbel (1939c), Grant (1940), Gumbel (1941a,b), Beard (1942), Gumbel (1942c), Kimball (1942), Gumbel (1943a,b, 1945a,b), Lane & Lei (1949).

* Lidstone, G. J. (1930). Notes on Professor Steffensen's first lecture. *Journal of the Institute of Actuaries* **61,** 340-353. (JFM 56, 1095).

**Summary:** The author comments on three main topics discussed by Steffensen (1930) in his first lecture: (1) the limit of life; (2) mortality tables giving the same policy values; and (3) frequency constants. Only the first of these is relevant to a study of order statistics. The following comments are pertinent: "in para. 5 of his first lecture the Professor deals rather severely with the classical text-book doctrine . . . that there is an age $\omega$ (the 'limit of life') at and beyond which $\ell_x = 0$. He in fact considers that 'this point of view contains a fallacy'." (p. 340). "Our conclusion is that the classic doctrine of $\omega$ is not to be stigmatized as a fallacy, but is properly to be regarded as prescribing the most suitable form of the idealistic continuous curve which—in spite of the inherent difficulties to which reference has been made—we find it theoretically convenient to adopt for the representation of $\ell_x$." (p. 343).

**References:** De Morgan (1847), Bertrand (1889), Poincaré (1896), Brunt (1917), Whittaker & Robinson (1924), Steffensen (1930).

**Citations:** Steffensen (1931), Insolera (1933).

March, Lucien (1930). *Les Principes de la Méthode Statistique.* Librairie Félix Alcan, Paris.

**Summary:** Chapter XI (pp. 264-288) deals with various types of averages, including the arithmetic mean, the median, the quadratic mean, the cubic mean, the biquadratic (quartic) mean, the harmonic mean, the logarithmic mean, and the geometric mean, as well as means of ratios, means of grouped data, and weighted means. The author discusses their properties and their relations to each other. Chapter XII (pp. 289-322) deals with measures of variability, including the interquartile range, the mean difference, the standard deviation, and the mean deviation. He defines not only the usual positional measures (quantiles, such as median, quartiles, deciles, percentiles, etc.) which divide the number of individuals possessing a character into equal parts according to the amount of it they possess, but also the corresponding values (quantals, such as medial, quartal, etc.) which divide the aggregate amount possessed into equal parts. For example, the median income is that income such that the same number of individuals have greater income as have less, while the medial income is that income such that the aggregate income of those individuals having greater income is the same as that of those having less. The difference between the median and the medial is called the intermediant. In Chapter XIX (pp. 503-524), the author considers comparisons according to the order of magnitude of the elements, both qualitative (ranks) and quantitative (order statistics). Among the latter, he discusses the interquantile and interquantal distances and their ratios to the median or the medial.

**Reference:** Gini (1912).

**Citations:** von Bortkiewicz (1931), Gini (1932), Julin (1933), Anderson (1935), Castellano (1935).

* de Montessus de Ballore, R. (1930). Moyennes. *Revue Générale des Sciences Pures et Appliquées* **41,** 49-53. (JFM 56, 448).

**Summary:** The author discusses various averages or means, including the arithmetic mean, the mode, the median, the geometric mean, and the harmonic mean. He defines the median in the usual way and shows how to compute the median for grouped data by linear interpolation between the limits of the class containing it. He states, however, that no analytic method is known for determining the median rigorously, even in the simplest cases. He observes that treatises on statistics give the empirical relation $M_0 + 2M_a = 3M$ between the mode $M_0$, the arithmetic mean $M_a$, and the median M, but points out that precise calculations cannot be based on that relation. He states that the sum of the absolute values of the deviations is a minimum when taken from the median, but opines that this fact is of much less practical interest than the analogous proposition concerning [the sum of squares of deviations from] the arithmetic mean. He defines the quartiles and other quantiles, and computes the former for a set of grouped data.

* Peterson, R. E. (1930). Fatigue tests of small specimens with particular reference to size effect. *Transactions of the American Society for Steel Treating,* **18,** 1041-1053; discussion, 1053-1056.

**Summary:** The author describes a small fatigue testing machine weighing ten pounds and taking a test specimen fifty thousandths of an inch in diameter at the critical section. He compares the results of fatigue tests on this machine with those of similar tests on larger machines using larger specimens. For ordinary steels he finds, contrary to what one would expect from the theory developed by Griffith (1920), that there is no appreciable size effect up to two-inch diameter. For cast iron specimens, he finds, however, that there is a considerable size effect; in the discussion, R. L. Templin reports a similar result for cast aluminum.

**Reference:** Griffith (1920).

**Citations:** Fowler (1945), Oding (1946).

* Rhodes, E. C. (1930). Reducing observations by the method of minimum deviations. *Philosophical Magazine* (7) **9,** 974-992. (JFM 56, 477).

**Summary:** The author states (pp. 974-978): "The writer recently was desirous of smoothing out the fluctuations in [a] series of figures [17 pairs of values of $x = -8(1)+8$ and y]. . . . A parabola was fitted by the method of Least Squares . . . . The result was not considered altogether satisfactory. . . . It was considered that the parabola was a bad fit. Two reasons suggested themselves for this: first, the original data from which the series was obtained did not involve absolutely random fluctuations; second, the parabola might not be the best curve for use in smoothing . . . . It was . . . decided to concentrate on the first consideration, which meant that although we had obtained the parabola of best fit by the criterion of least squares, yet it might not really be the best parabola which would smooth out the fluctuations in the series. This led us to the question of what other methods of fitting there were available, and Edgeworth's description of the use of medians in this connexion led to the attempt to fit by the method of Minimum Deviations. This method may be briefly described. Suppose the equation to the parabola is $y = a_0 + a_1x + a_2x^2$, and the given y's are $y_{-8}, y_{-7}, \ldots, y_{-1}, y_0, y_1, \ldots, y_8$, then instead of, as in the method of Least Squares, making $S_{x=-8}^{+8} (a_0 + a_1x + a_2x^2 - y_x)^2$ a minimum, we make $S_{x=-8}^{+8} |a_0 + a_1x + a_2x^2 - y_x|$ a minimum. His [Edgeworth's] description of the method and his arguments in its favour are briefly summarized [by] Bowley [(1928)], pp. 103 et seq., and are exposed by Edgeworth [(1888, 1923)]. Unfortunately Edgeworth confined himself in the working of the method to a reliance on a diagram (he

used as illustrations the problem of two variables), which means in practice a rather laborious piece of work, and apparently did not notice that the method could be applied in a more simple manner. . . . The simpler method is as follows:—Suppose we are dealing with a series of deviations, say, involving three unknowns, $A_1u+B_1v+C_1w+D_1, A_2u+B_2v+C_2w+D_2, \ldots A_nu+B_nv+C_nw+D_n$, and we want to find values of u, v, w which make $S_{s=1}^{n}|A_su+B_sv+C_sw+D_s|$ a minimum. First, find for what values of v and w the expression is a minimum when u is given by $-(B_rv+C_rw+D_r)/A_r$, i.e. find a local minimum point in the plane $A_ru+B_rv+C_rw+D_r=0$, where r is any one of the values of s from 1 to n. This reduces the problem to one involving two variables only, i.e. what values of v and w will make $S_{t=1}^{n-1}|E_tv+F_tw+G_t|$ a minimum, where the E's, F's, G's are obtained from the A's, B's, C's, D's. To solve this, find for what value of w the expression is a minimum when v is given by $-(F_pw+G_p)/E_p$, i.e. find a local minimum point in the line $E_pv+F_pw+G_p=0$, where p is any one of the values of t from 1 to n−1. This reduces simply to the problem of finding a weighted median. . . . Then [the] point of intersection of . . . three planes [$A_\ell u+B_\ell v+C_\ell w+D_\ell=0, A_mu+B_mv+C_mw+D_m=0, A_nu+B_nv+C_nw+D_n=0$] is the true minimum point, and the values u, v, w obtained from solving these equations make $S_{s=1}^{n}|A_su+B_sv+C_sw+D_s|$ a minimum.

**References:** Edgeworth (1888, 1923), Bowley (1928).

**Citations:** Bruen (1938), Singleton (1940).

* Ross, Frank Alexander (1930). Editor's note on the center of population and point of minimum travel. *Journal of the American Statistical Association* **25,** 447-452.

**Summary:** The author summarizes correspondence concerning the paper by Eells (1930) which has come to the Editor's office. He begins (pp. 447-448) with a letter from Professor Corrado Gini calling attention to an article by Gini & Galvani (1929) in which "the problem of the determination of point of minimum aggregate travel for a population . . . has been fully discussed and the general solution has been given" and "it is also noted that the designation of 'median point' should properly be given to the point of minimum aggregate travel, and not to the intersection of two orthogonal median lines, which intersection is not invariant with respect to the system of coordinates to which the points of the plane are referred." He continues (pp. 448-450) with a translation by A. J. Lotka of the relevant portion of the article by Gini & Galvani (1929). He reports correspondence indicating that the fallacy pointed out by Professor Eells was fully understood by the close of 1926, and that the erroneous statements will be eliminated from the Census Bureau report on the 1930 census. He notes that Hayford (1902) stated that the location of the median point might vary according to the pair of lines chosen as axes. He reports general solutions by E. B. Wilson of the problem of minimum distance for the special cases of any three or any four points. In closing, he states (p. 452): "There is still needed a practical method of procedure for locating the point of minimum travel for the case of actual multi-point populations, as Professors Gini and Galvani have not as yet published anything beyond the original formulation, and since their splendid and scholarly treatment does not give us means for locating the point".

**References:** Hayford (1902), Sloane (1923), Gini & Galvani (1929), Eells (1930).

**Citations:** Galvani (1933), Gini, Boldrini, Galvani & Venere (1933), Griffin (1933), Scates (1933).

* Scarborough, James B. (1930). *Numerical Mathematical Analysis.* Johns Hopkins Press, Baltimore. Second edition, 1950; fourth edition, 1958.

**Summary:** In Chapter XIV, the author discusses the normal law of error and the principle of least squares. He points out at the outset that all measurements are subject to three kinds of errors: constant or systematic errors, mistakes or blunders, and accidental errors. The mathematical theory of errors deals only with accidental errors, which are those whose causes are unknown and indeterminate. They

are usually small, and follow the laws of chance. Often (but not always) they are well approximated by the normal law of error, which the author discusses at length, along with the associated principle of least squares, which leads to choice of the arithmetic mean as the best average of a number of observations, with weights proportional to the squares of the precision indices $h_i = 1/\sigma_i\sqrt{2}$. In Chapter XV, the author discusses the precision of measurements, which can be estimated by the [root] mean square error [standard deviation $\sigma$], the probable error (P.E.), or the average [absolute] error. In Art. 141, he gives the following rule [cf. Wright (1884)] for rejecting outlying observations: "Find the mean of all the measurements (including the 'wild' one) and find the residual for each. Compute the P.E. of a single measurement .... Reject any measurement whose residual exceeds 5 times the P.E. of a single measurement." In Chapter XVI, which deals with empirical formulas, the author compares three fitting methods: the graphic method (method of selected points), the method of averages, and the method of least squares. He does not recommend the method of selected points except for obtaining approximate values or in cases where the results obtainable by this method are as accurate as the data used. The method of averages involves grouping the residual equations into as many groups as there are constants in the assumed formula, with each group containing as nearly as possible the same number of residuals, then setting the sum of the residuals in each group equal to zero, and solving the resulting equations. The residual equations can be grouped in several ways, each giving a different result. The best formula is obtained by grouping in consecutive order of the values of [one of] the independent variable[s]. The author states that when the number of residual equations is large enough to allow three or more to each group, the method of averages can be depended on to give good results, but that otherwise we should always use the method of least squares, which gives only one formula, which is always the best possible one [if the errors follow the normal law]. He discusses least squares regression in detail, including computational procedures, weighted residuals, non-linear formulas, both variables subject to error, and finding the best type of formula.

**Note:** All references in the above summary to specific chapters and articles are to those in the second (1950) edition.

**References:** Chauvenet (1863), Whitworth (1867).

* Steffensen, J. F. (1930). *Some Recent Researches in the Theory of Statistics and Actuarial Science.* Cambridge University Press, Cambridge, England. (JFM 56, 1095).

**Summary:** This monograph consists of three lectures delivered at the University of London. The relevant material is found in that part of Lecture I dealing with the "oldest age", concerning which the author writes (pp. 5-7): "What the textbooks say about the function $\ell_x$ is as a rule not expressed with sufficient reserve, and is even in many cases positively misleading. Without going into details about the various forms of vagueness or inaccuracy I have observed, I think it safe to assert that an ordinary student reading a textbook for the first time may be led to form the following opinions on the nature of the table of $\ell_x$. Assume that $\ell_0$ persons are born and that we follow these persons from their birth till their death, the number of those who are alive after x years being denoted by $\ell_x$. They will certainly all die within a finite time. There is therefore an "oldest age" $\omega$ such that $\ell_x > 0$ for $x < \omega$, and $\ell_x = 0$ for $x \geq \omega$. No question of convergence arises, for we have simply $\int_x^\infty \ell_x\,dx = \int_x^\omega \ell_x\,dx$, the limits of integration being in reality finite. If an analytical expression is assumed for $\ell_x$, it may safely be assumed that $\int_\omega^\infty$ is so small that it can be neglected. Our student is confirmed in this view when he discovers that the table of $\ell_x$ is stated in integral numbers, commencing with, say, 100,000 persons at the lowest age and terminating with 0 at an age about 100 years. But this point of view contains a fallacy. It is true that any given *finite* number of persons must all die within a finite time or, to put it more precisely, the probability that one or more of them will survive indefinitely is zero. This follows from the ... fact that ${}_tp_x[= \ell_{x+t}/\ell_x] \to 0$ when $t \to \infty$. But the greater the number of persons is, the greater will also be the number of them who may still be alive at age 100 or any other assigned age, and it is therefore quite impossible to maintain the existence of a definite 'oldest age' $\omega$ within which everybody must certainly die. Let us, to put it

mathematically, assume for a moment that $\omega$ is the upper limit which human life can attain. Then we have admitted that it is possible that a person may be alive at the age $\omega - \eta$, where $\eta$ is a quantity which we may choose *as small as we please*, for instance equal to one second. But at the exact age $\omega$, or one second later, that person must, according to hypothesis, necessarily be dead. Does anyone really believe that there is an age $\omega$ with this miraculous peculiarity? . . . An entirely different point of view from the one we have discussed is that in actually constructing a table of $\ell_x$ it is necessary to stop at a certain age which we may still call $\omega$, although it has nothing to do with an 'oldest age' in the abstract sense of the word."

**References:** Cramér (1928).

**Citations:** Lidstone (1930), Insolera (1931), Shewhart (1931), Steffensen (1931), Gumbel (1933f), Insolera (1933), de Finetti (1934), Gumbel (1934g, 1935d), Insolera (1935), Gumbel (1936c, 1936-37, 1937a), Kendall (1943, 1946).

* d'Addario, R. (1931). La curve dei redditi. *Rivista Italiana di Statistica* **3,** 140-161.

**Summary:** The author points out that the factor $a \log e/(x-h)$ should be included on the right-hand side of the equation $y=(1/\sqrt{\pi})e^{-[a \log (x-h)+b]^2}$ given by Gibrat (1930) as the probability density function of the law of proportional effect (log-normal law). He states that this analytic error vitiates Gibrat's treatment at the base. Using the correct form $y=\{(a \log e)/[(x-h)\sqrt{\pi}]\}\, e^{-[a \log(x-h)+b]^2}$, the author calculates the mode $v=h+10^{-1/2a^2 \log e - b/a}$, the arithmetic mean $A=h+10^{1/4a^2 \log e - b/a}$, and the median $M=h+10^{-b/a}$, and notes that $v<M<A$. He also calculates the quartiles $Q_1=h+10^{-0.4769/a - b/a}$ and $Q_3=h+10^{0.4769/a - b/a}$ and the standard deviation $\sigma=(A-h)(10^{1/2a^2 \log e}-1)^{1/2}$. Since the distribution law under consideration is obtained from the normal law $y=(1/\sqrt{\pi})\, e^{-z^2}$ by making the transformation $z=a \log (x-h)+b$, the author takes three values $z_1$, $z_2$, and $z_3$ of z in arithmetic progression (with common difference $d=z_2-z_1=z_3-z_2$) and the corresponding values of x, which are related by the equations $z_i=a \log (x_i-h)+b$, $i=1,2,3$ and proceeds to solve for the values of the parameters, obtaining $a=d/[\log (x_3-x_2) - \log(x_2-x_1)]$, $h=(x_1x_3=x_2^2)/(x_1+x_3-2x_2)$, $b=z_i-a \log(x_i-h)$ for i = 1, 2, or 3. He prefers this analytic method of estimating the parameters to Gibrat's graphical method. He applies his method to several sets of data on incomes, and shows that in each case the lognormal distribution gives a better fit to the data than does the Pareto distribution, often used for incomes.

**References:** Pareto (1897), Bowley (1901), Kapteyn (1903), Amoroso (1925), Lévy (1925), Jordan (1927), Gibrat (1930).

**Citations:** Gibrat (1931, 1932b), d'Addario (1934a).

* von Bortkiewicz, L. (1931). Die Disparitätsmasse der Einkommenstatistik. (French summary). *Bulletin de l'Institut International de Statistique* **25** (3), 189-298; discussion, 299-320M.

**Summary:** The author discusses various measures of the disparity [concentration] of income. In his first section, he considers the Lorenz curve and the relative mean difference $\eta = u/m$, where u is the mean difference and m is the arithmetic mean. He shows that $\eta = 4\lambda$, where $\lambda$ is the area of concentration between the Lorenz curve and the line of equidistribution. He gives formulas for computing u (and, from it, $\eta$), and traces the use of $\eta$ as a measure of disparity back to Furlan (1911). He notes that Gini (1914a) was aware of the relation between $\eta$ and $\lambda$, and that Czuber (1891a, 1899, 1903 [1914]) traced the use of the mean difference back to Jordan (1869) and pointed out that Bréger (1881) (apparently independently) had the same idea. In his second section, he treats the relative mean deviation $\theta$ [the ratio of the mean deviation (from the arithmetic mean) to the arithmetic mean] and shows that $\lambda = \eta/4 < 2\theta - \theta^2/2$. He also considers the quadratic mean deviation from the arithmetic mean (standard deviation $\sigma$) and

the quadratic mean difference $\gamma$, which satisfy the relation $\gamma = \sigma\sqrt{2}$. In his third section, he deals with the Pareto constant $\alpha$ as a measure of disparity and the true meaning of the Pareto formula. He points out that the measures $\eta$ and $\theta$ do not assume any particular distribution of income, whereas $\alpha$ assumes a Pareto distribution. Under this assumption, he shows that $\eta = 2/(2\alpha - 1)$. Gini's index of concentration is relegated to a footnote (p. 226). His fourth section is devoted to the case of specific distributions [exponential, straight line (trapezoidal) and polar (double spike)] and his fifth to intrafractional and interfractional mean differences. If incomes of the population under consideration lie between the limits h and k and if $h < x_1 < k$, then the mean difference $u_1$ between incomes less than $x_1$ and the mean difference $u_2$ between incomes greater than $x_1$ are called *intrafractional* mean differences and the mean difference $u_{1,2}$ between pairs of incomes, one greater than $x_1$ and the other less than $x_1$, is called the *interfractional* mean difference. His sixth and final section (supplements and conclusions) includes a discussion of the Holmes (1892) measure of concentration of wealth (or income). Let x′ be the usual median of income (the value such that the *numbers* of incomes above and below x′ are equal) and let x″ be the income such that the *total amounts* of incomes above and below x″ are equal, which Holmes also calls a median [and March (1930) calls the medial]. Then x″ − x′ [called the intermediant by March] is Holmes' measure. The author defines the relative measure $\epsilon = (x'' - x')/2m$, where m is the arithmetic mean, and shows that, under the assumption of a Pareto distribution, $\epsilon = [(\alpha-1)/\alpha]\,(\sqrt[\alpha-1]{2} - \sqrt[\alpha]{2})$, where $\alpha$ is the Pareto constant. He also discusses the index of concentration $\delta = \alpha/(\alpha-1)$ of Gini (1910), the relative quartile deviation $(x_3 - x_1)/m$, where $x_3$ and $x_1$ are the upper and lower quartiles, the measure of relative variability $\rho = (x_3 - x_1)/(x_3 + x_1)$ of Bowley (1901) [1920], and the measure of disparity $\omega = (x'_3 - x'_1)/(x'_3 + x'_1)$, where $x'_1$ and $x'_3$ are quartals defined analogously to the medial. The discussion consists of comments by Gini, Savorgnan and Pietra and a rejoinder by the author, followed by a long note by Gini and a short one by Pietra. Gini points out that several results presented by the author as new were given by him [Gini] in a paper [Gini (1912)] which the author does not mention and in others which he does, and that Czuber's statement, quoted by the author, about Bréger (1881) is incorrect—what Bréger actually considered was the mean *successive* difference.

**References:** Jordan (1869), Bréger (1881), Czuber (1891a), Holmes (1892), Pareto (1897), Czuber (1899), Bowley (1901) [1920], Czuber (1903) [1914], Holmes (1905), Lorenz (1905), Watkins (1905, 1908), Gini (1910), Furlan (1911), Gini ({1912}, 1914a), Savorgnan (1915), Ricci (1916), {Weinberg (1916)}, Dalton (1920), {Julin (1921)}, Winkler (1924), {Gini (1926)}, Gumbel (1927, 1928), {March (1930)}.

**Citations:** Gini (1932), Pietra (1932a,b), Yntema (1933), Anderson (1935), Castellano (1935), Pietra (1935a), Bresciani-Turroni (1937, 1939), Davis (1941), Pizzetti (1941a).

* Edgett, G. L. (1931). Frequency distributions with given statistics which are not all moments. *Metron* **9** (2), 25-35. (JFM 57, 1484). Abstract, *Bulletin of the American Mathematical Society* **37,** 677.

**Summary:** The author finds analytical representations for the following frequency curves having specified values (the same as for a set of observed data) of certain statistics [parameters] which are not all moments: (1) Pearson curve with specified mean, median, mode, and standard deviation; (2) Pearson Type III curve with specified mean, median, and mode; (3) Edgeworth series with specified mean, median, third quartile, and standard deviation. He compares the theoretical and observed frequencies, using Pearson's test of goodness of fit. The fit is excellent in the first and third cases, poor in the second.

**Comments:** The population median and third quartile are not order statistics [not statistics at all in the modern sense, though the author uses that name for them]. This paper is relevant to our study only because the sample median is involved in each case, and the sample third quartile in the third case. The poor fit in the second case presumably indicates that the data may not have come from a Pearson Type III distribution, not that there is anything wrong with the method of fitting used by the author.

**Citation:** Kendall (1946).

* de Finetti, Bruno (1931a). Sui metodi proposti per il calcolo della differenza media. *Metron* **9** (1), 47-52.

**Summary:** The author notes that, while other authors [Jordan (1869, 1872), von Andrae (1869, 1872), Helmert (1876a)] used the mean difference in calculating probabilities relative to the theory of errors of observation, Gini (1912) was the first to propose its use as an *index of variability*. He reviews four methods of calculating the mean difference or $\Sigma_{i,j=1}^{n} |x_i - x_j| = n(n-1)\,\Delta = n^2\Delta_R$, where $\Delta$ and $\Delta_R$ are the mean differences without and with repetition, for an ordered sample $x_1 \leq x_2 \leq \ldots \leq x_n$ of size n: (1) The method of Gini (1912). Formula: $\Sigma_{i,j=1}^{n} |x_i - x_j| = \Sigma_{i=1}^{n} d_{i,n-i+1} |x_i - x_{n-i+1}|$, where $d_{i,n-i+1} = |n+1-2i|$. [See Note below]. (2) Method of de Finetti & Paciello (1930). Formula: $\Sigma_{i,j=1}^{n} |x_i - x_j| = 2\,\Sigma_{i=1}^{n-1} i(n-i)(x_{i+1} - x_i)$. (3) Method of Gini (1914a) and Czuber (1914). Formula: $\Sigma_{i,j=1}^{n} |x_i - x_j| = 2(S' - S)$, where $S = \Sigma_{i=1}^{n} (n-i+1)x_i$, $S' = \Sigma_{i=1}^{n} i\,x_i$. Since $S + S' = (n+1)\,\Sigma_{i=1}^{n} x_i = n(n+1)M$, where M = arithmetic mean of $x_1, x_2, \ldots, x_n$, the following identities hold: $2(S' - S) = 2(S' + S) - 4S = 2n(n+1)\,M - 4S$; $2(S' - S) = 4S' - 2(S' + S) = 4S' - 2n(n+1)M$. (4) The graphical method of Gini (1914a). Formula: $\Delta_R = [(n-1)/n]\,\Delta = 2MR$, where R is determined graphically as the area of concentration (area between the Lorenz curve and the line of equidistribution). The author offers the opinion that the most convenient formula for calculating the mean difference is $\Sigma_{i,j=1}^{n} |x_i - x_j| = 4\,\Sigma_{i=1}^{n} i\,x_i - 2n(n+1)M = n(n-1)\,\Delta = n^2\Delta_R$.

**Note:** Gini (1912) [see also de Finetti & Paciello (1930)] incorrectly defined $d_{i,n-i+1}$ to be $n+1-2i$, which would give $\Sigma_{i,j=1}^{n} |x_i - x_j| = 0$, since there would be $[n/2]$ pairs of nonzero summands, the numbers of each pair being equal in magnitude but opposite in sign.

**References:** von Andrae (1869), Jordan (1869), von Andrae (1872), Jordan (1872), Helmert (1876a), Gini (1912), Czuber (1914), Gini (1914a), Pietra (1915, 1925), de Gleria (1929), de Finetti & Paciello (1930), de Gleria (1930).

**Citations:** Anderson (1935), Castellano (1935), Wold (1935), Kendall (1946).

* de Finetti, Bruno (1931b). Sul concetto di media. *Giornale dell'Istituto Italiano degli Attuari* **2** (3), 369-396. (JFM 57, 609).

**Summary:** The author studies various types of mean or average and their properties. He defines internal means, monotone means, associative means, homogeneous means, and translative means. He shows (p. 381) that the median is monotone but not associative and (p. 392) that the power means are associative and homogeneous. He points out that the power mean of order m is the arithmetic mean for $m=1$, the harmonic mean for $m=-1$, the quadratic mean for $m=2$, and that its limit as $m \to 0$ is the geometric mean.

**Comments:** The author defines the $m^{th}$ power mean of a set of non-negative real numbers as the $(1/m)^{th}$ power of the sum of their $m^{th}$ powers. This definition differs from that of Fechner (1874), which is the usual one.

**Citations:** Barral Souto (1938), Dodd (1940), Boldrini (1942).

* Galvani, L. (1931). Contributi alla determinazione degli indici di variabilità per alcuni tipi di distribuzione. *Metron* **9** (1), 3-46.

**Summary:** The author reviews the theory of various measures of central tendency, variability and concentration and applies this theory to continuous distributions in general and to linear distributions in particular. Measures of central tendency considered include the generalized geometric mean $G = \{\Sigma_a^b\, x^r\, \phi(x)\, dx\}^{1/r}$ of Dunkel (1909), where $\phi(x)$ is the p.d.f., which is positive for $a < x < b$ and zero otherwise. For $r = -1, 0, 1, 2$, the value of G coincides with the harmonic mean, the geometric mean, the arithmetic mean A, and the quadratic mean Q, respectively. As $r \to -\infty$, $G \to a$ and as $r \to +\infty$, $G \to b$. The

author also considers the median M, which is the value from which the mean (absolute) deviation is a minimum, a result which he attributes to Laplace. In addition to the median, he considers quantiles (in particular quartiles, quintiles and deciles) as measures of position. Measures of variability considered include the quadratic mean deviation [standard deviation] (from the arithmetic mean) $\sigma$, the mean (absolute) deviation from the arithmetic mean ($\eta_A$) or from the median ($\eta_M$), the simple mean difference $\triangle$, and the quadratic mean difference ${}^2\triangle$. He shows that ${}^2\triangle^2 = 2\sigma^2$, a relation first proved by Gini (1912), and that the maximum value of $\triangle$ for a set of n positive values with mean A is 2A, which occurs when all but one of the values are concentrated at 0 and the other one is at nA. Measures of concentration considered include the area of concentration and the ratio of concentration R = 2 (area of concentration) = $\triangle/2A$. If $\Phi(x)$ denotes the proportion of the population having wealth not greater than x, and $\Psi(x)$ denotes the proportion of the wealth possessed by that part of the population, then a plot of $\Psi$ (vertically) vs. $\Phi$ (horizontally) yields the lower curve of concentration and a plot of $\Phi$ (vertically) vs. $\Psi$ (horizontally) yields the upper curve of concentration. In either case the area between the curve of concentration and the line of equidistribution is the area of concentration. The author tabulates the arithmetic mean A, the quadratic mean Q, the median M, the quartiles and the deciles, also the simple mean (absolute) deviation from A and from M, the standard deviation $\sigma$, the variance $\sigma^2$, the simple mean difference $\triangle$, the quadratic mean difference ${}^2\triangle$, the ratio of concentration R = $\triangle/2A$ and the simple mean deviation relative to A, $\eta_A/2A$ for m = $-2$ (0.2) 2 for a linear (right triangular, trapezoidal or uniform) distribution of the form $\phi(x) = mx - (m-2)/2$, $0 < x < 1$. He also calculates the various measures of central tendency, variability and concentration for actual data from the 1921 census of England and Wales. Finally, he considers the corresponding weighted measures (where the weights may be the frequencies for grouped data) and gives examples of their use.

**References:** Lorenz (1905), Dunkel (1909), Gini (1912, 1914a), Pietra (1915), Galvani (1927), Gini (1930).

**Citations:** Gini (1932), Pietra (1932a, 1935a), Wold (1935), Gini (1936), Kendall (1946).

* Gibrat, Robert (1931). Une loi des répartitions économiques. *Rivista Italiana di Statistica* **3,** 332-334.

**Summary:** The author admits that the passage from his earlier paper [Gibrat (1930)] reproduced and criticized by d'Addario (1931) could, if considered out of context, lead to the impression that in the integral $(1/\sqrt{\pi}) \int_{-\infty}^{z} e^{-z^2} dz$, in replacing the variable z by x, where $z = a \log (x-h) + b$, he neglected to replace the differential dz by a log e $\cdot dx/(x-h)$. But he says it is astonishing to him that d'Addario would assume that he had made such an error without considering the rest of his memoir, which shows clearly that he had made the correct change of variable, but inadvertently omitted the factor a log e/(x−h) from the equation in question. As evidence of this, he cites the facts that (1) on p. 472 of his earlier paper he gave the differential equation $y'/y = (a + b \log x)/x$ for the curve type; (2) his graphical adjustment (curve fit) on the same page is consistent with the correct formula; and (3) most astonishing of all, on page 473 he gave expressions for the mean, the median and the mode which are absolutely identical with those given by d'Addario (1931), pp. 143-147, except that $x_0$ was used instead of h.

**References:** Gibrat (1930), d'Addario (1931).

* Hojo, Tokishige (1931). Distribution of the median, quartiles, and interquartile distance in samples from a normal population. *Biometrika* **23,** 315-360; appendix by Karl Pearson, 361-363. (JFM 57, 633, 634).

**Summary:** In defining the sample median, M, and the sample quartiles, $Q_1$ and $Q_3$, of a sample of size n, the author distinguishes four cases according as (i) $n=4m$, (ii) $n=4m+1$, (iii) $n=4m+2$, (iv) $n=4m+3$. His definitions for these cases are as follows: (i) $Q_1 = (x_m + x_{m+1})/2$, $M=(x_{2m}+ x_{2m+1})/2$, $Q_3=(x_{3m}+ x_{3m+1})/2$; (ii) $Q_1=(x_m + x_{m+1})/2$, $M =x_{2m+1}$, $Q_3=(x_{3m+1}+ x_{3m+2})/2$; (iii) $Q_1 = x_{m+1}$, $M=(x_{2m+1} + x_{2m+2})/2$, $Q_3= x_{3m+2}$; (iv) $Q_1 = x_{m+1}$, $M=x_{2m+2}$, $Q_3= x_{3m+3}$, where $x_i$ denotes the $i^{th}$ order statistic. He summarizes his results as follows (pp. 359-360): "The object of this paper has been to examine the sampling variation of the median, quartiles and interquartile distance in samples from a normal population, and to show how their distributions in the case of very small samples tend towards the limiting forms. The sampling distribution of the median is very nearly normal in all cases; it provides a somewhat less reliable estimate of the position of the population mean than the sample mean does. In this connection . . . comparison of three 'central estimates' is of interest. . . . The mean is definitely the most reliable; for very small samples there is little to choose in accuracy between the median and the centre [midrange]. For large samples however the centre becomes relatively increasingly unreliable, while the standard error of the median has almost attained its limiting value by the time n=20. . . . The distribution of the quartile is not so closely normal as that of the median. . . . The results . . . suggest that by the time n=10 the assumption of normality would not lead to serious error. The standard error of the quartile is somewhat greater than that of the median, tending to a limiting value of $1.3626\tilde{\sigma}/\sqrt{n}$ as against $1.2533\tilde{\sigma}/\sqrt{n}$ [$\tilde{\sigma}$= population standard deviation]. . . . The distribution of the interquartile distance is definitely skew in small samples; the longer tail of the distribution being towards the larger values of the measure. . . . But the distribution tends to the normal slowly as n increases. A method [unbiased] of estimating the population standard deviation, $\tilde{\sigma}$, from the interquartile distance, Q, has been considered . . . . It is not, however, as reliable an estimate as that which can be obtained from the sample standard deviation s, and in large samples its standard error tends to be 1.6495 × S.E. of s." In his appendix, Pearson considers a different approximation than that used by Hojo to the standard error of the median.

**Comments:** The compiler prefers to define the median and the quartiles so that they divide the population into four intervals each with equal probability 1/4, viz. $Q_1= x_{(n+1)/4}$, $M=x_{(n+1)/2}$, $Q_3= x_{3(n+1)/4}$, where a fractional subscript indicates interpolation between adjacent order statistics. This definition agrees with Hojo's except for the quartiles in cases (i) and (iii) above, for which it yields: (i) $Q_1= (3x_m+ x_{m+1})/4$, $Q_3=(x_{3m}+ 3x_{3m+1})/4$; (iii) $Q_1 = (x_m+ 3x_{m+1})/4$, $Q_3= (3x_{3m+2} + x_{3m+3})/4$.

**References:** K. Pearson (1920), Tippett (1925), E. S. Pearson (1926), Pearson & Adyanthāya (1928).

**Citations:** Pearson & Pearson (1931, 1932), Hojo (1933), Romanovsky (1933), Nair (1936), Paulson (1940), Kendall (1943, 1946), Bickerstaff (1947), Jacobson (1947), Pillai (1948), Godwin (1949b), Howell (1949).

* Horst, Paul (1931). A proof that the point from which the sum of the absolute deviations is a minimum is the median. *Journal of Educational Psychology* **22,** 463-464.

**Summary:** The author gives the following proof that the point from which the sum of the absolute deviations is a minimum is the median: "We have given a statistical series, $X_1, X_2, X_3, \ldots, X_m, X_{m+1}, \ldots X_n$ arranged in order of magnitude, where all the values up through $X_m$ lie below any point P, and all the values above $X_m$ lie above this point. Then the average deviation about P will be $AD_P=(1/N)[\Sigma_1^m (P-X_i) - \Sigma_{m+1}^n (P-X_i)] = (1/N)[mP-\Sigma_1^m X_i - (n-m)P + \Sigma_{m+1}^n X_i]$ [which gives ] $AD_P=(1/N) [\Sigma_{m+1}^n X_i - \Sigma_1^m X_i - (2m-n)P]$ (1). . . . Now we wish to determine P in (1) so that $AD_P$ will be a minimum. To do this we simply differentiate (1) with respect to P and equate the derivative to zero. Thus $d(AD_P)/dP = -(2m-n)/N = 0$ (2). Then solving for m in terms of n we have simply $m=(1/2)n$ (3). That is, half the values are below P and half above, which is, of course, the definition of the median. Hence that point from which the sum of the absolute deviations is a minimum is the median."

**Comments:** The author's notation would be improved by replacing N by n, since both denote the number of items in the series.

**Reference:** Yule (1911).

**Citation:** Bruen (1938).

* Insolera, Filadelfo (1931). On the oldest age. *Giornale di Matematica Finanziaria* **13** [(2) **1**], 39-44. (JFM 57, 651).

**Summary:** The author begins (p. 39) with the one-sentence summary: "The A. [author] shortly shows some remarkable consequences of the definition of oldest age, given in a previous paper of his [Insolera (1930)]". He continues (pp. 39-42): "According to some authors, the oldest age would be that value $\omega$ to which all the possible ages of life have tendency, without ever reaching it. The oldest age, as extreme, would, therefore, not be a part of the succession of the true ages of life, but it would represent a proper upper limit to such succession: that is to say, it would be the least of the values not possible. Therefore, as one cannot affirm, in a general abstract way, which may be the last of ages possible—like Steffensen [(1930)] well observes—it may, consequently, be permitted us to add that it will not be possible to say which is the first of the ages not possible. . . . We wish to avoid such imponderable notion of oldest age, that remains, in reality, far away from every satisfactory practical concentration. So much, the more we desire it, as the serious methodological inconvenience can easily be overcome if only one will recognize that the oldest age, by the fact of being extreme, is none the less a possible true and real age of life. By this, the oldest age, better than an upper limit of the possible ages, is to be considered as the highest age possible. . . . On the other hand, one cannot understand why to this conception must correspond a unique and sole finished value of the oldest age, for every group of population on all parts of the world and in all time of history: a conception which some ascribe to us, but of which we have really never thought. . . . It is just a logical synthesis of the former considerations, and admitting the continuity of the phenomenon, that we have found ourselves inducted [induced (?)] to propose as definition of the oldest age that value $\omega$ of age x such that in its closest left neighbourhood, the number of dead approaches to the number of living there considered, as x approaches to $\omega$. . . . As a first consequence of the previous definition of the oldest age, if it is admitted that the existence and continuity of a function $\ell(x)$ for persons alive at age $x<\omega$, results quite naturally, the existence of a right discontinuity of $\ell(x)$ in the point $x=\omega$, that is to say (1) $\mathrm{Lt}_{x\to\omega-0}\ell(x) = \ell(\omega) \neq 0$, (2) $\mathrm{Lt}_{x\to\omega+0}\ell(x) = 0$. . . . As a second consequence of the previous definition of the oldest age, we are able to demonstrate that force of mortality is always smaller than the unity, for $x<\omega$, and reachs [sic] the unity when x increases until $\omega$." In the remainder of the paper, the author gives a third, more complicated, consequence, and uses it to calculate the oldest age $\omega$ for each of six mortality tables.

**Reference:** Steffensen (1930).

**Citations:** Insolera (1933), de Finetti (1934), Insolera (1934, 1935), Gumbel (1936-37, 1937a).

* Kolmogoroff, A. N. (1931). La méthode de la médiane dans la théorie des erreurs. (Russian. French summary). *Recueil Mathématique de la Société Mathématique de Moscou* **38** (3-4), 47-50. (JFM 57, 1482).

**Summary:** "Let f(x) dx be the probability of an error between x and x+dx, m the median of this distribution and $m_n$ the median of the n (n=2k+1) observations. The distribution law for the quantity $\mu_n = (m_n - m)\sqrt{n}$ tends, as $n\to\infty$, toward the law of Gauss with the dispersion $\sigma_m = 1/[2f(m)]$, if only f(x) is continuous and $f(m)\neq 0$." (Compiler's translation of author's French summary, p. 50).

**Reference:** Haag (1924).

**Citation:** Gumbel (1937c).

* Logan, K. H.; Grodsky, V. A. (1931). Soil corrosion studies, 1930. Rates of corrosion and pitting of bare ferrous specimens. *Journal of Research of the National Bureau of Standards* **7,** 1-35. (Research Paper RP329).

**Summary:** The authors report the results of examinations of 1,300 specimens of ferrous pipe materials removed from 70 test locations during 1930 by the National Bureau of Standards after having been buried in various soils. The following portion (p. 5) of their report is relevant to our study: "Differences in size interfere with the comparison of materials. When small areas are concerned, other things being equal, it is probable that the deepest pit will be found on the largest area. This is illustrated in [a] table . . . which gives the average rates of penetration of the deepest pits on two materials for which both 1½-inch and 3-inch specimens were included. The averages cover specimens removed at the close of . . . 2, 4, 6, and 8 year periods, respectively. . . . In each case the 3-inch specimens showed deeper pits than the corresponding 1½-inch specimens."

**Comments:** The larger the area, the greater will be the number of pits (the sample size), and hence also the expected depth of the deepest pit (largest order statistic).

**Citation:** Logan (1936).

* von Mises, Richard (1931). *Vorlesungen aus dem Gebiete der Angewandten Mathematik. I Band. Wahrscheinlichkeitsrechnung und ihre Anwendung in der Statistik und Theoretischen Physik.* Franz Deuticke, Leipzig-Wien. Reprint, Rosenberg, New York 1945.

**Summary:** In a section (pp. 233-240) on elementary descriptive statistics, the author defines the median, quartiles, deciles, and percentiles. He defines the median of m observations as the $[(m+1)/2]^{th}$ ordered value if m is odd, the arithmetic mean of the $(m/2)^{th}$ and the $[(m+1)/2]^{th}$ ordered values if m is even. For a sample of size $m=4n$, he defines the first quartile as the arithmetic mean of the $n^{th}$ and $(n+1)^{th}$ ordered values [or sometimes simply the $n^{th}$], the second quartile as equal to the median, and the third quartile as the arithmetic mean of the $(3n)^{th}$ and the $(3n+1)^{th}$ ordered values [or the $(3n)^{th}$ ordered value itself]. On page 317, he proposes $\omega^2 = \int \lambda(x)\epsilon^2(x)dx$, where $\epsilon(x) = m\,[S(x) - V(x)]$, m is the sample size, and $\lambda(x)$ is a positive weight function, as a measure of the discrepancy between observed and theoretical cumulative distribution functions $S(x)$ and $V(x)$, respectively. On succeeding pages he finds the mean and the standard deviation of $\omega^2$. Thus he extends the work of Cramér (1928), apparently unknown to him at the time, in two ways: (1) he allows unequal weights; and (2) he finds the standard deviation as well as the mean. He goes on to show that the distribution of $\omega^2$ approaches normality as the sample size m increases, and illustrates the use of $\omega^2$ as a criterion of goodness of fit (agreement between observed and theoretical distributions) competing with the $\chi^2$-criterion of Karl Pearson. There is also a chapter on the theory of errors and adjustment of observations in which the author follows the approach of Gauss (1809, 1823).

**References:** Gauss (1809, 1823), Helmert (1872) [1907], Charlier (1910) [1920].

**Citations:** Czuber (1921), Gumbel (1933g), Kolmogoroff (1933), Rosin & Rammler (1934), Wold (1935), Smirnoff (1936), Gumbel (1937a,b), Neyman (1937a), Smirnoff (1937a), Arley & Buch (1940), von Schelling (1941), Gumbel (1942a), Wilks (1943), Smirnov (1944), Cramér (1946), Freudenthal (1946), Kendall (1946), von Mises (1947), Feller (1948).

* Pearson, Karl (editor) (1931). *Tables for Statisticians and Biometricians, Part 2.* Cambridge University Press (for the Biometrika Trustees), Cambridge, England.

**Summary:** Among the numerous tables, each with accompanying introductory material, included in this volume, the following are relevant to our subject: Tables XVIII-XX, Criteria for Rejection of Outlying Observations (Probability Integral, Chauvenet's and Irwin's Methods), pp. cv-cx (introduction) and 159-161 (tables); Tables XXI, XXI bis, and XXII, The Distribution of the Extreme Individuals and of the Range in Samples from a Normal Population, pp. cx-cxix and 162-166; and Tables XXIII-XXIV, Tables of the Distribution of Range, Median and Mid-point between Extreme Observations ('Centre') in Small Samples, pp. cxix-cxxi and 167-168.

**References:** Chauvenet (1863), Irwin (1925b), Tippett (1925), E. S. Pearson (1926), "Student" (1927), Pearson & Adyanthāya (1928), "Sophister" (1928).

**Citations:** Pearson & Pearson (1931), Shewhart (1931), Jeffreys (1932b), Pearson (1932), Rider (1933), Davies & Pearson (1934), McKay (1935), Pearson (1935), Pearson & Chandra Sekar (1936), Neyman (1937a), Snedecor (1937), Newman (1939), Simon (1941), Thompson (1941), Hartley (1942), Pearson & Hartley (1942), Knudsen (1943), Baker (1946), Cramér (1946), Kendall (1946), Mosteller (1946), Eisenhart, Hastay & Wallis (1947), Gumbel (1947), Lord (1947), Plackett (1947), Cox (1948), Epstein (1948a), Cox (1949), Shone (1949).

* Pearson, Karl; Pearson, Margaret V. (1931). On the mean character and variance of a ranked individual, and on the mean and variance of the intervals between ranked individuals. Part I. Symmetrical distributions (normal and rectangular). *Biometrika* **23,** 364-397. (JFM 57,633).

**Summary:** The authors summarize their results as follows (pp. 396-397): "This paper is to be looked upon as an Appendix to the paper by Professor Hojo [(1931)]. Approximate formulae have been obtained for finding the mean value of any rank in samples of size n taken from a normal population [and also for the standard deviation and higher moments of any rank and for the correlation of two ranks]. . . . We are thus able to give a general solution of Galton's problem [Galton (1902)], namely that of finding the mean interval between any two individuals in a ranked series, and of finding the standard error of that interval. . . . [Exact formulae are given for the above quantities for samples from a rectangular population.] The ordinary definition of the quartile when it does not fall on a rank, i.e. to take it as half the sum of the characters in the adjacent ranks, is shown to be unsatisfactory, and a method of improving it is provided. At the same time there is no special advantage in the use of the quartile, and the discovery of its standard error where it does not fall on a rank is very laborious. It is part of the old and rather vicious practice of using the 'probable error', which has given to the quartile a prominence in ranking which it does not deserve. Any pair of mirror-ranks in its neighborhood will give practically as good a value for the median, and we can go a good way out from the quartiles and continue to better our estimate of the standard deviation. The memoir also enables us to determine, when the mean is known, the error we are not likely to exceed in supposing the character of the individual in the $s^{th}$ rank to be that of the corresponding individual in the sampled population."

**Comments:** The definition of the quartiles proposed by the authors is that of the compiler [see comments on the paper by Hojo (1931)] with (n + 1) replaced by n. Either of these definitions gives more consistent results than does Hojo's definition.

**References:** Pearson (1895), Galton (1902), Pearson (1902, 1920), Tippett (1925), Neyman & Pearson (1928), Hojo (1931), Pearson (1931).

**Citations:** Pearson & Pearson (1932), Hojo (1933), Romanovsky (1933), Nair (1940b), Rajalakshman (1941), Gumbel (1942a, 1943a), Kendall (1943), Rajalakshman (1943), Kendall (1946), Bickerstaff (1947), Godwin (1949b).

* Reinkober, O. (1931). Die Zerreissfestigkeit dünner Quarzfäden. *Physikalische Zeitschrift* **32**, 243-250.

**Summary:** "In the systematic investigation of the tensile strength of thin quartz glass threads ranging in diameter from 100 to 1$\mu$ a strong increase in strength [per unit cross-sectional area] with decreasing diameter becomes evident, that is comprehended through accumulation of quantitative measurements that point to a decisive influence of surface tension on the increase in strength. The strength measured for a single thread increases with decreasing thread length, which is explained by the fact that in the structure of the material numerous derangements are present, of which the most marked tends to break first. Freshly produced threads show an appreciably greater tensile strength than old ones. The influence of surface tension is explained by the fact that these materials softened by drawing first solidify undisturbed and therefore possess a greater strength than the later solidifying and disturbed-spot permeated interior of the thread." (Compiler's translation of author's summary, p. 250.)

**Reference:** Griffith (1920).

**Citations:** Kontorova (1940), Kontorova & Frenkel (1941).

* Shewhart, W. A. (1931). *Economic Control of Quality of Manufactured Product*. D. van Nostrand Company, Inc., New York.

**Summary:** In view of the later prominence of the sample range in statistical quality control, it is somewhat surprising that the range and other measures based on order statistics play only a small role in this pioneering work on the subject. On pp. 197-199 the author discusses the use of the median and the midrange as measures of central tendency instead of the arithmetic mean. On pp. 201-205 he discusses the use of the range as a measure of dispersion instead of the standard deviation. He also studies differences between successive ordered sample values. To fill gaps in the distribution theory for these substitute measures of central tendency and dispersion, he obtains empirical distributions of them for 1000 samples each of size 4 from a normal population and compares them with theoretical and empirical distributions of the mean and standard deviation, giving mean, standard deviation, skewness, and flatness (kurtosis or elongation) for each distribution. He observes (p. 205) that control limits based on the sample range are less robust to departures from normality of the underlying distribution than are those based on the standard deviation.

**References:** Jevons (1874), Bowley (1901) [1926], Edgeworth (1905, 1911), Yule (1911) [1930], Brunt (1917), Keynes (1921), Fisher (1922), Whittaker & Robinson (1924), Coolidge (1925), Tippett (1925), Burgess (1927), Jordan (1927), Neyman & Pearson (1928), Rider (1929), Steffensen (1930), Pearson (1931).

**Citations:** Dodge (1932, 1933), Davies & Pearson (1934), Pearson (1935), Pearson & Haines (1935), Eisenhart (1939), Grant (1940), Simon (1941), Wilks (1941, 1942), Deming (1943), Knudsen (1943), Wilks (1943), Wing, Price & Douglass (1944), Grant (1946), Kendall (1946), Howell (1949).

* Steffensen, J. F. (1931). Notes on the life table and the limit of life. *Journal of the Institute of Actuaries* **62** (1), 99-104; discussion, 104-108. (JFM 57, 653).

**Summary:** In this paper the author responds as follows (pp. 99-101) to criticism by Lidstone (1930) of his earlier lectures [Steffensen (1930)]: ". . . In order to distinguish clearly between the life table $\ell_x$ and a number of *persons* followed from their birth to their death, I shall denote the latter by $\Lambda_x$; so that $\ell_x$ is a theoretical function but $\Lambda_x$ a number of real people and therefore necessarily integral and discontinuous. I propose first to investigate the relation between $\Lambda_x$ and $\ell_x$. I presume that Mr. Lidstone and I can agree about the following definition of the probability of death: If out of $E_x$ exposed $\theta_x$ die within a year, then $\theta_x/E_x$ is an observed relative frequency of death; and $q_x$ is the limit to which this relative frequency is supposed to tend if the number of exposed $E_x$ increases indefinitely. Assuming then that we agree about the definition of 'probability', the function $\ell_x$ may be derived from the numbers $\Lambda_x$ as follows. Out of $\Lambda_0$

new-born individuals, $\Lambda_x$ are according to the definition of $\Lambda_x$ alive after x years. It is important to note that it is not at all necessary for x to be an integer in order that $\Lambda_x$ shall exist. The quotient $\Lambda_x/\Lambda_0$ is a relative frequency which, for increasing $\Lambda_0$, is supposed to tend to a limit. This limit is the probability that a new-born baby will be alive after x years. Hence we have, in symbols ${}_xp_0 = \lim_{\Lambda_0 \to \infty} \Lambda_x/\Lambda_0$ (1). The function $\ell_x$ is essentially the same thing as ${}_xp_0$. But as in the applications not $\ell_x$ itself but only $\ell_{x+t}/\ell_x$ matters, we may multiply $\ell_x$ by an arbitrary constant and write $\ell_x = k\ {}_xp_0$ (2). Hence $\ell_x$ is the *mathematical expectation* of lives existing after x years out of k new-born individuals, and there is evidently no reason why this expectation should always be an integral number. But seeing that k may be chosen at liberty, it has become customary in England (though not in the writer's country [Denmark]) to choose this constant so that the radix of the table becomes a convenient integer, say 100,000. In that case $\ell_x$ may, throughout the greater part of the table, with sufficient *approximation* be represented by an integer. But as far as I can see, this is just what happens. On p. 341 of Mr. Lidstone's notes we find as assertion (1): 'It is impossible for $\ell_x$ to be a continuous curve diminishing by infinitesimal decrements;' and immediately after, as assertion (2), it is considered an objection that if an 'Oldest Age' does not exist, 'so that $\ell_x$ exists however great x may be, then we must eventually reach a point at and beyond which $\ell_x$ represents a fractional number of lives.' Assertion (1) would have been correct, if instead of $\ell_x$ had been written $\Lambda_x$. It would, in my opinion, also be correct, if after $\ell_x$ the word 'not' were inserted. As it stands, it is not difficult to show that it is incorrect, if the above definition of $\ell_x$ is accepted, as I think it must be. . . . The presence of an *arbitrary factor* k in (2) is a permanent reminder of the fact that we have no longer to do with a group of people actually dying out; for by a suitable choice of k we can always arrange that the value of $\ell_x$ at any specified age is any number we please. This remark has a certain bearing on the question of the existence of an 'Oldest Age' which can under no circumstances be exceeded. For while a table finishing off with, say $\ell_{200} = 1$, $\ell_{201} = 0$ may on the surface look a possible one, a table terminating with $\ell_{200} = 1{,}000{,}000$, $\ell_{201} = 0$ would look quite impossible, even if it is in reality *the same table*, only with the radix multiplied by one million. . . ." In his rejoinder, Lidstone (p. 104) remarks: "There is no doubt room for both doctrines, and I wish only for liberty to adopt whichever seems the more suitable."

**References:** Lidstone (1930), Steffensen (1930).

**Citations:** Gumbel (1935d, 1936-37, 1937a).

* "Student" [W. S. Gosset] (1931). Yield trials. *Baillière's Encyclopaedia of Scientific Agriculture* **2,** 1342ff. Reprinted in *"Student's" Collected Papers* (ed. E. S. Pearson and John Wishart), pp. 150-168. Cambridge University Press, Cambridge, England, 1942.

**Summary:** Most of this article is irrelevant to our study, but the following passage (pp. 164-165 of reprint) is relevant: "It should be noticed . . . that the formula which we have used in comparing [strains] B and D is that which it is correct to use when there are but two means to compare: it would be right to use it, for example, if we have a number of trials of B and D in different places or seasons, and we wish to examine the whole series of comparisons of B and D. If, however, there is no particular reason why we should compare these two rather than any other pair, it is clear that the chance of obtaining a large difference between some pair or other is greater the larger the number of possible pairs. To meet this Fisher suggests that the mean of each strain should be compared with the general mean, and that the strains should be divided in this way into (a) those significantly greater than the mean; (b) those not differing significantly from the mean; and (c) those significantly less than the mean. The appropriate standard deviation to use, if $\sigma$ be the standard deviation of the random error of a single plot and there are n repetitions and m strains, is $\sigma\sqrt{m-1}/\sqrt{mn}$."

**Comments:** The passage quoted above shows that the author had already an awareness of the problem of multiple comparisons, but as yet no clear idea as to how to solve it—see Pearson (1939).

**Reference:** Yule (1911).

**Citations:** Pearson (1939), Kendall (1946).

* d'Addario, Raffaele (1932). Intorno alla curva dei redditi di Amoroso. *Rivista Italiana di Statistica* **4,** 89-108.

**Summary:** The author discusses the equation for the distribution of income proposed by Amoroso (1925): $y = ke^{-\gamma(x-h)^{1/s}}(x-h)^{(p-s)/s}$ ($h < x < \infty$; k, h, $\gamma$, p essentially positive constants). He recalls that Amoroso pointed out that if $\gamma = h = 0$ and $s = 1$, Amoroso's equation reduces to the first approximation of Pareto; if $s = 1$, it reduces to Pearson's type III, which corresponds to Pareto's second approximation; if $s = -1$, it reduces to Pearson's type V. If $p - s \leq 0$, the curve is monotone decreasing; if $p - s > 0$, it has a maximum (mode) at the point $\theta = h + [(p-s)/\gamma]^s$. Let N be the total number of incomes, C be their aggregate amount, and $A = C/N$ be the mean income. Amoroso showed that $N = k|s|\Gamma(p)/\gamma^p$ and $A = h + \Gamma(p+s)/\gamma^s\Gamma(p)$. If $p - s > 0$, an index of asymmetry or of *concentration of income* is given by $\rho = (A - \theta)/(A - h) = 1 - (p - s)^s\Gamma(p)/\Gamma(p + s)$, which varies between a lower limit of 0 and an upper limit of 1. Amoroso, in order to have an index valid for $p - s \leq 0$ as well as for $p - s > 0$, defined a second quantity $q = 1 - 2^{(p-s)}[\Gamma(p)\Gamma(p+s)]^{1/s}$, which he called *determinant of the curve*. The author refers to p and q as measures of the concentration of income earners (Ital., redditieri) in contrast to the measures of concentration of income (Ital., redditi) $R = \pm[(A-h)/A][1-2H(p,s)]$ (Gini's ratio of concentration) and $S = \pm[(A-h)/A][I(\epsilon,p+s-1) - I(\epsilon,p-1)]$, where H(p,s) is the value defined by Amoroso and $I(\cdot,\cdot)$ is the incomplete Gamma function ratio.

**Comments:** Expressing S in terms of the incomplete Gamma function ratio is one of the few original contributions of the author; most of the remainder of the paper is merely a summary (albeit an excellent one) of the work of Amoroso (1925).

**References:** Pareto (1897), Lorenz (1905), Persons (1909), Gini (1914a), Pietra (1915), Ricci (1916), Amoroso (1925), d'Addario (1930).

**Citation:** Castellano (1935).

* Craig, Allen T. (1932a). On the distributions of certain statistics. *American Journal of Mathematics* **54,** 353-366. (JFM 58, 1163). Abstract, *Bulletin of the American Mathematical Society* **37** (1931), 26. (JFM 57, 664—listing only).

**Summary:** The author proves the following results (pp. 364-365) concerning the distributions of sample median, first quartile, and range: "Let f(x) be the probability function of the variable x and let $\phi_4(\xi)$ be the probability function of the median $\xi$ of samples of $n = 2m + 1$. If f(x) is a probability function of the third kind [range (0,a)], $\phi_4(\xi)$ is a probability function of the third kind and $\phi_4(\xi) = [(2m+1)!/(m!)^2][\int_0^\xi f(t)dt]^m[\int_\xi^a f(t)dt]^m f(\xi)$. If f(x) is a probability function of the first or second kind [range $(-\infty,\infty)$ or $(0,\infty)$], $\phi_4(\xi)$ remains unchanged except that the limits of integration are $-\infty$, $\xi$ and $\xi$, $\infty$ respectively for a function of the first kind and 0, $\xi$ and $\xi$, $\infty$ respectively for a function of the second kind. The probability function of the lower quartile $\eta$ of samples of $n = 4m + 1$ drawn from a universe represented by f(x) where f(x) is of the second kind is clearly $\phi_5(\eta) = [(4m+1)!/(m!)(3m!)][\int_0^\eta f(t)dt]^m[\int_\eta^x f(t)dt]^{3m} f(\eta)$. Similar forms of $\phi_5(\eta)$ hold for probability functions of the first and third kinds. It is obvious that the distribution of any statistic which is defined as that value of the variate which exceeds, and is exceeded by, specified numbers of items in the sample, may be determined in this manner. . . . Let f(x) be the probability function of the variable x and let $\phi_6(W)$ be the probability function of the range W of samples of n items. If f(x) is a probability function of the third kind, $\phi_6(W)$ is a probability function of the third kind and $\phi_6(W) = n(n-1)\int_W^a f(x)f(x-W)[\int_{x-W}^x f(x)dx]^{n-2}dx$."

**Comments:** The author does not state his definition of the lower quartile, but it is obvious from the expression he gives for its probability function that he defines it, for samples of size $4m + 1$, as the $(m+1)^{th}$

order statistic. For alternate definitions, see the summaries of and comments on the papers by Hojo (1931) and Pearson & Pearson (1931).

**References:** Laplace (1812), Helmert (1876b), Czuber (1891a), Dodd (1922), Neyman & Pearson (1928), Rider (1929).

**Citations:** Olds (1935), Paulson (1940), Mood (1941), Wilks (1943), Bickerstaff (1947), Wilks (1948).

* Craig, Allen T. (1932b). Distributions of functions of middle items (abstract). *Bulletin of the American Mathematical Society* **38,** 813. (JFM 58, 580—listing only).

**Summary:** "In the present paper, laws of probability are determined in accord with which various functions of the middle n items of samples consisting of (2k+n) items are distributed. Among the functions considered are the arithmetic mean, the average of the extreme values, and the range. An experimental study of certain aspects of this problem has been made by E. S. Pearson [& Adyanthāya (1928)]." (Abstract, p. 813).

**Reference:** Pearson [& Adyanthāya] (1928).

* Dodge, H. F. (1932). Statistical control in sampling inspection. *American Machinist* **76,** 1085-1088, 1129-1131.

**Summary:** After explaining the conventional variables criterion based on the sample mean and the sample standard deviation, the author continues (p. 1130): "There are many other variables criteria that might be chosen, involving the average alone or the average and standard deviation jointly. For example, one that has proved to be quite satisfactory . . . and easy to use in practice provides a maximum allowable 'range' for each magnitude of observed average, where the range is the difference between the largest and the smallest observation in the sample and is used as an estimate of the standard deviation." He also presents an attributes criterion which accepts a lot if no test specimen falls below the minimum limit.

**Reference:** Shewhart (1931).

**Citations:** Dodge (1933), Daly (1946), Grant (1946).

* de Finetti, Bruno (1932). Sulla legge di probabilità degli estremi. *Metron* **9** (3-4), 127-138. (JFM 57, 1485).

**Summary:** "We consider the law of probability of the maximum (minimum) value among those of n independent random variables following the same law, and study especially its asymptotic behavior as n increases." (Compiler's translation of author's summary, p. 127). In Table I (pp. 135-136) the author gives the probability density function of the maximum value of n independent random variables following the reduced normal law for n=1,2,...,10. In Table II (p. 137) he gives the median value of the maximum among n independent random variables following the reduced normal law for n=1(1)50(10)500(100)1000, (1,1.2,1.5,2,2.5,3,4,5,6,7,8,9) $\times$ $10^{3(1)8}$.

**References:** Bertrand (1889), Lévy (1925).

**Citations:** de Finetti (1934), Gumbel (1934g, 1935a, 1935-36), Smirnoff (1935), Gnedenko (1943), Kendall (1946).

* Galvani, L. (1932). Sulle curve di concentrazione relative a carratteri non limitati e limitati. (French summary). *Metron* **10,** 61-73.

**Summary:** The author writes (French summary, p. 73, as translated by the compiler): "Professor Gini [(1930, 1932)] has shown that, given a certain distribution of a quantitative character, its concentration curve, and therefore also the concentration ratio, vary according to whether one assumes that the character in question is unbounded, or whether it is bounded below or above or on both sides. The same author has shown that in the first case the concentration ratio is determined by $\Delta/2A$, where $\Delta$ represents the mean difference and A the arithmetic mean of the character, whereas in the last three cases, in order to calculate the value of R exactly, it is necessary to apply to the denominator 2A a suitable correction coefficient. In this note, we first formulate several observations on the influence which the concentration ratio undergoes when one applies certain transformations to the given distribution. A particular study is then explained on the analytical and geometric problem of transforming the concentration curve, relative to a bounded character, into the corresponding reduced curve, that is to say, into a concentration curve having the same concentration ratio, but which one can consider as relating to an unbounded quantitative character."

**References:** Lorenz (1905), Gini (1914a, 1930, 1932).

**Citations:** Castellano (1933a), Wold (1935), Kendall (1946).

* Garver, Raymond (1932). Concerning the limits of a measure of skewness. *Annals of Mathematical Statistics* **3,** 358-360. (JFM 58, 1165).

**Summary:** The author gives a different proof than that of Hotelling & Solomons (1932) that the measure of skewness s, defined by s=(mean−median)/ standard deviation, cannot be greater than unity in absolute value.

**Reference:** Hotelling & Solomons (1932).

* Gibrat, Robert (1932a). Sur l'ajustement mathématique des courbes de débits d'un cours d'eau. *Comptes Rendus de l'Académie des Sciences de Paris* **194,** 843-845; *Revue Générale de l'Électricité* **31,** 664-665. (JFM 58, 1161).

**Summary:** Let t be the number of days during a period of T days for which the observed discharge of a stream is less than or equal to q. The author proposes, as an adjustment formula for the cumulative distribution function of the discharge, $t/T = (1/\sqrt{\pi}) \int_{-\infty}^{z} e^{-z^2} dz$, $z = a \log(q-q_0) + b$, where a,b, and $q_0$ are three constants which are determined so as to make the adjustment the best possible. The author calls this the law of proportional effect, but it is really nothing but the log-normal law; the author himself observes that the frequency curve of the logarithms of discharges in excess of the minimum discharge $q_0$ is a normal curve. The author proposes as an index of irregularity of a stream the quantity 100/a. He states that it is easy to show that one has the relation $\log [(Q-q_0)/(S-q_0)] = 1/4a^2 \log e$, where Q is the mean discharge and S is the median discharge. He gives an example (daily discharges during the year 1917 of the Truyère at the bridge of Cadène) in which the law of proportional effect fits the data much better than the usual parabolic law $q = A+Kt^n$.

**Reference:** Kapteyn (1903), Gibrat (1930).

**Citations:** Gibrat (1932b), Coutagne (1937).

* Gibrat, Robert (1932b). Aménagement hydroélectrique des cours d'eau: Statistique mathématique et calcul de probabilités. *Revue Générale de l'Électricité* **32** (15-16), 493-501.

**Summary:** The author applies the methods of mathematical statistics and probability to problems posed by hydroelectric harnessing of streams. He calls attention to the fact that the life of a dam depends on the maximum flood which it can withstand without damage. Another problem he considers is the storage capacity that must be provided on a river in order to assure, with a given probability, a specified [minimum] production of energy. He shows that data on daily discharges of a stream and monthly rainfall, as well as other types of data (economic, demographic, etc.) are well fitted by the law of proportional effect (lognormal law). He shows how to fit the three constants $a,b,q_0$ of this law [see Gibrat (1932a)] and apply it to problems in hydraulic engineering. Important roles are played by the median discharge and the characteristic largest and characteristic smallest (low-water mark) discharges. He defines the latter as the values which the daily discharge exceeds or falls below ten days per year; they are sometimes approximated by taking the arithmetic means of the 30 highest and of the 30 lowest daily discharges during the year.

**References:** Gibrat (1930), d'Addario (1931), Gibrat (1932a).

**Citations:** Foster (1934), Grassberger (1936), Gumbel (1937-38, 1941a, 1943c).

* Gini, Corrado (1932). Intorno alle curve di concentrazione. (French summary). *Metron* **9** (3-4), 3-76.

**Summary:** The author discusses various measures of concentration, including (1) the mean difference divided by twice the arithmetic mean, (2) the mean deviation from the arithmetic mean divided by twice the arithmetic mean, and (3) the mean deviation from the median divided by twice the arithmetic mean. He gives a number of references on measures of variability (including the mean difference) and of concentration.

**References:** Bowley (1901) [1920], Dettori (1912), Gini (1912), Czuber (1914), Gini (1914a), Pietra (1915), Savorgnan (1915), Ricci (1916), Dalton (1920), Julin (1921), Amoroso (1925), Gumbel (1927, 1928), Gini (1930), March (1930), von Bortkiewicz (1931), Galvani (1931).

**Citations:** Galvani (1932), Pietra (1932a), Castellano (1933a, 1935), Pietra (1935a,b), Wold (1935), Gini (1936), Nair (1936), Davis (1941), Kendall (1946).

* Grassberger, Hans (1932). Die Anwendung der Wahrscheinlichkeitsrechnung auf die Wasserführung der Gewässer. *Die Wasserwirtschaft* (Vienna) **25,** 16-22, 31-33, 55-60.

**Summary:** The author shows, as other writers have done, that the lognormal distribution provides a good fit to data on the discharge of a large river. He goes a step further than previous writers by introducing the hypothesis that the largest value of a sample from a lognormal distribution itself follows a lognormal distribution. He does not attempt a mathematical proof of this hypothesis, but he does verify empirically that the lognormal distribution fits quite well the data for the annual floods (largest daily discharges) of three Austrian rivers during the period 1897-1930.

**References:** Fechner (1897), Czuber (1903).

**Citations:** Grassberger (1933, 1936), Gumbel (1936a), Coutagne (1937), Gumbel (1937-38).

* Gumbel, E. J. (1932). L'âge limite. (Résumé). *Bulletin de la Société Mathématique de France* (Paris) **60** (3-4), 34-35.

**Summary:** "The problem of the limiting age, for the solution of which very artificial methods have been proposed, admits a solution based on the calculus of probabilities when one regards the life table as a distribution of deaths according to the ages at the moment of death and assumes only that this

distribution tends asymptotically to zero. By applying a formula established by von Mises, one calculates the probable value of the limiting age by $\ell(x) = 1/N$ where $\ell(x)$ is the number surviving and N is proportional to the number of observations which have been employed in the construction of the table. But, moreover, one can calculate by the same formula an approximate value for the most probable limiting age. Then one can assume that the distribution of the limiting age, that is that the probability that this age has a given value, is Gaussian. By applying the theory of Lexis to the life table, that is by considering the end of the distribution of deaths according to ages as a Gaussian distribution, one can calculate the probable value of the limiting age and its mean error. The limiting age increases very slowly with N, while its mean error decreases. When one compares these theoretical values with the observations, one obtains very satisfactory results. This theory can also be applied to the probable age of the youngest of the m oldest, for example of the ten oldest persons. For the distribution of this limiting age, the probable value is again equal to the most probable value. The calculation of the mean error of this age leads to the law of rare events. The mean error decreases as the number m increases. Finally, by reversing the question, one can calculate the probable number of those who, for a given population, attain the age of 100 years, ordinarily considered as the limiting age. In all these calculations, one uses only the number of observations of the normal age $\xi$, the probability that a new-born will attain this age $\ell(\xi)$, and the expectation of life at this age $E(\xi)$, values which are given in every conveniently calculated life table." (pp. 34-35, compiler's translation).

**Citations:** Gumbel (1933f,g, 1934d,f,g, 1935a,c, 1937a, 1939f).

* Hotelling, Harold; Solomons, Leonard M. (1932). The limits of a measure of skewness. *Annals of Mathematical Statistics* **3,** 141-142. (JFM 58, 1165).

***Summary:*** The authors show that the value of the measure of skewness s=(mean−median)/(standard deviation) necessarily lies between $-1$ and $1$.

**Citation:** Garver (1932).

* Jeffreys, Harold (1932a). An alternative to the rejection of observations. *Proceedings of Royal Society of London* (A) **137,** 78-87. (JFM 58, 569).

**Summary:** "A method is given for dealing with sets of observations, appreciable fractions of which are affected by abnormal errors. The probability of an error is taken to be given jointly by two normal [Gaussian] laws, one for the normal and the other for the abnormal errors, and a method of solution for the five unknowns is provided. An approximate solution may also be obtained by a method of weighting, the weight of an observation being a continuous function of its deviation." (Author's summary, p. 87). The author states (p. 79): "The distribution of the probability of error is . . . taken to be $f(x_r)=(1-m)(h/\sqrt{\pi})\,e^{-h^2(x_r-x)^2} + m(k/\sqrt{\pi})e^{-k^2(x_r-x-y)^2}$. This expresses the conditions (1) that if x is the true value the probability, given x, that an observation will lie between $x_r$ and $x_r + dx_r$ is $f(x_r)dx_r$; (2) the probability that an observation will be affected by the abnormal error is m; (3) the normal observations follow a normal law with modulus of precision h; (4) the abnormal ones follow another normal law with modulus k, but are subject also to an unknown systematic error y. Our problem is to find the most probable value of x, and incidentally y, from a given set of observed values $x_1, x_2, \ldots, x_n$." The author's method makes no use of the order of the observations; he does, however, discuss the methods of Peirce and Chauvenet, and he recommends the latter in cases in which there is a single discordant observation.

**Remarks:** Rider (1933) states (p. 19): "The principal advantage of Jeffreys' method seems to be that large deviations have little weight in determining the value of x, and that small deviations have little weight in determining x + y. Also the weight of an observation is a continuous function of its deviation."

**Reference:** Chauvenet (1863).

**Citation:** Rider (1933).

* Jeffreys, Harold (1932b). On the theory of errors and least squares. *Proceedings of the Royal Society of London* (A) **138,** 48-55. (JFM 58, 1160).

**Summary:** The following portion (pp. 48-49) is relevant: "The normal law of error is supposed to hold, but the true value x and the precision constant h are unknown. Two measures are made: what is the probability that the third observation will lie between them? The answer is easily seen to be one-third. For the law says nothing about the order of occurrence of errors of different amounts, and therefore the middle one in magnitude is equally likely to be the first, second, or third made (provided, of course, that we know nothing about the probable range of error already)." The author proceeds to find the distribution of prior probability which will give this result.

**Reference:** K. Pearson (1931).

**Citations:** Bartlett (1933), Fisher (1933), Jeffreys (1933), Fisher (1934), Bond (1935), Thompson (1936).

* Luyten, Willem J. (1932). Notes on stellar statistics. V: On the use of the first Laplacean error curve. *Proceedings of the National Academy of Sciences (USA)* **18,** 360-365. (JFM 58, 569).

**Summary:** The author points out that the Gaussian law of error, which was once accepted without question, is no longer taken for granted. Wilson & Hilferty (1929), for example, have shown that data originally used by C. S. Peirce to show that the normal law does hold cannot be satisfactorily represented by the normal law. The author considers data on the differences of pairs of measures of the distance of double stars. He computes mean, standard deviation, median, quartiles, and quartile deviation, plots the data on probability paper, and fits normal, Charlier Type A, Pearson Type IV, and Laplace's first error curves to the data. He concludes that Laplace's first error curve fits much better than his second (the normal curve). He points out (p. 365) that, as a corollary to the use of the first Laplacean curve, it is no longer the arithmetic mean and the standard deviation but the median and arithmetic mean error [mean deviation from the median] that are the significant constants of the distribution.

**Reference:** Wilson & Hilferty (1929).

* Pearson, Egon S. (1932). The percentage limits for the distribution of range in samples from a normal population. ($n \leq 100$). *Biometrika* **24,** 404-417. (JFM 58, 1169).

**Summary:** The author begins (p. 404) with the following summary: "The Table A given on p. 416 below represents an attempt to summarize in most convenient form for practical use the recent work on the distribution of range in samples from a normal population. It deals only with the case of samples of 100 or less. The accompanying discussion may be divided into three parts: (1) The method of computation of Table A. (See p. 416). (2) Experimental checks on the adequacy of the approximations involved. (3) Illustrations of the use of Table A. In addition to the value of mean and standard deviation, of which the former has already been completely and the latter partially tabled, the present Table gives certain percentage limits, namely the values of the range which will (a) not be attained, and (b) be exceeded, in 0.5%, 1%, 5%, and 10% of random samples. The unit is throughout the population standard deviation." On pp. 405-407, the author adds: "The complete set of values of $\bar{w}$ was given by Tippett [(1925)], but $\sigma_w$ had only been computed for $n = 2,3,4,5,6,10,20,60,100$. Three additional values were therefore computed at $n = 30$, 45 and 75 by the same process of cubature as that employed by Tippett. . . . From this

framework the intermediate values of $\sigma_w$ shown in Table A [3DP] were readily obtained . . . [and] finally the limits, $w_p$."

**References:** Tippett (1925), E. S. Pearson (1926), "Student" (1927), K. Pearson (1931).

**Citations:** Dodge (1933), McKay & Pearson (1933), Rider (1933), Davies & Pearson (1934), McKay (1935), Pearson & Haines (1935), Sukhatme (1936), Snedecor (1937), Newman (1939), Hartley (1942), Pearson & Hartley (1942), Kendall (1943, 1946), Egudin (1947), Elfving (1947), Grubbs & Weaver (1947), Lord (1947).

* Pearson, Karl; Pearson, Margaret V. (1932). On the mean character and variance of a ranked individual, and on the mean and variance of the intervals between ranked individuals. Part II. Case of certain skew curves. *Biometrika* **24,** 203-279. (JFM 58, 1170).

**Summary:** After some preliminaries on the hypergamma and hyperbeta functions [including a 12DP table of dibeta, tribeta, tetrabeta, and pentabeta functions for n = 1(1)100], a detailed study is made of the moments and product moments of order statistics, intervals between order statistics, medians, quartiles, and interquartile distances for the exponential distribution. This is followed by a study, in somewhat less detail, of the same quantities for various members of the Pearson family of frequency curves satisfying the biquadratic $\beta_1(8\beta_2 - 9\beta_1 - 12)(\beta_2+3)^2 - (10\beta_2 - 12\beta_1 - 18)^2(4\beta_2 - 3\beta_1) = 0$. For the exponential distribution, finite series are given for the moments; e.g., the mean of the $q^{th}$ order statistic of a sample of size n is given as $1/n + 1/(n-1) + \ldots + 1/(n-q+1)$. On p. 278 the authors state: "Certain broad principles . . . arise from our work. For example: (a) The correlation between adjacent rank-variates is high, but the correlation between adjacent (and a fortiori non-adjacent) rank-intervals is small, and for many purposes negligible. (b) The partial correlation of any two rank-variates, or any two rank-intervals for a constant rank-variate or a constant rank-interval lying between them is zero. (c) The order of the variabilities of rank-intervals as measured by their standard deviations is much the same as the order of the intervals themselves. There is equality in the case of the exponential curve, and this property extends approximately for a considerable range on either side of it. (d) Galton's Ratio, namely 2 to 1, for the ratio of the first rank-interval to the second in the case of the end of the curve with lesser frequency [exact for the exponential distribution] is approximately true for a large number of curves. (e) In samples from a curve of finite range the correlations of adjacent interrank intervals are negative; in samples of one with unlimited range they are positive. (f) In cases where there is much predominance of mediocrity, the interval between the first and second ranks may be ten or more times the interval between mediocre individuals. . . ."

**References:** Whitworth (1867) [1901], Pearson (1895), Galton (1902), Pearson (1920), Neyman & Pearson (1928), Hojo (1931), Pearson & Pearson (1931).

**Citations:** Nair (1940b), Gumbel (1942a, 1943a), Rajalakshman (1943), Kendall (1946).

* Pietra, Gaetano (1932a). Nuovi contributi alla metodologia degli indici di variabilità e di concentrazione. *Atti del Reale Istituto Veneto di Scienze, Lettere ed Arti* **91** (2), 989-1008.

**Summary:** The author begins with a review of the controversy between von Bortkiewicz on the one hand and Gini, Savorgnan and himself on the other [see discussion of the paper by von Bortkiewicz (1931)] concerning priority of work on the mean difference and related measures of variability and concentration. Then he gives graphical representations of the simple mean deviation from the arithmetic mean $'S_A$, the simple mean deviation from the median $'S_M$ and the mean difference with repetition $\triangle_R = [(n-1)/n]\triangle$, where $\triangle$ is the mean difference without repetition. He presents the simplified formula of de Gleria (1929) for computing $\triangle_R$ (and hence also $\triangle$). He points out that the graphical representations do not extend to the case of grouped data and that de Gleria's formula requires modification in this case [see de Gleria (1930)]. He then turns to consideration of the concentration curve and the concentration ratio.

Let $y_1 \leqslant y_2 \leqslant \ldots \leqslant y_n$ be a series of ordered values and let $\Sigma_{i=1}^{r} y_i / \Sigma_{i=1}^{n} y_i = q_r$ and $r/n = p_r$. If the points $(p_r, q_r)$ $(r = 1, 2, \ldots, n)$ are plotted in Cartesian coordinates and connected by a smooth curve, this curve is the concentration curve. If n is sufficiently large and if the values of y are given by $y = f(x)$, where $x = i/n$, then the concentration curve is given by $\phi(x) = \int_0^x f(x)\,dx / \int_0^1 f(x)\,dx$. If $\Delta$ is the mean difference of the values $y_i$ and A is the arithmetic mean, then $\Delta/2A = 1 - 2\int_0^1 \phi(x)\,dx$ and this ratio, called the concentration ratio R, is equal to twice the area between the concentration curve and the line of equidistribution. The author gives three expressions R, R′ and R″ for the concentration ratio of grouped data; R is the exact value under the assumption that all the values in each class are concentrated at the class mark, R′ is an approximation under the assumption that the sum of the values in each class is known, and R″ is an approximation under the assumption that the values in each class are equally spaced between the class limits. He also gives a graphical method of approximation to R. He points out that the tangent to the concentration curve at the arithmetic mean A is parallel to the line of equidistribution (slope 1) and that the slope of the tangent at the median M is equal to M/A. He applies his results to data on real estate transfers in various countries.

**References:** Gini (1912), Czuber (1914), Gini (1914a), Pietra (1915), Savorgnan (1915), Ricci (1916), Amoroso (1925), Gumbel (1927), de Gleria (1929, 1930), von Bortkiewicz (1931), Galvani (1931), Gini (1932), Pietra (1932b).

**Citations:** Dominedò (1934), Pietra (1935a,b), Kendall (1946).

* Pietra, G. (1932b). Contributions à la méthodologie du calcul du rapport de concentration. *Bulletin de l'Institut International de Statistique* **26** (2), 568-574.

**Summary:** This article is essentially a condensed version of the preceding one [Pietra (1932a)], with the polemics and certain other material omitted and some additional material on the construction of the concentration curve and on the graphical method of approximating the concentration ratio R included.

**References:** Gini (1914a), Pietra (1915), Ricci (1916), von Bortkiewicz (1931).

**Citations:** Pietra (1932a, 1935a).

* Alexandrov, A. P.; Žurkov, S. N. (1933). *The Phenemenon of the Brittle Rupture (Russian). Moscow-Leningrad.*

**Summary:** The authors offer a qualitative explanation of the fact that the average value of the brittle strength does not remain constant, but depends in a regular way upon the size of the specimens tested, diminishing as the latter increases; namely, that the dependence of the strength of a fiber upon its diameter is due to the inhomogeneous structure of the material.

**Comments:** The compiler has been unable to obtain a copy of this book, and has therefore had to depend on information given by other writers, especially Kontorova & Frenkel (1941) [Frenkel & Kontorova (1943)], who credit the authors with being the first to remark that the observed mechanism [noted by earlier writers, including Griffith (1920)] can be interpreted in a purely statistical way: the larger the size of the specimen, the more frequently will particularly "dangerous" defects [surface cracks] be encountered in it.

**Citations:** Kontorova (1940), Kontorova & Frenkel (1941), Davidenkov, Shevandin & Wittmann (1947).

* Bartlett, M. S. (1933). Probability and chance in the theory of statistics. *Proceedings of the Royal Society of London* (A) **141**, 518-534. (JFM 59, 1207—listing only).

**Summary:** The author recalls that Jeffreys (1932b) has stated that the probability that the third of three observations from a normal population shall lie between the other two, when the population mean and standard deviation are unknown, is 1/3, while Fisher (1933) has commented that *"for any particular population*, the probability will generally be larger when the first two observations are far apart than when they are near together," but that the probability that $x_3$ lies between $x_1$ and $x_2$, averaged over all possible values of $x_1$ and $x_2$, is 1/3. The author observes that Jeffreys and Fisher are not talking about the same thing, and their results bear little relation to each other. He proceeds to comment further concerning Jeffreys' results, as follows (p. 527): "For he [Jeffreys] says 'the law says nothing about the order of occurrence of errors of different amounts, and therefore the middle one in magnitude is equally likely to be the first, second, or third made (provided, of course, that we know nothing about the probable range of error already).' But even if we know all about the distribution of error beforehand, the three observations will be different in value, and it is still equally likely that the middle observation should happen to be the first, second, or third made. Thus, whether we know the range of error or not, the probability that the *middle* one in magnitude of three observations was the *third* one made is 1/3. But this tells us nothing about the probability that, given two observations, a *third* observation will be the *middle* one; as we have seen, if we know the range of error, this probability will be a function of the values of the first two observations; if we do not know the range of error, its value is still undetermined. In a footnote (pp. 527-528), the author states that Jeffreys agrees that his original argument is hardly sufficient, and gives his own interpretation of the essential point of Jeffreys' argument, which is that if the range of error is completely unknown, the probability that $x_3$ lies between $x_1$ and $x_2$ is independent of $x_1$ and $x_2$ (and in particular of $x_1 - x_2$), which means that "if we were completely ignorant of the range of error beforehand, the apparent magnitude of $x_1 - x_2$, since it depends on what scale we have adopted, cannot give us any information whether $x_3$ is likely to fall within the interval $(x_1, x_2)$ or not."

**References:** Gauss (1809), Venn (1866) [1888], Keynes (1921), Fisher (1922), Jeffreys (1932b), Fisher (1933).

**Citations:** Fisher (1934), Jeffreys (1934), Thompson (1936), Kendall (1946).

* Baticle, Edgar (1933a). Le problème de la répartition. [I]. *Comptes Rendus de l'Académie des Sciences de Paris* **196,** 1945-1946. (JFM 59, 508).

**Summary:** The problem considered is that of determining the probability $\omega_k^\alpha$ that k of n non-negative variables $x_i$ $(i = 1,2, \ldots, n)$ such that $x_1 + x_2 + \ldots + x_n = a$ are greater than a quantity $\alpha a$ $(\alpha < 1)$. Instead of continuous variables, the author initially considers m objects and seeks the probability $\bar{\omega}_k^p$ that, these m objects having been divided into n lots, k of these lots contain more than p objects. After passing to continuous variables by setting $p = \alpha m$, where $m \to \infty$ the author obtains [see note below] the following system of equations giving the probabilities $\omega_k^\alpha, \omega_{k+1}^\alpha, \omega_{k+2}^\alpha, \ldots$ of having k, k+1, k+2, ... variables greater than $\alpha a$: $\binom{n}{k}(1-k\alpha)^{n-1} = \omega_k^\alpha + \binom{k+1}{k}\omega_{k+1}^\alpha + \binom{k+2}{k}\omega_{k+2}^\alpha + \ldots, \binom{n}{k+1}[1-(k+1)\alpha]^{n-1} = \omega_{k+1}^\alpha + \binom{k+2}{k+1}\omega_{k+2}^\alpha + \ldots, \binom{n}{k+2}[1-(k+2)\alpha]^{n+1} = \omega_{k+2}^\alpha + \ldots, \ldots$. By setting $k = 1$ in the first of these equations, one finds that the expected number of the n variables which exceed $\alpha a$ is $\bar{k} = n(1-\alpha)^{n-1}$.

**Note:** The compiler has corrected an obvious error in the first equation in the system.

**Comments:** The result sought is the probability that if an interval (0,a) of length a is divided into n segments by (n−1) points chosen at random, k of the n segments will be of length greater than $\alpha a$ $(\alpha < 1)$. Since the points of division are the order statistics of a sample of size (n−1) from a uniform distribution on the interval (0,a), this note is relevant to our subject.

**Citations:** Baticle (1933b, 1935), Garwood (1940).

* Baticle, Edgar (1933b). Le problème de la répartition. [II]. *Comptes Rendus de l'Académie des Sciences de Paris* **197,** 632-634. (JFM 59, 508).

**Summary:** The author rewrites the system of equations of the preceding note [Baticle (1933a)], dropping the superscript $\alpha$ and beginning with the equation in $\omega_n$, as follows: $\omega_n = (1 - n\alpha)^{n-1}$, $\omega_{n-1} + \binom{n}{n-1}\omega_n = \binom{n}{n-1}[1-(n-1)\alpha]^{n-1}$, $\omega_{n-2} + \binom{n-1}{n-2}\omega_{n-1} + \binom{n}{n-2}\omega_n = \binom{n}{n-2}[1-(n-2)\alpha]^{n-1}$,.... By making successive substitutions, he finds $\omega_k = \binom{n}{k}\{(1-k\alpha)^{n-1} - \binom{n-k}{1}[1-(k+1)\alpha]^{n-1} + \binom{n-k}{2}[1-(k+2)\alpha]^{n-2} - \ldots\}$. When n is large and $\alpha$ is small, a good approximation is given by $\omega_k = \binom{n}{k}[(1-\alpha^{n-1}]^k[1-(1-\alpha)^{n-1}]^{n-k}$.

**Reference:** Baticle (1933a).

**Citations:** Baticle (1935), Garwood (1940).

* Cantelli, F. P. (1933). Sulla determinazione empirica delle leggi di probabilità. *Giornale dell'Istituto Italiano degli Attuari* **4,** 421-424. (JFM 59, 1166).

**Summary:** The author shows how to extend the results of Glivenko (1933) to the case in which the theoretical cumulative distribution function has discontinuities.

**References:** Lévy (1925), de Finetti (1933), Glivenko (1933).

**Citation:** Kendall (1946).

* Castellano, Vittorio (1933a). Sulle relazioni fra curve di frequenza e curve di concentrazione e sui rapporti di concentrazione corrispondenti a determinate distribuzioni. *Metron* **10,** (4), 3-60.

**Summary:** In the first section (pp. 4-20), the author discusses various generalities concerning the curve of concentration and the ratio of concentration. Given a statistical distribution of a character, let q be the minimum fraction of the total number of elements n, which defines q as a function that is unique, continuous and increasing with p. If one assumes p as the abscissa and q as the ordinate of a point of a Cartesian system, the point (p,q) describes a curve, as p varies, which is called the curve of concentration, and which lies entirely within the right triangle whose vertices are (0,0), (1,0) and (1,1). The area between this curve and the line of equidistribution joining (0,0) and (1,1) is called the area of concentration. Let $\ell$ and L be the lower and upper limits of the value of the character and let m be the total amount of the character for all n elements. The area of maximum concentration, for every distribution, is the area of the triangle determined by the line of equidistribution and the extreme tangents of the curve of concentration, whose point of intersection has abscissa $p = [1-(n/m)L]/[(n/m)(\ell - L)]$. For $L = \infty$, the area of maximum concentration is $(1/2)[1-(n/m)\ell]$; for L finite, it is $(1/2)[1-(n/m)\ell]\,[(L-m/n)/(L-\ell)]$. If one sets m/n = A (the average amount of the character per element), the area of maximum concentration is given by Gini's formula, $(1/2)(1-\ell/A)\,[(L-a)/(L-\ell)]$, which can also be written in the form $(1-\ell/A)(1-A/L)/2(1-\ell/L)$. Then the ratio of concentration R is given by $R = [2A(L-\ell)/(A-\ell)(L-A)]S$, where S is the area of concentration. The second section (pp. 20-59) deals with analytical determination of the curve of concentration relative to various types of distributions [uniform, linear (trapezoidal), polygonal, circular, elliptic, hyperbolic, Pareto, exponential and normal] and calculation of the ratio of concentration.

**References:** Gini (1914a), Pietra (1915), Amoroso (1925), Gumbel (1927, 1928), Gini (1930), Galvani (1932), Gini (1932).

**Citations:** Castellano (1933b), Pietra (1935b), Wold (1935), Gini (1936), Davis (1941), Kendall (1946).

* Castellano, V. (1933b). Sulla interpretazione dinamica del rapporto di concentrazione. *Giornale dell' Istituto Italiano degli Attuari* **4,** 268-274. (JFM 59, 1169); *Atti del Riunione della Società Italiana per il Progresso delle Science* **21** (2), 149-151. (JFM 59, 1206).

**Summary:** The author discusses the deformation which the curve of concentration undergoes as the law of distribution of the character changes and, in the light of its significance, he arrives at the following dynamic interpretation of the ratio of concentration: The ratio of concentration is the integral of the flux when one sets the maximum value which it can assume equal to 1. It represents the proportion of the income (or other transferrable character) which must be transferred from the richest to the poorest in order to obtain a uniform distribution.

**References:** Dalton (1920), Castellano (1933a).

**Citations:** Castellano (1935), de Vergottini (1940), Kendall (1946).

* Craig, Allen T. (1933). On the correlation between certain averages from small samples. *Annals of Mathematical Statistics* **4,** 127-142. (JFM 59, 1178).

**Summary:** The author gives expressions in terms of double integrals for the joint p.d.f. of the arithmetic mean $\bar{x}$ and the range W for samples of sizes 3 and 4 from populations having probability functions of the first, second, and third kinds [i.e. having ranges $(-\infty, +\infty)$, $(0, +\infty)$, and $(0, k), k > 0$, respectively]. He gives similar expressions for the joint p.d.f.'s of the arithmetic mean $\bar{x}$ and the median $\xi$ for samples of sizes 3 and 5 and of the median $\xi$ and the range W for samples of size $(2m + 1)$ for probability functions of the three kinds; also for the joint p.d.f. of the median $\xi$ and the lower quartile $\eta$ for samples of size $(4m + 1)$ for probability functions of the second kind. As illustrations (second and third kinds, respectively), he considers the exponential density $f(x) = e^{-x}$ $(0 \le x \le \infty)$ and the uniform density $f(x) = 1/k$ $(0 \le x \le k)$. For the former, for samples of three items, the regression of W on $\bar{x}$ is linear with correlation $r = \sqrt{15}/5$, the regression of $\xi$ on $\bar{x}$ is linear with $r = 5\sqrt{267}/89$, while the regression of W on $\xi$ is non-linear. For the latter, for samples of three items, the regression curves of W on $\bar{x}$ and of $\xi$ on $\bar{x}$ are continuous, but the regression is non-linear for $k/3 \le x \le 2k/3$, while $\xi$ and W are uncorrelated.

**References:** Neyman & Pearson (1928), Rider (1929).

* Crowe, P. R. (1933). The analysis of rainfall probability. A graphical method and its application to European data. *Scottish Geographical Magazine* **49,** 73-91.

**Summary:** The author proposes a graphical method, based on the median and the quartiles, of representing the distribution of monthly rainfalls. He compares the median with the mean and the mode, and the quartile deviation with the standard deviation and the mean deviation, giving advantages and disadvantages of each from the viewpoint of the climatologist.

**Reference:** Bowley (1901).

**Citations:** Crowe (1936), Mathews (1936), Savur (1937a, b).

* Dodge, H. F. (Chairman); other members of Manual Committee of Subcommittee IX on Interpretation and Presentation of Data, of Committee E-1 on Methods of Testing, American Society for Testing Materials (1933). *ASTM Manual on Presentation of Data.* American Society for Testing Materials, Philadelphia; Supplements A and B, 1935. Revised 1951 by ASTM Technical Committee E-11 on Quality Control of Materials (H.F. Dodge, chairman) as *ASTM Manual on Quality Control of Materials.*

**Summary:** The revision is divided into three parts which are respectively revisions of the original manual and of the two supplements: Part I (Presentation of Data); Part II (Presenting ± Limits of Uncertainty of an Observed Average); and Part III (Control Chart Method of Analysis and Presentation of Data). The range is introduced in Part I and is used extensively in Part III. Table B2 (p. 115) gives factors for computing control limits based on the range for samples of size n=2(1)25.

**References:** Yule (1911), Hazen (1914), Whittaker & Robinson (1924), Tippett (1925), Shewhart (1931), Dodge (1932), E. S. Pearson (1932, 1935), {Simon (1941), Dudding & Jennett (1942), Grant (1946), Hoel (1946), Mosteller (1946)}.

**Citations:** Simon (1941), Wing, Price & Douglass (1944), Grant (1946), Tanenhaus (1947), Peterson (1949a).

* Eddington, A. S. (1933). Notes on the method of least squares. *Proceedings of the Physical Society* (London) **45,** 271-282; discussion, 282-287. (JFM 59, 544).

**Summary:** The author endeavors, without introducing inverse probability, to controvert two prevalent ideas: (1) that the method of least squares is only justified if the errors of observation have a Gaussian distribution, and (2) that the orthodox theory of errors does not sanction any of the other methods of combining observations that are often used in practice. He criticizes the common assumption that the object of the theory of combination of observations is to find *the most probable value* of the quantity x that is under discussion, and asserts that what the physicist desires is *the most accurate determination* of x, or sometimes merely one which is *accurate enough*, i.e. which reaches some specified degree of accuracy. He adopts the mean square error of x as the best general criterion of the accuracy of x. He employs the method of Laplace (1811a) to find the most accurate determination of x from a set of equations of condition $a_r x + b_r y + c_r z = \mu_r (r=1,2, \ldots, n)$, where $\mu_r$ has an error derived from any symmetrical error-distribution. His result agrees with that given by the method of least squares, and is a linear combination of all the $\mu_r$'s, though he admits that, in the case of only one unknown, the median, which is not of this form, may be more accurate than the arithmetic mean, e.g. when the error-distribution is Laplace's first. He asserts that the method of least squares is justified even when the error-distribution is non-Gaussian, because any required Gaussian assumption applies to the error of x and not to the errors of observation. The following statement (pp. 276-277) is of interest: "It is well worth while to place our theory of combination of observations on a basis which is not restricted to observations obeying the Gaussian law, for exceptions to this law are not entirely trivial. It is well known that there is a widely prevalent cause tending to make observational errors follow the Gaussian law, viz. the combination of a large number of sources of small errors; but it is not so well known that there is a widely prevalent cause tending to bring about systematic deviation from the Gaussian law, viz. heterogeneity of observational material. This last cause operates when the data are supplied by a number of observers of different skill or by the same observer working under conditions which have varied. It gives an error law with an excess of large and small errors and a defect of intermediate errors, in comparison with the Gaussian distribution; analytically, the ratio of the fourth moment to the second moment is greater than in the Gaussian law." The author also considers simplified methods, e.g. the method of thirds—fitting a straight line by connecting the centroids of the first third and the last third of the data points, arranged in order of increasing abscissas, and ignoring the middle third.

**Reference:** Laplace (1811a).

**Citation:** Hulme & Symms (1939).

* Faulhaber, R. (1933). Über den Einfluss des Probestabdurchmessers auf die Biegeschwingungsfestigkeit von Stahl. *Mitteilungen des Forschungsinstitut der Vereinigte Stahlwerke Aktiengesellschaft* **3,** 153-172.

**Summary:** "The influence of the test-rod diameter on the vibratory bending strength of rods with various surface defects is investigated. For tests of polished rods a smaller vibratory strength is found with increasing diameter, and so much the more so, the greater the tensile strength is. The ratio of the vibratory bending strength to the tensile strength, established at about 0.5 for the hitherto usual measurements of test rods under 13 mm. in diameter, falls as low as 0.35 for tests of eutectic carbon steel rods of 27 mm. diameter. All values hitherto established with small rods for the vibratory strength under bending and clearly also under torsion are therefore to be regarded only as comparative values, in no case as absolute indices of the material, and used for measurement only with caution. For tests of bars with notched surfaces, graphs show the bending strength in relation to the kernel diameter, the depth of the notch on ordinary and logarithmic scales, and the ratios of the notch depth to the kernel diameter or to the outer diameter with constant kernel diameter for each particular curve. The vibratory bending strength for large notch depths and sharp notches tends to decrease asymptotically to a minimum value clearly characteristic of the material." (From the author's summary, p. 170, as translated by the compiler).

**Citations:** Faulhaber, Buchholtz & Schulz (1933), Moore & Morkovin (1942-44)

* Faulhaber, Richard; Buchholtz, Herbert; Schulz, Ernst Hermann (1933). Einfluss des Probendurchmessers auf die Biegeschwingungsfestigkeit von Stahl. *Stahl und Eisen* **53,** 1106-1108.

**Summary:** "The vibratory bending strength [of steel] is found to have smaller values with increasing diameter of the test specimen; thus no absolute constant of the material is exhibited. For notched specimens it tends with increasing diameter for all notch forms to the same minimum value, evidently characteristic of the steel." (Compiler's translation of authors' summary, p. 1108).

**Comments:** The results reported in this paper and the preceding one [Faulhaber (1933)] are in accordance with the "weakest link" (or "largest flaw") concept.

**References:** Faulhaber (1933).

**Citations:** Moore & Morkovin (1942-44), Oding (1946), van Meer & Plantema (1949).

* de Finetti, B. (1933). Sull'approssimazione empirica di una legge di probabilità. *Giornale dell'Istituto Italiano degli Attuari* **4,** 415-20. (JFM 59, 1166).

**Summary:** The author gives another proof of the result of Glivenko (1933) that the cumulative distribution function $F_n(x)$ of a sample of size n converges stochastically to the population cumulative distribution function $F(x)$ under very general conditions.

**References:** Lévy (1925), Glivenko (1933), Kolmogoroff (1933).

**Citations:** Cantelli (1933), Kendall (1946).

* Fisher, R. A. (1933). The concepts of inverse probability and fiducial probability referring to unknown parameters. *Proceedings of the Royal Society of London* (A) **139,** 343-348. (JFM 59, 1207—listing only).

**Summary:** In his summary (p. 348) the author states: "The argument of Jeffreys [(1932b)] in favour of a particular frequency distribution *a priori* for the precision constant of a normally distributed variate rests on the fallacy that the probability of the last of three observations, lying between the previous two, should be one-third, irrespective of the distance apart of the two previous observations."

**Comments:** Jeffreys' argument is valid if and only if his hypothesis "that we know nothing about the probable range already" is literally true. If the observer does not know whether the standard deviation is one or one million, he does not know whether two observations which differ by a thousand are, relatively speaking, near to each other or far apart, and he cannot improve on the value one-third for the probability that a third observation will lie between them. In support of Fisher's position, however, it should be added that rarely, if ever, does an observer start an investigation in such complete ignorance.

**Reference:** Jeffreys (1932b).

**Citations:** Bartlett (1933), Jeffreys (1933), Davies & Pearson (1934), Fisher (1934), Thompson (1936), Snedecor (1937), Wilks (1938), Eisenhart (1939), Wilks (1943), Cramér (1946), Kendall (1946).

* Galvani, Luigi (1933). Sulla determinazone del centro di gravità e del centro mediano di una popolazione, con applicazioni alla popolazione italiana censita il l° dicembre 1921. *Metron* **11** (1), 17-47. (JFM 59, 1174).

**Summary:** The author considers two-dimensional generalizations of the arithmetic mean and of the median. His generalization of the arithmetic mean is the center of gravity, whose coordinates are the arithmetic means of the coordinates of the data points. His generalization of the median is the point such that the sum of its distances from the data points is less than for any other point. The author shows geometrically that such a point exists, and that its position can be found by successive approximation.

**Comments:** Another two-dimensional generalization of the median, probably more widely known than that proposed by the author [see Gini & Galvani (1929), Ross (1930)] is the point whose coordinates are the medians of the coordinates of the data points. This generalization is based on order statistics, but the author's is not. Both generalizations are useful concepts, but both have shortcomings, retaining some but not all of the properties of the one-dimensional median. It is not clear to the compiler which, if either, of these generalizations should be called the two-dimensional median.

**References:** Sloane (1923), Gini & Galvani (1929), Ross (1930), Griffin (1933), Linders (1933), Scates (1933).

**Citation:** Gini, Boldrini, Galvani & Venere (1933).

* Gini, C.; Boldrini, M.; Galvani, L.; Venere, A. (1933). Sui centri della popolazione e sulle loro applicazioni. *Metron* **11** (2), 3-102. (JFM 59, 1174).

**Summary:** After having recalled the definitions and the properties of the center of gravity (or mean center) and of the median center [as defined by Gini & Galvani (1929)], the authors explain their applications to the determination of centers of population for the distribution of the inhabitants of a territory. They also consider the mean latitude and longitude and the median latitude and longitude. The former are the coordinates of the center of gravity (mean center), but the latter are not, in general, the coordinates of the median center as defined by the authors. Moreover, the point of intersection of orthogonal lines dividing the population equally (median lines) varies with changes in the coordinate system. Lastly, the authors consider the mean altitude and the median altitude (above sea level) of (the places of residence of) the population.

**References:** Hayford (1902), Sloane (1923), Gini & Galvani (1929), Eells (1930), Ross (1930), Galvani (1933), Griffin (1933), Linders (1933), Scates (1933).

* Glivenko, V. (1933). Sulla determinazione empirica delle leggi di probabilità. *Giornale dell'Istituto Italiano degli Attuari* **4**, 92-99. (JFM 59, 1166).

**Summary:** "Relative to a series of independent determinations $x_1, x_2, \ldots x_n$, that constitute the values of a random variable x, it is very often interesting to know whether or not the variable x obeys a specified probability law. To ascertain if it does, one habitually forms an empirical probability law by assigning to each of the values $x_1, x_2, \ldots, x_n$ the probability 1/n and compares the theoretical law in question with the empirical law thus obtained, considering how improbable it is that the latter deviates as much from the law which the variable x in fact obeys. However, the indicated hypothesis of improbability is only stated, never proved. In this note it is verified." (Compiler's translation of author's summary, p. 92). Let the cumulative distribution function of the theoretical probability law be $V(t)=P\{x\leq t\}$ and that of the observations be $U(X_n,t)$. The author shows (1) that the probability that $m(X_n)>\lambda$, where $m(X_n)=$ sup $|U(X_n,t) - V(t)|$, $(-\infty<t<+\infty)$, $\lambda$ being an arbitrary positive number, becomes infinitesimally small as n increases without limit and (2) that the probability that $m(X_n)$ does not tend to zero, when n increases without limit, is zero. He also finds the asymptotic expression, as $n\to\infty$, of the probability that $m(X_n)$ exceeds $\lambda_n$.

**Reference:** Kolmogoroff (1933).

**Citations:** Cantelli (1933), de Finetti (1933), Kolmogoroff (1933), Kozakiewicz (1937), Waschakidse (1938), Smirnov (1939a, 1944), Kendall (1946).

* Grassberger, Hans (1933). Untersuchungen über die Hochwässer des Jangtsekiang. *Die Wasserwirtschaft* (Vienna) **26** (6), 65-70.

**Summary:** The author studies the highest stage (meters) and the highest discharge rate ($m^3$/sec) on the Yangtze River at Hankow for each of the years 1865-1928 and 1931. He computes medians as well as arithmetic means, and also fits a lognormal curve to the data. This curve appears to fit quite well, even though a very extraordinary flood occurred in 1931.

**References:** Czuber (1903) [1924], Grassberger (1932).

**Citations:** Grassberger (1936), Coutagne (1937), Gumbel (1937-38).

* Griffin, F. L. (1933). The center of population for various continuous distributions of population over areas of various shapes. *Metron* **11** (1), 11-15. (JFM 59, 1174).

**Summary:** The author states (p.11): "Instead of the definition of Center of Population used by the U. S. Census Bureau (which, admittedly, has a great practical convenience in numerical computation, but is logically unsatisfactory), I adopt the definition of the median point advocated by Professors Gini and Galvani viz., *the point at which the entire population could assemble with the minimum aggregate travel, each individual travelling directly.*" Before considering various special cases, he outlines the general theory as follows (p.12): "Given a population P distributed over a plane region R so that the density of population D persons per square unit varies in some continuous manner from point to point. The fragment of population on any element of area (dx dy, or r dr d$\theta$ etc.) located at any point (x,y) or ($r,\theta$), etc., is thus D dx dy, or D r dr d$\theta$ and the aggregate distance travelled by all such fragments in reaching any chosen meeting point ($x_1,y_1$) or ($r_1,\theta_1$), etc. is $T = \int\int \sqrt{(x-x_1)^2 + (y-y_1)^2}\, D\, dx\, dy$, or $T = \int\int \sqrt{r^2 + r_1^2 - 2rr_1\cos(\theta-\theta_1)}\, D\, r\, dr\, d\theta$, etc., the integrations being extended over the region R. To minimize T, its partial derivatives with respect to the two parameters ($x_1$ and $y_1$, or $r_1$ and $\theta_1$), are equated to zero. This gives two equations for the determination of the two parameters, and thus the Center of Population is known. In general each equation involves some definite integrals. In particular cases, when R and D are specified, it may not be possible to effect the integrations exactly, and adequate approximations must be used. In any event it is often preferable to find the partial derivatives at the very outset, by differentiating under the integral sign, rather than after integrating to obtain T."

**References:** Gini & Galvani [(1929)], Ross (1930).

**Citations:** Galvani (1933), Gini, Boldrini, Galvani & Venere (1933).

* Gumbel, E. J. (1933a). La plus petite valeur parmi les plus grandes. *Comptes Rendus de l'Académie des Sciences de Paris* **196,** 1857-1859. (JFM 59, 519).

**Summary:** "Let $w(x)$ be the distribution of a statistical variable x for which $w(x) \to 0$ and let $W(x)$ be the probability of a value less than x. What will be the smallest value among the m largest of $N(>>m)$ observations. Mr. von Mises [(1923)] has given the conditions under which the mathematical expectation $\bar{x}_w$ for the largest value among N observations can be deduced from the equation $W = 1 - 1/N$. This result is only the special case $m=1$ of our question: one can easily show that the mathematical expectation $\bar{x}_{\omega,m}$ can be calculated, provided the same conditions are fulfilled, from the analogous equation (1) $W = 1 - m/N$. In dealing with the distribution of the statistical variable $x_{\omega,m}$ (2) $w = \binom{N}{m} m W^{N-m} (1-W)^{m-1} W$, its most probable value is given by the same formula on condition that (3) $w'/w = -w/(1-W)$, which is identical according to L'Hopital's rule provided $w'(x) \to -\sigma$. This equality between the probable value [arithmetic mean] and the most probable value [mode] of the variable $x_{\omega,m}$ can be used to calculate a first approximation to the mean error $\sigma_m$ of the distribution w considered as Gaussian by (4) $\sigma_m = 1/[\sqrt{2\pi}\, N w(x_{\omega,m})\, \psi(m)]$, where $\psi(m)$ is the well known [Poisson] law of rare events. As a first approximation the mean error decreases proportionally to the square root of m. If in particular $w(x)$ is a Gaussian distribution with mean value $\xi$ and precision h, the conditions of von Mises and condition (3) are fulfilled. One obtains the probable and the most probable value $\bar{x}_{\omega,m}$ of the smallest value among the m largest of N observations from (5) $\bar{x}_{\omega,m} = \xi + t_{\omega,m}/h$, where $t_{\omega,m}$ is calculated from (6) $N/2m = 1/[1 - \Phi(t_{\omega,m})]$ with the aid of the Gaussian integral $\Phi$. As a first approximation one obtains for the largest values of N: $t_{\omega,m} = \sqrt{\ln N - \ln m}$. The mean error is given by (7) $\sigma_m = 1/[2\sqrt{2\pi} m \Psi(m)\, h\, t_{\omega,m}]$. The asymptotic equality between the arithmetic mean and the most probable value [mode] rests upon the condition $2t^2_{\omega,m} >> 1$. Thus the last three formulas are valid only if the number of observations is at least of the *order of magnitude* $10^5$." *(Compiler's translation).*

**Reference:** von Mises (1923).

**Citations:** Gumbel (1933b,g, 1934f, 1935a, 1939f).

* Gumbel, E. J. (1933b). La plus petite valeur parmi les plus grandes et la plus grande valeur parmi les plus petites. *Comptes Rendus de l'Académie des Sciences de Paris* **197,** 965-967. (JFM 59, 519).

**Summary:** "Let x be the initial distribution of a statistical variable. . . . The smallest value among the m largest [of N values] has a distribution $w_m$ and a mode $\tilde{u}_m$. Then . . . the largest value among the m smallest will have the distribution $w_{N-m+1}$ and the mode $\tilde{u}_{N-m+1}$. . . . The distribution of the smallest value among the m largest is $w_m = \binom{N}{m} m\, W^{N-m}(1-W)^{m-1} w$, while the distribution of the largest value among the m smallest is $w_{N-m+1} = \binom{N}{m} m W^{m-1}(1-W)^{N-m} w$, where W is the probability of a value smaller than x. . . . The mode $\tilde{u}_m$ of the smallest value among the m largest and the mode $\tilde{u}_{N-m+1}$ of the $m^{th}$ value are calculated from $W(\tilde{u}_m) = 1 - m/N$, $W(\tilde{u}_{N-m+1}) = m/N$ $(m = 1,2, \ldots)$. . . . The probability densities of the modes become, for large values of N and small values of m, $w_m(\tilde{u}_m) = N\psi(m) w(\tilde{u}_m)$, $w_{N-m+1}(\tilde{u}_{N-m+1}) = N\psi(m) w(\tilde{u}_{N-m+1})$, where $\psi(m) = m^m e^{-m}/m!$ is the probability of the mean value in the well known [Poisson] law of rare events. For all values of m, one will have then $w_m(\tilde{u}_m)/w_{N-m+1}(\tilde{u}_{N-m+1}) = w(\tilde{u}_m)/w(\tilde{u}_{N-m+1})$. The quotient of the probability density of the mode of the smallest value among the m largest by the probability density of the mode of the largest value among the m smallest is thus equal to the quotient of these values themselves in the initial distribution. . . . For $x \gtrless \tilde{u}_m$, one has $w_{m+1} \lessgtr w_m$ and, for $x \gtrless \tilde{u}_{N-m+1}$, one has $w_{N-m} \lessgtr w_{N-m+1}$. . . ." (Compiler's translation).

**Reference:** Gumbel (1933a).

**Citations:** Gumbel (1933c,d), Eyraud (1934b), Gumbel (1934a,e,f, 1935a,b,c,d, 1939f).

* Gumbel, E. J. (1933c). La distribution limite de la plus petite valeur parmi les plus grandes. *Comptes Rendus de l'Académie des Sciences de Paris* **197,** 1082-1084. (JFM 59, 519).

**Summary:** "For an unlimited initial distribution $w(x)$ and for N observations, the distribution $w_m$ of the smallest value among the m largest [Gumbel (1933b)] can be written (1) $w_m = \binom{N-1}{m-1} (1/W-1)^{m-1} dW^N/dx$, where (2) $W = \int_{-\infty}^{x} w(z)dz$ is the probability of a value less than x. For an initial distribution unlimited in both directions and symmetric this will also be the distribution of the absolute value of the largest value among the m smallest. The mode $\tilde{u}_m$ will be calculated by (3) $W(\tilde{u}_m) = 1-m/N$, provided one has (4) $w'(\tilde{u}_m)/ w(\tilde{u}_m) = -w(\tilde{u}_m)/ [1-W(\tilde{u}_m)]$. In order to arrive at the simplest form of the distribution (1), which in general is quite complicated, we first expand $W(x)$ about the mode $\tilde{u}_m$: (5) $W(x) = W(\tilde{u}_m) + [(x-\tilde{u}_m)/ 1!] w(\tilde{u}_m) + [(x-\tilde{u}_m)^2/2!] w'(\tilde{u}_m) + \ldots = 1-m/N + (m/N)[(x-\tilde{u}_m)/ 1!](N/m)w(\tilde{u}_m) - (m/N)[(x-\tilde{u}_m)^2/2!] w^2(\tilde{u}_m) (N^2/m^2) + \ldots$, according to (3) and (4). These are the first three terms of the expansion of (6) $V(x) = 1 - (m/N) e^{-(x-\tilde{u}_m)(N/m) w(\tilde{u}_m)}$, provided that it is allowable to set approximately (7) $w^{(\nu-1)}(\tilde{u}_m) = (-1)^{\nu} (N^{\nu-1}/ m^{\nu-1}) w^{\nu}(\tilde{u}_m) = (-1)^{\nu} w^{\nu}(\tilde{u}_m)/ [1-W(\tilde{u}_m)]^{\nu-1}$ ($\nu=2,3, \ldots$) according to (3). The formula (4) is only the particular case of (7) for $\nu=2$. We will now have approximately $W^N(x) = e^{-me^{-y_m}}$, by introducing the new variable $y_m$ by the final transformation (8) $y_m = (N/m) w(\tilde{u}_m)(x-\tilde{u}_m)$. One will have (9) $dW^N(x)/ dx = Nw(\tilde{u}_m) e^{-y_m - me^{-y_m}}$. By applying the same expansion to the first factor of (1) one arrives at $(1/W-1)^{m-1} = \{[1/(1-me^{-y_m}/ N)] - 1\}^{m-1} = (m^{m-1}/ N^{m-1}) e^{-(m-1)y_m}$ for sufficiently large values of N. But for these values one has $(m^{m-1}/ N^{m-1}) \binom{N-1}{m-1} = m^m/m!$. Thus the limiting form of the distribution of the smallest value among the m largest will be (10) $w_m = [m^m/(m-1)!](N/m) w(\tilde{u}_m) e^{-my_m - me^{-y_m}}$, where the mode $\tilde{u}_m$ and the probability $w_m(u_m)$ of the distribution (1) are retained. For $m=1$, one obtains the final distribution of the largest value (11) $w_1 = N w(\tilde{u}_1) e^{-y_1 - e^{-y_1}}$, the formula deduced by R. A. Fisher [& L. H. C. Tippett (1928)] by imposing the condition that the variable in the distribution shall be a linear function of the variable in the initial distribution. The curves (10) are asymmetric. They decrease to the left more rapidly than to the right. The calculation of these values is simpler since the curves are contained in quite narrow intervals, which tighten very quickly for increasing values of m. The distribution of the smallest value among the m largest can be represented for a sufficient number of observations by the doubly exponential formula (10), provided that the initial distribution behaves asymptotically as an exponential. The formula becomes exact if the initial distribution is exponential." (Compiler's translation).

**Note:** Obvious typographical errors have been corrected in equations (5) and (9).

**References:** Fisher & Tippett (1928), Gumbel (1933b).

**Citations:** Gumbel (1933d, 1934a,e,f,g, 1935a, 1937a, 1939f).

* Gumbel, E. J. (1933d). La distribution limite de la plus grande valeur parmi les plus petites. *Comptes Rendus de l'Académie des Sciences de Paris* **197,** 1381-1382. (JFM 59, 519-520).

**Summary:** The author adapts the results of the preceding paper [Gumbel (1933c)] to find the limiting distribution of the largest value among the smallest values when the initial distribution is unlimited to the left. It is not necessary to give the details, since, as the author states (p. 1382): "All of the formulas are in strict analogy with those which we have found for the least value among the largest, from which they can be derived by replacing $y_m$ by $-{}_m y$ [where $y_m$ denotes the $m^{th}$ largest value and ${}_m y$ the $m^{th}$ smallest]. For symmetric initial distributions the distributions of the smallest values among the largest and the distributions of the largest values among the smallest are thus simply symmetric with respect to the axis of ordinates."

**References:** Gumbel (1933b,c).

**Citations:** Gumbel (1934a,e,f, 1935a,c,d, 1939f).

* Gumbel, E. J. (1933e). Die Gauss'sche Verteilung der Gestorbenen. *Jahrbücher für Nationalökonomie und Statistik* **138** [(3) **83**], 365-389.

**Summary:** This paper deals with the question of fitting a Gaussian distribution to data on the lives of humans (after a transformation due to Lexis) and of electric lamps. The only part relevant to our study is an allusion to applications to the question of the oldest age (p. 389): "Finally the Lexis theory allows a complete clarification of the question of the existence of a limiting age." This question is discussed more fully in the next two papers [Gumbel (1933f,g)].

**References:** Fechner (1897), Czuber (1903) [1924].

**Citations:** Gumbel (1933f,g, 1934g, 1935a, 1937a, 1939f).

* Gumbel, E. J. (1933f). Das Alter des Methusalem. *Zeitschrift für Schweizerische Statistik und Volkswirtschaft* **69** (4), 516-530.

**Summary:** "We reject the usual theory of the limiting age and the previously made suggestions for an approximate determination. We assume that the distribution of mortality extends to infinity and approaches zero only asymptotically. Accordingly, we define the highest age as that age which the highest one of a given finite number of persons is expected to attain. Therefore if a certain formula for this calculation is to be applicable, the vital force in it must go asymptotically to zero. This is true especially for the Lexis formula, which leads to the following procedure: For a certain mortality table one ascertains the maximum number of deaths during one year and its difference from the two neighboring values. Thus one obtains the normal age $\xi$. Further one ascertains the corresponding probability of survival $\ell(\xi)$ and life expectancy $E(\xi)$. For the number of deaths on which the mortality table is based, one calculates $N = 2D\ell(\xi)$. Then the table of the Gaussian integral gives the corresponding reduced limiting age $\bar{t}_\omega$. The corresponding expected limiting age is given by $[\bar{x}_\omega = \xi + \bar{t}_\omega / h]$ and the standard error by $[\sigma = 0.54222/h\,\bar{t}_\omega]$. Whereas all previous suggestions determine the limiting age with the help of naturally uncertain observations at the highest ages, our theory has the advantage of utilizing only the information in the vicinity of the normal age, where the observed number can by all means be regarded as more certain. Although the problem ultimately tapers to the life of a single person, thus apparently — but only apparently — standing outside the realm of probability theory, in spite of the simplicity of the analytical formulation and in spite of the considerable neglect undertaken therein, the agreement between theory and practice is excellent. Thus the numerical examples show that the apparently paradoxical conception, according to which there is no finite age limit as such, at which the survival series terminates, agrees well with the theory. Thereby all arguments are refuted which rest upon the existence of such an age." (Compiler's translation of author's summary, p. 530).

**References:** Czuber (1903) [1924], von Bortkiewicz (1922a,b), von Mises (1923), Steffensen (1930), Gumbel (1932, 1933e, 1934g).

**Citations:** Gumbel (1934f, 1935a, 1936c, 1937a, 1939f).

* Gumbel, E. J. (1933g). Die Verteilung der Gestorbenen um das Normalalter. *Aktuárské Vědy* **4** (2), 65-69. (JFM 59, 518).

**Summary:** This paper, like a preceding one [Gumbel (1933e)], deals with the question of fitting a Gaussian distribution to data on the lives of humans (after a transformation due to Lexis) and of electric lamps; this one also considers data on the lives of fruit flies (Drosophila). The only part relevant to a study of order statistics is the following passage (p. 95) on the question of the oldest age: "The conception based on this theory of the mortality table as a distribution of deaths allows a solution of the existence and the value of the limiting age $\omega$, for which till now only a series of very artificial suggestions have

been made. For this question is identical with that of the largest value which is to be expected among a large number of observations from an unlimited distribution. Accordingly if one defines the limiting age as that age for which one would expect that out of D observations which form the basis of the mortality table, exactly one survives to it, then the extraordinarily simple analytical equation reads $D\ell(\omega)=1$, where it is assumed only that the mortality table satisfies certain conditions which empirically always occur. For the Gaussian distribution employed here, the calculation of the limiting age is especially simple. It is directly tied to the value of the normal age treated here, its empirical probability and its life expectancy."

**References:** Fechner (1897), Czuber (1903) [1924], von Mises (1931), Gumbel (1932, 1933a,e).

**Citations:** Gumbel (1934f, 1935a,d, 1936-37, 1937a, 1939f).

* Hojo, Tokishige (1933). A further note on the relation between the median and the quartiles in small samples from a normal population. *Biometrika* **25,** 79-90. (JFM 59, 1173).

**Summary:** "In a previous paper [Hojo (1931)] I have considered certain properties of the median and quartiles in small samples from a normal population, and have investigated methods of linking these up with the limiting values in large samples. The latter problem has since been further investigated by K. Pearson [& M. V. Pearson (1931)]. . . . In the earlier paper the following comparisons were made: (1) Standard error of mean with that of median. . . . (2) Standard error of an estimate of $\bar{\sigma}$ obtained from the sample standard deviation with that of another estimate obtained from the interquartile distance. . . . In the present paper I shall add to these previous results: (3) Standard error of the mid-point between the two quartiles, i.e. of (1/2) $(q_1+q_2)$. (4) Standard error of an estimate of $\bar{\sigma}$ obtained from the distance between a quartile and the median, i.e. from $q_1-m$ or $m-q_2$. The first result follows from calculations previously carried out; for the second it is necessary to determine the coefficient of correlation $r_{q,m}$ between quartiles and median." (Introduction, p. 79).

**References:** K. Pearson (1920), Hojo (1931), Pearson & Pearson (1931).

**Citation:** Kendall (1946).

* Hotelling, Harold (1933). Analysis of a complex of statistical variables into principal components. *Journal of Educational Psychology* **24,** 417-441, 498-520. (JFM 59, 1182-1183). Abstract, *Bulletin of the American Mathematical Society* **38** (1932), 812.

**Summary:** The author gives the following description (pp. 420-421) of the method of principal components: ". . . We begin with a component $\gamma_1$ whose contributions to the variance . . . have as great a total as possible; . . . we next take a component $\gamma_2$, independent of $\gamma_1$, whose contribution to the residual variance is as great as possible; and . . . we proceed in this way to determine the components, not exceeding n in number, and perhaps neglecting those whose contributions to the total variance are small." On pp. 424-425 he describes the procedure for finding the principal components as follows: "The problem of finding a component $\gamma_1$ which will account for as large as possible a part of the total variance is thus solved by finding the largest root $k_1$ of the characteristic equation (18)

$$f(k)=\begin{bmatrix} 1-k & r_{12} & r_{13} & \dots & r_{1n} \\ r_{21} & 1-k & r_{23} & \dots & r_{2n} \\ & & \dots & & \\ r_{n1} & r_{n2} & r_{n3} & \dots & 1-k \end{bmatrix}=0 \quad ,$$

substituting in (17)

$$\begin{bmatrix} (1-k)\,a_1 + r_{12}\,a_2 + \dots + r_{1n}\,a_n = 0 \\ r_{12}\,a_1 + (1-k)\,a_2 + \dots + r_{2n}\,a_n = 0 \\ \dots \\ r_{1n}\,a_1 + r_{2n}\,a_2 + \dots + (1-k)a_n = 0 \end{bmatrix} ,$$

finding any solution $a_1, a_2, \ldots, a_n$ of these linear equations and dividing these last values by the sum of their squares and multiplying by $\sqrt{k_1}$. The resulting quantities are the coefficients of $\gamma_1$ in the expression (3) [$z_i = \Sigma_j a_{ij}\gamma_j$] giving the test scores in terms of the independent components. . . . When the coefficients of the component $\gamma_1$ which make the largest contribution to the total variance, or of the q components which equally make maximum contributions, have been determined, the next problem is to find a component making a maximum contribution to the residual portion of the variance. The argument and procedure are virtually the same as before. . . ." The author continues (p. 434): "The exact distribution of the roots of the characteristic equation, or of their ratio, can be found at once when $n=2$. . . . For $n>2$, this same distribution may be used as a close approximation for differentiating between any two roots."

**References:** Fisher (1922), Whittaker & Robinson (1924).

**Citations:** Girshick (1936), Hotelling (1936a,b), Aitken (1937), Bartlett (1938), Fisher (1939), Girshick (1939), Wilks (1943), Tintner (1946), Bartlett (1947b, 1948), Geary (1948).

* Insolera, Filadelfo (1933). Probabilità e sopravivenza. *Atti dell' Istituto Nazionale delle Assicurazioni* **5**, 165-182.

**Summary:** The author notes that Bohlman (1909) stated two postulates underlying the mathematical theory of survival: 1. For a given positive number $\epsilon$ and for every x and y such that $\epsilon \leq x \leq y \leq \omega$, where $\omega$ represents the extreme (oldest) age, there exists a probability $p(x,y)$ that an individual of age x survives to age y. 2. The probability $p(x,\omega)$ is zero. The author continues the discussion of the oldest age begun in his earlier papers [including Insolera (1931)], referring also to the lectures of Steffensen (1930) and the notes of Lidstone (1930) on the first of them. The author [Insolera (1931)] proposed as definition of the oldest age that value $\omega$ of age y such that in its closest left neighborhood, the number of dead approaches to the number of living there considered, as y approaches to $\omega$, and showed that this definition has the following consequences: (1) $\lim_{y \to x-0} p(x,y) \neq 0$ and $\lim_{y \to x+0} p(x,y) = 0$; (2) The force of mortality is less than unity for $x < \omega$ and approaches unity as $x \to \omega$. The author asserts that the oldest age $\omega$ is not a fixed constant, but depends on biological characteristics of the population (varying from place to place and from time to time, according to race, sex, etc.) and on the population size — the larger the population considered, the greater the probability that at least one individual in the population survives to age y, and hence the greater the value of $\omega$. He illustrates this point by a discussion of mortality tables, which usually start with 100,000 persons alive at some age $\varepsilon$, and give the number $\ell_x$ (to the nearest integer) surviving to age x. If 0.95 persons out of every 100,000 in the population survive to age 102 and only 0.25 out of every 100,000 to age 103, the table would show $\ell_{102} = 1$ and $\ell_{103} = 0$, so that the table would indicate an oldest age of 102 years for a population of 100,000 persons. However, if one were to construct a mortality table based on 1,000,000 persons starting at age $\varepsilon$, it would show 2 or 3 of them (depending on rounding) still alive at age 103 years.

**References:** Bohlmann (1909), Lidstone (1930), Steffensen (1930), Insolera (1931).

**Citations:** de Finetti (1934), Insolera (1935), Gumbel (1936-37, 1937a), Insolera (1937-38).

* Jeffreys, Harold (1933). Probability, statistics, and the theory of errors. *Proceedings of the Royal Society of London* (A) **140**, 523-535. (JFM 59, 1207 — listing only).

**Summary:** The author attacks the method used by Fisher (1933) in his criticism of the author's earlier paper [Jeffreys (1932b)]. On page 532, he writes: "Fisher proceeds to reduce my theory to absurdity by integrating with respect to all values of the observed measures. This procedure involves a fundamental confusion, which pervades the whole of his statistical work, and deprives it of all meaning. The essential distinction in the problem of inference is the distinction between what we know and what we are trying

to find out: between the data and the proposition whose probability on the data we are trying to assess. If we have made two observations, . . . those are our observations, and there is no more to be said. To integrate with respect to them and average a function of them over the range of integration is an absolutely meaningless process."

**References:** Keynes (1921), Fisher (1922), Jeffreys (1932b), Fisher (1933).

**Citations:** Fisher (1934), Jeffreys (1934), Thompson (1936).

Julin, Armand (1933). Médiane-médiale-dominante. I. La médiane. *Barometro Economico Italiano* **5** (10), 580-584.

**Summary:** Part I deals with the median. Most of it is almost identical with the treatment given in the author's book [Julin (1921), pp. 405-415]. The author adds a discussion of the calculation of the median from grouped data. He gives two graphical methods, the one proposed by Galton et al. (1881) and one based on plotting the cumulative frequency on graph paper and drawing a horizontal line from the point N/2 (N = total frequency) on the vertical axis to the point of intersection with the cumulative frequency curve and then, from that point, a vertical line to the horizontal axis, the abscissa of the point of intersection being the median. He also gives an arithmetic method. In closing, he points out that the median, like the arithmetic mean, enjoys certain mathematical properties, the most important being that it is the value from which the *sum* of the (absolute) deviations of the observations is a minimum. By contrast, he states that the mean is the value from which the *product* of the (absolute) deviations is a minimum.

**Comments:** The statement that the product of the absolute deviations is a minimum when deviations are taken from the arithmetic mean is false; if deviations are taken from any observation, their product is zero, and the mean may not coincide with any one of the observations. It is the *sum of the squares* of the deviations that is a minimum when deviations are taken from the mean.

**Note:** Part I deals only with the median. The medial and the mode are apparently discussed in the subsequent part(s), which the compiler has not seen. The mode is discussed by the author in his book [Julin (1921)], but the medial [see March (1930)] is not.

**References:** Galton et al. (1881), Galton (1889), Fechner (1897), Bowley (1901) [1920], Yule (1911), March (1930), Yang (1933).

* Kolmogoroff, A. (1933). Sulla determinazione empirica di una legge di distribuzione. *Giornale dell'Istituto Italiano degli Attuari* **4**, 83-91. (JFM 59, 1166).

**Summary:** Let $F(x)$ be the cumulative distribution function of a probability distribution and $F_n(x)$ be the corresponding cumulative distribution function of a sample of size n from the given distribution. Let $\Phi_n(\lambda)$ be the probability that $m_n = \lim \sup |F_n(x) - F(x)| < \lambda/\sqrt{n}$. The author shows that $\Phi_n(\lambda)$ is independent of $F(x)$ and that $\lim_{n\to\infty}\Phi_n(\lambda) = \Phi(\lambda) = \sum_{-\infty}^{+\infty}(-1)^k e^{-2k^2\lambda^2}$. A short table, computed by N. Kogeknikov, gives the values of $\Phi(\lambda)$ [4 to 8 DP] for $\lambda = 0.0(0.2)2.8$.

**Comments:** The statistic proposed by the author is an alternative to the Cramér-von Mises statistic $\omega^2$. The author refers to the work of von Mises (1931), but apparently both he and von Mises were unaware of the earlier work of Cramér (1928).

**References:** von Mises (1931), Glivenko (1933).

**Citations:** de Finetti (1933), Glivenko (1933), Kozakiewicz (1937), Smirnoff (1937a), Waschakidse (1938), Smirnov (1939a, 1939b), Kolmogoroff (1941), Wald & Wolfowitz (1941), Scheffé (1943), Smirnov (1944), Kendall (1946), Feller (1948), Wilks (1948), Chung (1949), Doob (1949), Kac (1949), Maniya (1949), Wolfowitz (1949).

* Linders, F. J. (1933). Über die Berechnung des Schwerpunktes und der Trägheitsellipse einer Bevölkerung. *Metron* **11** (1), 3-10. (JFM 59, 1174).

**Summary:** Given n points $(x_i,y_i)$ $(i=1,2,\ldots,n)$ in a plane, the distance of point i from the point (X,Y) is given by $\rho_i$, where $\rho_i^2=(X-x_i)^2+(Y-y_i)^2$. The author considers the problem of choosing the point (X,Y) so as to minimize the sum of the $s^{th}$ powers of the distances of the point (X,Y) from the n given points. He works out only the case $s=2$, for which the point (X,Y) is the center of gravity.

**Comments:** In the case $s=1$, which the author does not work out, the point (X,Y) is the median point as defined by Gini & Galvani (1929).

**Citations:** Galvani (1933), Gini, Boldrini, Galvani & Venere (1933).

* McKay, A. T.; Pearson, E. S. (1933). A note on the distribution of range in samples of n. *Biometrika* **25**, 415-420. (JFM 59, 1172).

**Summary:** "In a recent paper [Pearson (1932)] one of the present writers has provided a table giving certain percentage limits for the distribution of range in samples from a normal population. These limits were obtained on the assumption that the distribution could be adequately represented by Pearson curves having the appropriate moment coefficients. The theoretical treatment . . . below, while following the general method of approach previously employed, leads to certain new results regarding the form of the range curve at the terminals, and also provides the exact distribution of range in the case of samples of 3 from a normal population. In this latter case, therefore, it makes possible a check on the accuracy of the published table." (p. 415). Also given are the exact distributions of the range in samples of n from rectangular and straight line (right triangular) universes. The former were given earlier by Neyman & Pearson (1928).

**References:** Neyman & Pearson (1928), Pearson (1932).

**Citations:** Pearson & Haines (1935), Neyman & Pearson (1936), Newman (1939), Hartley (1942), Pearson & Hartley (1942), Kendall (1946), Egudin (1947), Elfving (1947), Gumbel (1947), Lord (1947), Nair (1948b), Wilks (1948).

* Rhodes, E. C. (1933). *Elementary Statistical Methods*. George Routledge & Sons, Ltd., London. (JFM 59, 1162).

**Summary:** Chapter 7 (pp. 97-116) on graphical methods and Chapter 8 (pp. 117-135) on the median and measures of dispersion contain the material relevant to our study. On p. 113 the author states: "A cumulative diagram [ogive] is really an approximate [for grouped data; exact when the individual measurements are known and used] graphical representation of the group of values of the character when these are arranged in order of size and the number of cases summed successively." He concludes Chapter 7 as follows (p. 116): "A cumulative diagram shows us then, by means of vertical ordinates, the approximate number of the whole group possessing less than (or more than) any measure of the

character represented on the horizontal scale. In effect this kind of diagram puts in order of magnitude the individuals of the group, and is very helpful in assisting us to pick out measurements possessed by certain individuals. One of these of most value is the *Median* measurement, which is used as a representation of the group." In Chapter 8 he gives a detailed discussion of the median and its calculation from ungrouped and grouped data in tabular and graphical form. He also discusses various measures of dispersion, including three (range, mean deviation from the median, and quartile deviation) which are based on order statistics and two (mean deviation from the arithmetic mean and standard deviation) which are not, along with their calculation from grouped and ungrouped data.

* Rider, Paul R. (1933). Criteria for Rejection of Observations. *Washington University Studies, New Series, Science and Technology,* No. 8.

**Summary:** After summarizing the history of the subject from Peirce (1852) to Jeffreys (1932a), the author draws the following conclusion (pp. 21-22): "From the various methods cited above it is easy to see that devices for rejecting discordant observations could be invented without number. The choice of which to use, if any, is largely an individual matter. If one is willing to subscribe to the hypothesis laid down in a given criterion, he should be willing to abide by the result of applying the criterion to a set of data. . . . In the final analysis it would seem that the question of the rejection or retention of a discordant observation reduces to a question of common sense. Certainly the judgment of an experienced observer should be allowed considerable influence in reaching a decisiion. This judgment can undoubtedly be aided by the application of one or more tests based on the theory of probability, but any test which requires an inordinate amount of calculation seems hardly to be worthwhile, and the testimony of any criterion which is based upon a complicated hypothesis should be accepted with extreme caution."

**References:** Bessel [& Baeyer] (1838), Peirce (1852), Gould (1855), Airy (1856), Winlock (1856), Chauvenet (1863) [1876], Stone (1868), Glaisher (1873), Stone (1873b), Glaisher (1874), Stone (1874), Peirce (1878), Schott (1878), Edgeworth (1883b), Wright (1884) [Wright & Hayford (1906)], Newcomb (1886), Edgeworth (1887a,d), Faye (1888), Czuber (1891a), Saunder (1903), Brunt (1917) [1931], Stewart (1920a,b), Fisher (1922), Archibald (1925), Coolidge (1925), Irwin (1925a,b), Tippett (1925), Pearson (1931), Jeffreys (1932a), Pearson (1932).

**Citations:** Thompson (1935), Pearson & Chandra Sekar (1936), Wing, Price & Douglass (1944), Kendall (1946).

* Romanovsky, V. (1933). On a property of the mean ranges in samples from a normal population and on some integrals of Prof. T. Hojo. *Biometrika* **25**, 195-197. (JFM 59, 1173).

**Summary:** The author states his principal result as follows (p. 195): "Studying the paper of Prof. K. Pearson [& M. V. Pearson (1931)], I discovered a property of the mean ranges in samples taken from a normal population which deserves some attention. This property may be formulated as follows. Let us put $\alpha_x = (1/\sqrt{2\pi}) \int_{-\infty}^{x} e^{-x^2/2} dx$; then $w_m = 2m \int_0^1 x \, \alpha_x^{m-1} \, d\alpha_x$, representing the mean range in samples of size m from an infinite normal population with zero as mean and unity as variance, verifies identically the relation $w_{2m+1}/(4m+2) - C_m^1 w_{2m}/4m + C_m^2 w_{2m-1}/(4m-2) - \ldots + (-1)^m w_{m+1}/(2m+2)$ [where $C_m^h = m!/h!(m-h)!$] for $m = 1,2,3, \ldots$" The author proves this property and states and proves certain other relations involving the integrals considered by Hojo (1931).

**References:** Hojo (1931), Pearson & Pearson (1931).

**Citations:** Kendall (1946), Egudin (1947).

* Scates, Douglas E. (1933). Locating the median of the population in the United States. *Metron* **11** (1), 49-66.

**Summary:** The author notes that a new interest has been aroused in the median point of population by a new definition given by Gini & Galvani (1929). Previously, the median of population had been regarded as the intersection of two orthogonal median lines, usually east-west and north-south lines. This definition suffers from the difficulty that it is not invariant with respect to changes in the orthogonal

directions of the median lines [see Hayford (1902)]. Chief interest has centered in the mean of the population or center of gravity, which is the intersection of two orthogonal mean lines and which *is* invariant with respect to changes in their orthogonal directions. It has been stated erroneously in publications of the U.S. Census Bureau [e.g. Sloane (1923)] that the center of gravity of population has the property that its aggregate distance from each person in the country is less than that of any other point; the fallacy of this statement has been pointed out by Eells (1930). The definition of the median of the population given by Gini & Galvani is such that the median, as defined by them, has the property incorrectly attributed to the center of gravity. This property is analogous to a property of the one-dimensional median, namely that the sum of the distances from it of all data points is less than from any other point. The author gives a working method, based on successive approximation, of finding the median as defined by Gini & Galvani, to be used pending the appearance of a more elegant solution. He applies the method to the 1930 U.S. census returns.

**References:** Hayford (1902), Sloane (1923), Gini & Galvani (1929), Eells (1930), Ross (1930).

**Citations:** Galvani (1933), Gini, Boldrini, Galvani & Venere (1933).

* Tricomi, F. (1933). Determinazione del valore asintotico di un certo integrale. *Rendiconti della Reale Accademia Nazionale dei Lincei* (6a) **17**, 116-119. (JFM 59, 426).

**Summary:** The author determines the asymptotic value of an integral which is proportional to the mathematical expectation of the largest value of a sample of size m from a Gaussian distribution, as $m \to \infty$. The result shows that for sufficiently large samples, the expected value of the largest observation in a sample from a normal distribution is proportional to the square root of the logarithm of the sample size.

**Citations:** Gumbel (1935a, 1935-36).

* Watkins, G. P. (1933). An ordinal index of correlation. *Journal of the American Statistical Association* **28**, 139-151; discussion (letter to the editor from Corrado Gini), **29** (1934), 200-202.

**Summary:** The author defines an ordinal index of correlation, $\{[\Sigma(x_{\ell d} - y_{\ell d}) - F/2] - [\Sigma(x_{\ell d} - y_{vd}) - F/2]\}/F$, where $\Sigma$ denotes summation with signs disregarded, $x_{\ell d}$ any item (rank) of one ordinal series, $y_{\ell d}$ the corresponding item of the paired ordinal series, $y_{vd}$ the item in the inverted y series that corresponds (after inversion) to item $y_{\ell d}$ of the direct series, and F the maximum possible sum of differences. In his letter to the editor, Gini points out that this formula can still be used if the x's and y's are order statistics instead of ranks, and that the result, $\{\Sigma|x_{\ell d} - y_{\ell d}| - \Sigma|x_{\ell d} - y_{vd}|\}/F$, is the index of cograduation which he proposed nearly twenty years earlier [see Gini (1916b)].

**References:** {Gini (1916b)}.

**Citations:** Castellano (1935), Kendall (1946).

* Weinberg, Wilhelm (1933). Die Güte der Prüffunctionen Geschwistermethode und Weinberg-Apert-Bernstein Methode verglichen mit der direkten Methode bei vollständigem oder vollständig repräsentativem Material. *Zeitschrift für Induktive Abstammungs- und Vererbungslehre* **63**, 420-424. (JFM 59, 1186).

**Summary:** The author uses the ratio of the standard deviation to the range of a proportion as a criterion for comparing three test functions applied to a study of hereditary diseases. He attributes this method to von Mises, but does not give a specific reference.

* Yang, Simon (1933). On partition values. *Journal of the American Statistical Association* **28**, 184-191; discussion, **29** (1934), 202-205. (JFM 59, 1207—listing only).

**Summary:** The author states (pp. 184-186): "The term 'partition values' is here used to comprise all such measures as the median, the quartiles, the deciles, the percentiles, and the like. If the definitions given by various authors for those measures be summarized, the general definition would be somewhat like this: The partition values are the magnitudes of those items in an array which divide the number of items thereof into some specified number of equal parts. Here, by *an array* is meant an arrangement of ungrouped data according to magnitude. . . . If the space between every two adjacent items be considered as a unit, the limits of the sample array are half a unit space beyond each of the terminating items . . . . Having known the limits of a sample array, we can immediately get the whole length of its base line, that is, the distance between the two limits . . . . Hence a new definition for partition values may be given as follows: The partition values are the magnitudes of those items in a sample array, actually existing or interpolated, which divide the whole length of its base line into . . . equal parts. . . . Hence the following general formula may be made: Let an array of n items be partitioned into p equal parts, then the $r^{th}$ partition value is the magnitude of the $(rn/p + 1/2)^{th}$ item." The author points out that this result agrees with, but is much simpler than, that given by Bowley (1901) [1920], but disagrees (except in the case of the median) with the result, viz. the $[r(n+1)/p]^{th}$ item, given by King (1912) [1924], Secrist (1917) [1925], and Burgess (1927). The discussion consists of open letters to the editor by Willford I. King and the author. King [wrongly, in the opinion of the compiler (see comments below)] backs away from his earlier result and agrees that the author's is correct.

**Comments:** The result given by King, Secrist, and Burgess is justified by the fact that if an $(n+1)^{th}$ item were drawn from the same population, the probability that it would fall below the smallest of the n sample values (or above the largest of them) is equal to the probability that it would fall in any one of the gaps between them (not to *half* this probability, as the author's formulation assumes).

**References:** Bowley (1901) [1920], King (1912) [1924], Secrist (1917) [1925], Jones (1921) [1927], Burgess (1927).

**Citation:** Julin (1933).

* Yntema, Dwight B. (1933). Measures of the inequality in the personal distribution of wealth or income. *Journal of the American Statistical Association* **28**, 423-433.

**Summary:** The author lists certain qualities a coefficient of inequality should have in order to be acceptable: (1) It must be independent of the number of persons in a distribution; (2) It must be independent of the unit in which wealth or income is measured; (3) It must characterize each distribution by a single definitive value; (4) It should not be difficult to compute; (5) It should be more or less adaptable to the limitations of existing data; (6) It should admit of interpretation in harmony with the connotation of the word "inequality"; and (7) It should have definite limits, preferably 0 and 1. The author lists and discusses eight plausible coefficients satisfying these conditions. Let Ma represent the arithmetic mean of the attribute magnitudes and S the standard attribute; i.e. the quadratic mean of the attribute magnitudes. The eight coefficients, of which the first six are measures of dispersion, are: (1) Relative mean deviation (mean deviation from Ma divided by Ma); (2) Relative mean difference, with repetitions (mean difference, with repetitions, divided by Ma); (3) Coefficient of variation (standard deviation, $\sigma$, divided by Ma); (4) Standard deviation devided by S; (5) Mean deviation of logarithms of attribute values; (6) Standard deviation of logarithms of attribute values; (7) $\alpha$ = Pareto's coefficient of inequality; and (8) $\delta$ = Gini's index of concentration. He notes that the relative mean difference, without repetitions, is double Gini's ratio of concentration R, which is the area between the Lorenz curve and the line of equidistribution divided by the corresponding area (1/2) representing complete inequality. He shows by a numerical example using actual income data that the various coefficients considered do not agree very well among themselves, so one is forced to choose among them, the choice being determined

largely by two factors: (1) the completeness of available data and (2) the tacit assumption that inequality consists in that aspect of personal distribution which a given coefficient measures.

**References:** Pareto (1897), Lorenz (1905), Gini (1909), Persons (1909), Gini (1910, 1912, 1914a), Ricci (1916), Dalton (1920), de Finetti & Paciello (1930), Gini (1930), von Bortkiewicz (1931).

**Citations:** Mortara (1934b), Castellano (1935), Pietra (1935a), Gini (1936), Davis (1941).

* d'Addario, R. (1934a). Intorno alla validità dei due teoremi paretiani sulla dinamica distributiva. *Atti dell'Istituto Nazionale delle Assicurazioni* **6**, 179-198.

**Summary:** The author discusses Pareto's law and its relation to the curve of concentration and Gini's index of concentration $\delta$. Let $N_x$ be the number of individuals having income not less than x and let $(h, +\infty)$, where $h>0$, be the range of variability of x. Then Pareto's first approximation is $N_x = k/x^{\alpha}$, where $k>0$ and $\alpha>1$, and his second approximation is $\overline{N}_x = k/(x + a)^{\alpha}$, where $k>0$, $\alpha>1$, and a is such that $h + a > 0$. By differentiation, one finds the equations of the corresponding frequency curves, $y\,dx = (\alpha k/x^{\alpha+1})\,dx$ and $\bar{y}\,dx = [\alpha k/(x + a)^{\alpha+1}]dx$. Let P and R be the cumulative number of incomes (from above) and the cumulative amount of the incomes (from above), respectively. For Pareto's first approximation, $P = k/h^{\alpha}$ and $R = [\alpha/(\alpha-1)]hP$; the corresponding values for the second approximation are $\overline{P} = k/(h + a)^{\alpha}$ and $\overline{R} = [(\alpha h + a)/(\alpha-1)]\overline{P}$. Gini, by expressing the cumulative frequency of individuals having incomes not less than x, $(1/P)\int_x^{\infty} y\,dx$, as a function of the total amount of income received by those individuals $(1/R)\int_x^{\infty} x\,y\,dx$, obtained the equation of the curve of concentration, $(1/P)\int_x^{\infty} y\,dx = (1/R^{\delta})[\int_x^{\infty} x\,y\,dx]^{\delta}$, where $\delta$ (the index of concentration) is a positive constant greater than unity. If income follows Pareto's law, the author shows that $\alpha = \delta/(\delta-1)$. He also investigates the situation in which income follows the lognormal law (law of proportional effect) [see Gibrat (1930, 1931) and d'Addario (1931)] instead of Pareto's law. He shows that if the minimum income h remains constant, an increase in the concentration of income produces an increase in the mean income and vice versa when Pareto's law holds, but that this is no longer true in general when the lognormal law holds.

**References:** Pareto (1897), Porru (1912), d'Addario (1931).

**Citations:** Castellano (1935), d'Addario (1936), Bresciani-Turroni (1939).

* d'Addario, R. (1934b). *Sulla Misura della Concentrazione dei Redditi*. Istituto Poligrafico dello Stato, Roma. Review by Benedetto Barberi, *Economia* (N.S.) **14** (1), 92-94; review by G. Santacroce and response by author, *Rivista Italiana di Statistica* **6**, 946-948; review by Guglielmo Tagliacarne, *Rivista Bancaria* **11** (1), .

**Summary:** The first part deals with the classical equation of Pareto, the equation of Amoroso, and the equation $[p - q = kp(1 - p)^{\beta}$, where p and q have a known meaning and k and $\beta$ are positive constants $\leq 1]$ recently proposed by the author, and their place in econometric theory. The second part deals with the quantities proposed for the measurement of the concentration of incomes, especially with the famous controversy between R (the concentration ratio proposed by Gini) and S (the mean deviation from the arithmetic mean divided by its upper limit) and the no less celebrated one between the indices $\alpha$ and $\delta$. The third and last part treats a method, based on semi-invariants, for determining the parameters of an equation and its limits and applications. The author appends to the review by Santacroce a response concerning the solidarity of the indices S and R, in the sense that they respond in the same way to changes in the concentration of income.

**Note:** The compiler has not seen the book itself or the review by Tagliacarne; hence the above summary is based solely on the reviews by Barberi and Santacroce and the author's response to the latter.

**Citations:** Dominedò (1934), Castellano (1935), Pietra (1935b), Bresciani-Turroni (1939), Boldrini (1942).

* Baidaff, B. J.; Barral Souto, J. (1934). Cinco valores interesantes de una media general. *Boletin Matemático* (Buenos Aires) **7**, 102-104.

**Summary:** Let $y = (\Sigma_{i=1}^{n} x_i^m/n)^{1/m}$, where $x_1, x_2, \ldots, x_n$ are n positive numbers arranged in non-decreasing order: $x_1 \leq x_2 \leq \ldots \leq x_n$. The authors show that $\lim_{m\to-\infty} y = x_1$ [the smallest of the $x_i$'s], $y|m = -1) = n/\Sigma_{i=1}^{n} (1/x_i)$ [the harmonic mean of the $x_i$'s], $\lim_{m\to 0} y = (\Pi_{i=1}^{n} x_i)^{1/n}$ [the geometric mean of the $x_i$'s], $y|(m=1) = \Sigma_{i=1}^{n} x_i/n$ [the arithmetic mean of the $x_i$'s], and $\lim_{m\to\infty} y = x_n$ [the largest of the $x_i$'s].

**Citations:** Baidaff & Barral Souto (1935), Barral Souto (1938).

* Davies, O. L.; Pearson, E. S. (1934). Methods of estimating from samples the population standard deviation. *Journal of the Royal Statistical Society, Supplement* **1**, 76-93. (JFM 63, 1098).

**Summary:** A comparison is made of the standard errors in estimating the population standard deviation from the variance, the standard deviation, the range, and the mean deviation (from the arithmetic mean) of a sample of size n and from the means of these statistics from k samples each of size n, for $n = 2(1)6, 10, 15, 20, 50$ and $k = 1(1)6, 10, 20, 50$. These estimates are given respectively by $\sigma_{e_1} = \sqrt{\Sigma_{i=1}^{n} (x_i - \bar{x})^2/(n-1)}$, $\sigma_{e_2} = (1/b_n)\sqrt{\Sigma_{i=1}^{n} (x_i - \bar{x})^2/n} = s/b_n$, $\sigma_{e_3} = (1/d_n)(x_n - x_1) = w/d_n$, $\sigma_{e_4} = (1/f_n)\Sigma_{i=1}^{n} |x_i - \bar{x}|/n = m/f_n$; $\sigma_{e_1} = \sqrt{\Sigma_{t=1}^{k} \Sigma_{i=1}^{n} (x_{ti} - \bar{x}_t)^2/(n-k)}$, $\sigma_{e_2} = \Sigma_{k=1}^{t} s_t/kb_n = \Sigma_{t=1}^{k} \sqrt{\Sigma_{i=1}^{n} (x_{ti} - \bar{x}_t)/n}/kb_n$, $\sigma_{e_3} = \Sigma_{t=1}^{k} w_t/kd_n = \Sigma_{t=1}^{k} (x_{tn} - x_{t1})/kd_n$, $\sigma_{e_4} = \Sigma_{t=1}^{k} m_t/kf_n = \Sigma_{i=1}^{n} |x_{ti} - \bar{x}_t|/nkf_n$. The standard errors of these estimates (3DP) are given in Table I for the values of n and k mentioned above. The factors $b_n$, $1/b_n$, $d_n$, $1/d_n$, $f_n$, and $1/f_n$ (4DP) are given in Table II for $n = 2(1)15$, and the moment constants $\beta_1$ and $\beta_2$ (2DP) are given in Table V for estimates $\sigma_{e_1}$, $\sigma_{e_2}$, and $\sigma_{e_3}$ for $n = 2(1)6, 20$ and $k = 1(1)4$. The authors draw the following conclusions (p. 89): ". . . The majority of the results quoted depend on the assumption that the variable character measured is normally distributed. This condition need not, however, be rigorously satisfied. $\sigma_{e_1}$ must be regarded as the most reliable of the four estimates. There appears little to be gained by the use of $\sigma_{e_2}$. If the number of observations in the samples or groups is not more than $10 (n \leq 10)$, the estimate $\sigma_{e_3}$ from range is undoubtedly of value in any case where time is a matter of importance. . . . The estimate from mean deviation [$\sigma_{e_4}$] appears to have no special advantage over any of the others. . . . In cases where the data are to be subjected to more detailed analysis, the estimate $\sigma_{e_1}$ will naturally be preferred, since it forms part of an Analysis of Variance procedure. But very frequently no detailed analysis is required and sometimes the choice is between a rapid estimate of standard deviation or none at all. In such cases the range estimate will be of value."

**References:** Helmert (1876a), Fisher (1920), Tippett (1925), E. S. Pearson & Adyanthāya (1928), K. Pearson (1931), Shewhart (1931), E. S. Pearson (1932), Fisher (1933).

**Citations:** Pearson (1935), Pearson & Haines (1935), Nair (1936), Bose (1938), Simon (1941), Dudding & Jennett (1942), Pearson & Hartley (1942), Kendall (1943), Knudsen (1943), Cramér (1946), Kendall (1946), Eisenhart & Solomon (1947), Grubbs & Weaver (1947), Lord (1947), Wilks (1948), Godwin (1949b).

* Dominedò, Valentino (1934). Sulla differenza media. *Giornale degli Economisti e Rivista di Statistica* **74**, 886-894.

**Summary:** The author notes that Gini and Pietra have illustrated the advantage and the propriety of using, as a measure of inequality, the mean difference and, consequently, the ratio of concentration which is an immediate corollary of it, while Mortara has favored the use of the simple mean deviation (from the arithmetic mean) and other authors (Ricci and d'Addario) have favored a measure associated with the Pareto distribution. The author shows, by a geometric proof, that the mean difference (with repetitions) $\Delta_R$ can be obtained by repeated application of the concept of the simple mean deviation S. Let OO′ be the line of equidistribution, M be the point on the Lorenz curve at which its tangent is parallel to

OO′, C be the foot of the perpendicular from M to the horizontal axis OA, and n be the number of observations. Then $\triangle_R$ is equal to $4/n^2$ times the area of concentration bounded by the Lorenz curve and the line of equidistribution, while S is equal to $4/n^2$ times the area of the triangle OO′M, which is inscribed in the area of concentration. The ordinate MC divides the data into two parts, comprising y and n−y observations, of which the simple mean deviations are equal to $4/y^2$ and $4/(n-y)^2$, respectively, times the areas of the triangles OGM and MHO′, where G and H are the points on the Lorenz curve at which the tangents are parallel to OM and MO′, respectively. Now form four more triangles whose bases are OG, GM, MH and HO′, whose vertices are the points on the Lorenz curve at which its tangents are parallel to the respective bases, and whose areas, multiplied by suitable constants, are the simple mean deviations of parts of the data. If one continues this process, the sum of the areas of all the triangles so formed approaches the area of concentration as a limit. The ratio of concentration is twice the area of concentration, and $\triangle_R$ is $4/n^2$ times the area of concentration ($2/n^2$ times the ratio of concentration).

**References:** Ricci (1916), Pietra (1932a), d'Addario (1934b), Mortara (1934b).

**Citations:** Mortara (1934a), Castellano (1935), Pietra (1935b), Boldrini (1942).

* Eyraud, H. (1934a). Sur quelques lois d'erreurs analogues aux erreurs systématiques. *Comptes Rendus de l'Académie des Sciences de Paris* **199**, 763-764. (JFM 60, 471-472).

**Summary:** "Let there be n measurements of a quantity, made under the same conditions with a given apparatus. Arrange these measurements in non-decreasing order of magnitude (1) $x_1 \leq x_2 \leq x_3 \ldots \leq x_n$. In the classical theory of Gauss, one assumes that the most probable value of the measured quantity is the mean of the n measurements. There are certainly cases where the law of error is different from the law of Gauss. In particular, there exist determinations for which only observations that are too low can be obtained. Such are for example: (1) The extreme limit of the length of human life; (2) The weight of an old coinage of which one has available only specimens altered by time. The fact of being thus in a necessarily constant direction brings such errors, which would otherwise be obviously of accidental character, nearer to systematic errors. Numerous cases can also occur where determinations which are too low are more numerous than determinations which are too large or vice versa. Let p be a function included between 0 and 1. Suppose n is large and, for simplicity, such that np is an integer; assume that the most probable value of the quantity being measured is an undetermined value in the interval (2) $x_{np} \leq x \leq x_{np+1}$. An analysis quite similar to that of Gauss leads to assigning to the error function $f(\omega)$ the value (3) $f(\omega) = h\sqrt{p^2+q^2} \cdot e^{(h/q)\sqrt{p^2+q^2}\cdot\omega}$ (for $\omega<0$), $f(\omega) = h\sqrt{p^2+q^2} \cdot e^{-(h/q)\sqrt{p^2+q^2}\cdot\omega}$ (for $\omega>0$), where one has set $q=1-p$. One easily determines, relative to this law of error: The median error (which one has a probability 1/2 of surpassing) (4) $\lambda = -p \log (2p)/h\sqrt{p^2+q^2}$. The mean error (5) $M = (q-p)/h\sqrt{p^2+q^2}$. The dispersion, or second moment about the mean, (6) $\sigma^2 = 1/h^2$. *Case of p=1:* – The limiting law (3) (abstraction made of passing to the limit) retains a sense if one sets $p=1$, $q=0$. Among n observations, the largest measurement is then found to be the most probable. This largest measurement admits a cumulative distribution function of which the expression is $F_n(\omega) = e^{nh\omega}$ for $\omega<0$, $F_n(\omega)=1$ for $\omega>0$; it has the same form as that which relates to each observation, but the precision is n times as great." (Compiler's translation, pp. 763-764).

* Eyraud, H. (1934b). Sur la valeur la plus précise d'un distribution. *Comptes Rendus de l'Académie des Sciences de Paris* **199**, 817-819. (JFM 60, 463).

**Summary:** "Let a random variable be continuous with cumulative distribution function (c.d.f.) u(x). We set (1) $u(x) = 1-w(x)$. We take at random n observations of the variable and arrange them in non-decreasing order (2) $x_1 \leq x_2 \leq x_3 \leq \ldots \leq x_r \leq \ldots \leq x_n$; we seek the c.d.f. of the $(r+1)^{th}$ value. Set (3) $s = n-r-1$. This c.d.f. will be (4) $V_{rs}(x) = \int_0^u [n!/r!s!]\, u^r v^s du$. For n very large and r,s of the same order as n, this function corresponds to a Gaussian distribution. The integral in the second member of (4) contains a principal term which is that for which one has (5) $r/u = s/v = n-1$. Now let (6) $r=(n-1)u_0$, $s=(n-1)v_0$;

(7) $u = u_0 + \sqrt{(u_0 v_0 / n)}\lambda$, and substitute into (4). One obtains, for n infinite, (8) $U = (1/\sqrt{2\pi}) \int_{-\infty}^{\lambda} e^{-\lambda^2/2} d\lambda$, that is, a Gaussian law relative to $\lambda$. If $x_0$ is the value of x for $u = u_0$, one has, assuming the validity of the Taylor expansion in its neighborhood, (9) $u = u_0 = (x - x_0)u'_0 + \ldots$, which gives, as a first approximation, (10) $x = x_0 + \sqrt{(u_0 v_0 / n)}\,(\lambda / u'_0)$; the standard deviation of the $(r+1)^{th}$ value is thus (11) $\mu = (1/u'_0)\sqrt{u_0 v_0 / n}$, the reciprocal of which measures the precision of the $(r+1)^{th}$ value. The precision is maximum when the ratio (12) $|u'_0| / \sqrt{u_0 v_0}$ is maximum, while the dominant [modal] value (that is, the value having maximum probability density) is the solution of $u''(x) = 0$ and the central [median] value (that which one has the same chance of exceeding as of not exceeding) makes the product uv a maximum. It may happen that the precision is uniform over the entire range of the distribution. Then the function u satisfies the differential equation (13) $du/dx = (1/\mu)\sqrt{u(1-u)}$. *Case where s is of smaller order than n.*—One sets, in place of (7): (14) $u = u_0 + \lambda \sqrt{s+1}/n$ and (8) is replaced by (15) $U = (s^s e^{-s}/s!)\sqrt{s+1} \int_{-\infty}^{\lambda} e^{\lambda(s+1)}(1 - \lambda\sqrt{s+1}/s)^s d\lambda$. One easily verifies that $\lambda$ is a random variable with standard deviation unity". (Compiler's translation, pp. 817-819).

**References:** Haag (1924), Gumbel (1933b).

**Citations:** Gumbel (1935a, 1937c, 1939a).

* de Finetti, Bruno (1934). Sul comportamento assintotico della mortalità. *Rendiconti del Circolo Matematico di Palermo* **58**, 359-366. (JFM 60, 1183).

**Summary:** The author writes (pp. 359-361): "Denote by $\ell(x)$ the probability at birth that an individual will die at age $>x$; . . . $\ell(x)$ is the distribution function of the random variable X = 'length of life of an individual' = 'age at death of an individual,' and is precisely $\ell(x) = \text{Prob.}(X > x)$ [the 'distribution function' is usually defined as $1 - \ell(x) = \text{Prob.}(X \leq x)$]. Obviously the function $\ell(x)$ never increases. . . . Assuming that $\ell(x)$ is differentiable, set $\mu(x) = -[1/\ell(x)]\, d\ell(x)/dx = -d \log \ell(x)/dx$. . . . $\mu(x)$ represents the intensity of mortality, and thus $\mu(x)\,dx$ is the probability that an individual, having survived to age x, dies in the interval $(x, x+dx)$. The popular opinion, as a general rule confirmed by experience, that mortality becomes worse as age increases is expressed mathematically in the presumption that $\mu(x)$ increases, if one excludes the period of infancy. . . . Assuming that $\mu(x)$ increases, there are three possibilities for the asymptotic behavior of $\ell(x)$ and $\mu(x)$ as x increases: (a) there exists an age $\omega$ such that $\ell(\omega) = 0$ [while $\ell(\omega - \epsilon) > 0$ for $\epsilon > 0$]; (b) $\ell(x) > 0$ for all x, and $\mu(x)$ increases without limit; (3) $\ell(x) > 0$ for all x, but $\mu(x)$ increases without exceeding an upper limit $\bar{\mu} = \lim_{x \to \infty} \mu(x)$. In cases (b) and (c) it is evident that $\lim_{x \to \infty} \ell(x) = 0$ (since $\int_0^x \mu(z)dz = -\log \ell(x)$ increases without limit). On investigation of the one remaining case prejudice usually strips it completely of any logical value whatever. Thus one sometimes considers the first form to be justified by the observation that the duration of human life cannot be infinite. On other occasions the second and third cases are excluded by an equally absurd claim, an absurdity which sounds totally illusory. Enough said when an author [Insolera (1934)] cannot comprehend how $\mu(x)$ can become greater than unity and therefore terminates the curve $\mu\ell(x)$ at the age where $\mu(x)$ becomes equal to 1. This evidently follows from an inexplicable misunderstanding in which a 'probability density' becomes confused with a 'probability': $\mu(x)dx$, not $\mu(x)$, is a probability. Moreover, it makes no sense to say that $\mu(x)$ is less than or greater than 'unity', since $\mu(x)$ is not a pure number, but a physical quantity with dimension $(\text{time})^{-1}$ . . . ." The author examines the three cases above, and rejects (a) and (b) because both lead to the absurd conclusion that for sufficiently great age x, the probability of surviving any length of time, however small (say one day) is zero. This leaves (c), and the author speculates that $\bar{\mu}$, the asymptotic value of $\mu$, lies between 0.25/year (probable life about three years) and 1.40/year (probable life about 1/2 year). The term "extreme age" or "age limit", which has a precise meaning for case (a), has no such meaning for cases (b) and (c), but the author cites the work of Gumbel (1934g) and Steffensen (1930), who give alternate definitions.

**References:** von Mises (1923), Fisher [& Tippett] (1928), Steffensen (1930, [Insolera (1931)], de Finetti (1932), [Insolera (1933)], de Franchis (1934), Gumbel (1934g), [Insolera (1934)].

**Citations:** de Franchis (1934), Insolera (1935), Gumbel (1936c, 1936-37, 1937a), Insolera (1937-38).

* Fisher, R. A. (1934). Probability likelihood and quantity of information in the logic of uncertain inference. *Proceedings of the Royal Society of London* (A) **146**, 1-8.

**Summary:** The author (p.1) quotes the following words of Jeffreys [(1932b), p. 48]: "Two measures are made. What is the probability that the third observation will lie between them? The answer is easily seen to be one-third." He points out that this statement is ambiguous and may have either of two distinct meanings, one true and the other demonstrably false: "(a) If sets of three independent observations are taken from any continuous distribution, the probability that the third observation of any set shall lie between the first two of the same set is one-third. . . . (b) If the first two observations are the same for all sets, and a third observation be chosen at random independently for each set, the probability that the third observation shall lie between the first two is one-third, for all values of the first two observations." He asserts that Jeffreys uses the false proposition (b) in his derivation, and proceeds with further criticism of Jeffreys' theory and exposition of his own notions of likelihood, quantity of information, maximum likelihood estimation, and sufficient statistics.

**References:** Keynes (1921), Jeffreys (1932b), Bartlett (1933), Fisher (1933), Jeffreys (1933).

**Citations:** Thompson (1936), Kendall (1946).

* Foster, H. Alden (1934). Duration curves. *Transactions of the American Society of Civil Engineers* **99**, 1213-1235; discussion, 1236-1267.

**Summary:** The author begins (p. 1213) with the following synopsis: "The method of analyzing statistical data by means of the duration curve [another name for the cumulative frequency curve (usually cumulated *downward*, with the variable under study plotted vertically and the cumulative percentage horizontally)] is discussed in a general way in this paper. The relation of the duration curve to other statistical curves is described. The computation and method of plotting the curve are outlined, and its mathematical and graphical characteristics are explained and illustrated, followed by a description of graphical methods for using it. Examples of its application to hydro-electric power problems have been included and worked out in detail. The paper concludes with a brief historical review." On page 1216 the author states: "The duration curve, as constructed from actual records, may have two distinct uses (1) If treated as a probability curve, it may be used to determine the probability of occurrence of future events; this use in hydraulic problems was first proposed by the late Allen Hazen [(1914)] . . .; and (2) it can also be used separately from its character as a probability curve, merely as a convenient tool for study of the data . . . . It is to this use . . . that the present paper is limited." The author uses $(2m-1)/n$ as the plotting position of the $m^{th}$ largest (smallest) of n values when cumulating downward (upward). The discussion includes contributions by several persons. C. R. Pettis discusses (pp. 1237-1240) estimation of floods having various return periods. The following statements (pp. 1248-1249) contributed by Howard L. Cook are of special interest: "There are three important statistics, or characterizing values, that should be calculated in all analyses of stream-flow records: (a) The arithmetic mean or average flow; (b) the median flow; and (c) the modal flow. . . . Note that, for a skew frequency curve, the median always lies between the mode and the mean. The calculation of the average flow is always made, of course. The median flow is often noted also, it being the 50% flow on the duration curve; that is, the flow that has one-half the flows greater and one-half smaller than itself. The modal flow, however, is very rarely calculated or used, although it is suggested that, for a number of purposes, it is the most valuable of all characterizing statistics. . . . In connection with the history of the duration curve the following may be of interest. The cumulative frequency curve is usually called an 'ogive' by the statisticians and has been used in statistical studies for many years. It came into prominence through being extensively used by Sir Francis Galton in his studies of inheritance. For this reason it is sometimes called Galton's ogive. The first published paper on the duration curve of stream flow seems to have been that of the late Clemens Herschel [(1878)] . . ., who stated that the use of the curve was suggested to him by the late Joseph P. Davis . . . ."

**References:** Herschel (1878), Yule (1911), Hazen (1914), Czuber (1921), Hall (1921), Foster (1924), Rietz (1924), Goodrich (1927), Hazen (1930), Gibrat (1932b).

**Citations:** Slade (1934), Foster (1936), Beard (1942), Lane & Lei (1949), Langbein (1949).

* de Franchis, Michele (1934). A proposito della precedente nota del Prof. Bruno di Finetti e in riposta ad un libello del Sig. Insolera. *Rendiconti del Circolo Matematico di Palermo* **58**, 367-369. (JFM 60, 1183).

**Summary:** This note is a criticism, quite similar to that of de Finetti (1934), of the statement by Insolera (1934) that the intensity of mortality $\mu = \mu(x)$ [x = age] cannot exceed unity. The author points out, as does de Finetti, that since $\mu$ is not a pure number, but has dimension (time)$^{-1}$, it can be made to take any numerical value whatever by proper choice of the unit of time. For example, if $\mu = 1$ for a one-year period, then $\mu = 5$ for a 5-year period.

**References:** de Finetti (1934), Insolera (1934).

**Citations:** de Finetti (1934), Insolera (1934, 1935), Gumbel (1936-37, 1937a).

* Gumbel, E. J. (1934a). L'espérance mathématique de la m[ième] valeur. *Comptes Rendus de l'Académie des Sciences de Paris* **198**, 33-35.

**Summary:** "The smallest (largest) value among the m largest (smallest) will be called the m$^{th}$ value from above (below). For an unlimited initial distribution w(x) and a very large number N of observations the final distributions of the m$^{th}$ values around their modes $\tilde{u}_m$ and ${}_m\tilde{u} = \tilde{u}_{N-m+1}$ can be written in the unique form (1) $w = [m^m/(m-1)!]\,\alpha e^{\mp my - me^{\mp y}}$, where the minus sign applies to the distribution from above, for which (2) $\alpha = \alpha_m = Nw(\tilde{u}_m)/m$, and the final variable is (3) $y = y_m = \alpha_m (x - \tilde{u}_m)$, while the plus sign applies to the distribution from below, (4) ${}_m w = w_{N-m+1}$ for which (5) $\alpha = {}_m\alpha = Nw({}_m\tilde{u})/m$, and the final variable is (6) $y = {}_m y = {}_m\alpha(x - {}_m\tilde{u})$. The mathematical expectation of the m$^{th}$ value expressed in terms of the final variable will be $\bar{u} = [m^m/(m-1)!] \int_{-\infty}^{+\infty} (y/\alpha + \tilde{u})\, e^{\mp my - me^{\mp y}}\, dy$. Then the difference between the mathematical expectation and the modal m$^{th}$ value will be, after the final transformations (3) and (6), (7) $\alpha(\bar{u} - \tilde{u}) = \bar{y}$, and one will have the mathematical expectation of the final variable of the distribution of the m$^{th}$ value $\bar{y} = [m^m/(m-1)!] \int_{-\infty}^{+\infty} y e^{\mp(m-1)y - me^{\mp y}} e^{\mp y} dy$. By introducing a new variable by (8) $e^{\mp y} = z/m$, one obtains $\bar{y} = \mp [1/(m-1)!] \int_0^{\infty} (\log z - \log m) z^{m-1} e^{-z}\, dz$. Let us introduce Gamma functions. One obtains $\bar{y} = \pm \log m \mp d \log \Gamma(m)/dm$. One will have, as a result of a well-known property of the Gamma function, (9) $\bar{y} = \pm \log m \pm \gamma \mp \sum_{\nu=1}^{m-1} (1/\nu)$, where $\gamma$ is Euler's constant. By introducing the index m one will have the relation (10) $\bar{y}_m = -{}_m\bar{y}$, which is a generalization of the relation $\tilde{u}_m = -{}_m\tilde{u}$ which holds only for symmetric initial distributions. The mathematical expectation of the final variable in the distribution of the m$^{th}$ value from above is equal to but of opposite sign from the same quantity for the m$^{th}$ value from below. One draws from that, with the aid of (2), (5), (7) for symmetric distributions, the evident relation (11) $\bar{u}_m = -{}_m\bar{u}$. For m = 1, one obtains the mathematical expectation of the last and of the first value by $\bar{y}_1 = \gamma$, ${}_1\bar{y} = -\gamma$, or by (7) (12) $\bar{u}_1 = \tilde{u}_1 + \gamma/Nw(\tilde{u}_1)$, ${}_1\bar{u} = {}_1\tilde{u} - \gamma/Nw({}_1\tilde{u})$. For very large values of m one can set approximately $\gamma = \sum_{\nu=1}^{m} (1/\nu) - \log m$. Consequently from (9) the mathematical expectation of the m$^{th}$ value approaches its mode if m increases sufficiently. That results naturally from the fact that the final distribution of the m$^{th}$ value narrows for increasing values of m. In general one will have for the mathematical expectation $\bar{u}_m$ and ${}_m\bar{u}$ of the m$^{th}$ values from above and from below as a result of (7) and (9): (13) $\bar{u}_m = \tilde{u}_m + [m/N\, w(\tilde{u}_m)]\,[\log m + \gamma - \sum_{\nu=1}^{m-1}(1/\nu)]$, (14) ${}_m\bar{u} = {}_m\tilde{u} - [m/Nw({}_m\tilde{u})]\,[\log m + \gamma - \sum_{\nu=1}^{m-1}(1/\nu)]$. The mathematical expectation of the m$^{th}$ value approaches the mode for increasing numbers of observations, provided that the factors $m/Nw(\tilde{u}_m)$ and $m/Nw({}_m\tilde{u})$ increase more slowly with the number of observations than the modes themselves. In order to calculate the mathematical expectation of the m$^{th}$ value with the aid of (13) and (14), one has only to calculate the mode with the aid of the formulas previously indicated." (Compiler's translation, pp. 33-35).

**References:** Gumbel (1933b,c,d).

**Citations:** Gumbel (1934b,c,f, 1935a,b,c,d, 1939f).

* Gumbel, E. J. (1934b). Les moments des distributions finales de la première et de la dernière valeur. *Comptes Rendus de l'Académie des Sciences de Paris* **198**, 141-143.

**Summary:** "For an initial distribution w(x) unlimited in both directions and for a large number of observations, the final distributions $_1w$ of the first and $w_1$ of the last value can be written in the unique form (1) $w=\alpha e^{\mp y}-e^{\mp y}$. It is necessary to add the index 1 to the values of $\alpha$ and y, to the left for the distribution of the first value, and to the right for the distribution of the last value, as we have shown . . . [Gumbel (1934a)]. In order to calculate the moments of order n about the expected values of the two distributions, it suffices to calculate the moments of the last value, which will be expressed in terms of $y_1$, (2) $\alpha_1^n \mu_{n,1} = \int_{-\infty}^{+\infty} \alpha_1^n (x-\tilde{u}_1+\tilde{u}_1-\tilde{u}_1)^n e^{-y_1-e^{-y_1}} dy_1 = \int_{-\infty}^{+\infty} (y_1-\gamma)^n e^{-y_1-e^{-y_1}} dy_1$ since $\gamma$ is the mathematical expectation of the last value. By introducing a new variable z, defined by (3) $e^{-y_1}=z$, one obtains $\alpha_1^n \mu_{n,1}=(-1)^n \int_0^\infty (\log z+\lambda)^n e^{-z} dz$, which can be written $\alpha_1^n \mu_{m,1}=(-1)^n [\int_0^\infty (\log z+\gamma)^n e^{(\log z+\gamma)(p-1)} e^{-z} dz]_{p=1}$. By introducing the $\Gamma$-function and by taking as the argument of this function $p=1+x$ (an expression which has nothing to do with the variable of the initial distribution) the $n^{th}$ moment of the final distribution of the last value becomes $\alpha_1^n \mu_{n,1}=(-1)^n (d^n/dx^n) [e^{\gamma x}\Gamma(1+x)]_{x=0}$. Finally by introducing (4) $e^{\gamma x}\Gamma(1+x)=f(x)$, one obtains (5) $\alpha_1^n \mu_{n,1}=(-1)^n f^{(n)}(0)$. Let us reduce these $n^{th}$ derivatives to logarithmic $n^{th}$ derivatives. One will have, by a well-known theorem of analysis, (6) $f^{(n)}=\sum_{\nu=0}^{n-1} \binom{n-1}{\nu} f^{(\nu)} \log^{(n-\nu)} f$, where the superscripts in parentheses signify the orders of the corresponding derivatives. But the values of the logarithmic derivatives of our function f(x) are known from the theory of $\Gamma$-functions. In fact, (7) $\log' f(0)=\gamma+[d \log \Gamma(1+x)/dx]_{x=0}$ vanishes, whereas for $\lambda \geq 1$, one will have (8) $\log^{(\lambda+1)} f(0)=\log^{(\lambda+1)} \Gamma(1+x)_{x=0}=(-1)^{\lambda+1}\lambda! S_{\lambda+1}$, where the factors $S_{\lambda+1}$ are $S_{\lambda+1}=\lim_{N\to\infty} \sum_{\nu=1}^{N} (1/\nu^{\lambda+1})$, well known numerical values. By introducing these values (7) and (8) into the sum (6) one obtains, with the aid of (5), $\alpha_1^n \mu_{n,1}=(-1)^n (n-1)! \sum_{\nu=0}^{n-2} (-1)^{\nu+n-\nu} (\alpha_1^\nu \mu_{\nu,1}/\nu!) S_{n-\nu}$. By separating the first two terms for which one has naturally $\mu_{0,1}=1$; $\mu_{1,1}=0$, one obtains finally (9) $\alpha_1^n \mu_{n,1}=(n-1)! S_n+(n-1)! \sum_{\nu=2}^{n-2} \alpha_1^\nu \mu_{\nu,1} S_{n-\nu}/\nu!$ For n=2,3,4 one will have the values already calculated by R. A. Fisher [and L.H.C. Tippett (1928)] by a very laborious process. As for the moments of the final distribution of the first value, one obtains them by replacing $\alpha_1$ by $_1\alpha$. The second member of (9) remains the same for the moments of even order and changes sign for the moments of odd order. In particular the standard deviation $\sigma=\sqrt{\mu_2}$ of the two distributions will be $\alpha_1 \sigma_1 = {}_1\alpha\, {}_1\sigma = \pi/\sqrt{6}$. Therefore, all the moments of the final distributions of the first and of the last value can be calculated from very simple formulas." (Compiler's translation, pp. 141-143).

**References:** Fisher [& Tippett] (1928), Gumbel (1934a).

**Citations:** Gumbel (1934f, 1935a,d, 1937a, 1939f).

* Gumbel, E. J. (1934c). Les moments des distributions finales de la m$^{ième}$ valeur. *Comptes Rendus de l'Académie des Sciences de Paris* **198**, 313-315.

**Summary:** "For an initial distribution w(x) unlimited in both directions and for a large number N of observations, the final distribution of the $m^{th}$ value can be written in a symbolic manner (1) $w=[m^m/(m-1)!]\alpha e^{\mp my-me^{\mp y}}$, a formula from which one draws the distributions of the $m^{th}$ value from above and from below by the substitutions which we have described in a previous communication [Gumbel (1934a)]. In order to calculate the $n^{th}$ moments about the mathematical expectations, it suffices to calculate the $n^{th}$ moments of the $m^{th}$ values from above, which will be expressed in terms of $y_m$, (2) $\alpha_m^n \mu_{n,m}=[m^m/(m-1)!] \int_{-\infty}^{+\infty} (y-\bar{y})^n e^{-(m-1)y-me^{-y}} dy$, a formula in which we have written y instead of $y_m$. By introducing a new variable of integration by (3) $e^{-y}=z/m$, one obtains $\alpha_m^n \mu_{n,m}=[(-1)^n/(m-1)!] \int_0^\infty (\log z-\log m+\bar{y})^n z^{m-1} e^{-z} dz$, which can be written $\alpha_m^n \mu_{n,m}=[(-1)^n/\Gamma(m)] [d^n/dp^n] (e^{(p-1)\bar{y}-\log m)} \int_0^\infty z^{m-1+p-1} e^{-z} dz]_{p=1}$. Upon introduction of the Gamma function and by setting for the variable of this function

$p=1+x$, the $n^{th}$ moment of the final distribution of the $m^{th}$ value from above becomes $\alpha_m^n\ \mu_{n,m}=[(-1)^n\ \Gamma(m)]\ [(d^n/dx^n)\ e^{x(\bar{y}-\log m)}\Gamma(m+x)]_{x=0}$. By introducing finally (4) $e^{x(\bar{y}-\log m)}\Gamma(m+x)/\ \Gamma(x)=\phi(x)$, one obtains (5) $\alpha_m^n\ \mu_{n,m}=(-1)^n\ \phi^{(n)}(0)$. We reduce these $n^{th}$ derivatives to logarithmic $n^{th}$ derivatives. One will have (6) $\phi^{(n)}=\Sigma_{\nu=0}^{n-1}\binom{n-1}{\nu}\ \phi^{(\nu)}\log^{(n-\nu)}\phi$. But the values of the logarithmic derivatives of our function are known from the theory of Gamma functions. In fact (7) $\log'\phi(0)=\bar{y}-\log m+[d\log\Gamma(m+x)/dx\ ]_{x=0}$ vanishes, since for $\lambda\geq 1$, one will have $\log^{(\lambda+1)}\phi(x)=\log^{(\lambda+1)}\Gamma(m+x)=[(d^{(\lambda+1)}/dx^{(\lambda+1)})\ \Sigma_{\nu=1}^{m-1}\log(\nu+x)+\log^{(\lambda+1)}\Gamma(1+x)]$. By introducing the logarithmic derivatives of the Gamma functions, one obtains $\log^{(\lambda+1)}\phi(0)=(-1)^\lambda\lambda!\ \Sigma_{\nu=1}^{m-1}(1/\nu^{\lambda+1})-(-1)^\lambda\lambda!\ \Sigma_{\nu=1}^{\infty}\ (1/\nu^{\lambda+1})$, which we are going to write (8) $\log^{(\lambda+1)}\phi(0)=(-1)^{\lambda+1}\ \lambda!\ S_{\lambda+1,m}$, by setting $\Sigma_{\nu=1}^{\infty}\ (1/\nu^{\lambda+1})=S_{\lambda+1,m}$. By introducing these values (7) and (8) into the sum (6), one obtains, with the aid of (5), $\alpha_m^n\ \mu_{n,m}=(-1)^n(n-1)!\ \Sigma_{\nu=0}^{n-2}(-1)^{\nu+n-\nu}\alpha_m^\nu\ \mu_{\nu,m}S_{n-\nu,m}/\nu!$. By separating the first two members, for which one naturally has $\mu_{0,m}=1$, $\mu_{1,m}=0$, one obtains finally (9) $\alpha_m^n\ \mu_{n,m}=(n-1)S_{n,m}+(n-1)!\Sigma_{\nu=2}^{n-2}\alpha_m^\nu\ \mu_{\nu,m}S_{n-\nu,m}/\nu!$. For $m=1$ one will have $S_{\lambda,m}=S_\lambda$, and the formula gives the moments of the final distribution of the last value. As for the moments of the final distribution of the $m^{th}$ value from below, one obtains them by replacing $\alpha_m$ by ${}_m\alpha$. Except for that, the second member of (9) remains the same for the moments of even order and changes sign for the moments of odd order. In particular, the standard deviation $\sigma_m=\sqrt{\mu_{2,m}}$ of the $m^{th}$ values from above and from below will be (10) $\alpha_m\ \sigma_m={}_m\alpha_{\ m}\sigma=\sqrt{S_{2,m}}$. Thus the moments of the final distributions of the $m^{th}$ values from above and from below can be derived by a procedure exactly analogous to the calculation of the moments of the final distributions of the last and the first values." (Compiler's translation, pp. 313-315).

**Reference:** Gumbel (1934a).

**Citations:** Gumbel (1935a,b,c, 1939f).

* Gumbel, E. J. (1934d). Le paradoxe de l'âge limite. *Comptes Rendus l'Académie des Sciences de Paris* **199,** 918-920.

**Summary:** "The expected value $\bar{u}$ of the largest value [von Mises (1923)] of a Gaussian variable having mean $\xi$ and standard deviation $\sigma$, is (1) $\bar{u}=\xi+\bar{t}\sigma/\sqrt{2}$. The value $\bar{t}$, calculated for a sufficiently large number N of observations by $\Phi(\bar{t})=1-2:N$, where $\Phi$ represents the integral of Gauss, tends, for large values of N, toward (2) $\bar{t}=\sqrt{\log N}$. One may use (1) for an approximate calculation of the limiting age $\bar{\omega}$ by treating the age at the time of death as a Gaussian variable, which is legitimate for high ages. Then $\xi$ will be the normal age where the density of mortality is maximum and (3) $\sigma\sqrt{2/\pi}=E(\xi)$ the corresponding life expectancy. One will have $N=2D\ell(\varepsilon)$, where $\ell(\varepsilon)$ is the probability of a newborn attaining the normal age and D the number of deaths used for the construction of the table. Then the limiting age will be [Gumbel (1932, 1934g)] (4) $\bar{\omega}=\xi+\bar{t}E(\xi)\sqrt{\pi}$. Moreover one arrives at an analogous expression by applying Gompertz' law to the mortality table. The limiting age increases with the normal age and its life expectancy. But these two influences may counterbalance each other. Because mortality tables having large values of the normal age have, in general, small values of life expectancy at that age. These tables may be considered as favorable since they will also have in general a large value of life expectancy of a newborn. For two tables, one more favorable than the other, having $\xi_1>\xi_2$; $E_1(\xi_1)<E_2(\xi_2)$ and the same probabilities $\ell(\xi)$, the limiting age, calculated for the same number of observations, will be larger for the less favorable table. Because, from a number of observations defined by (5) $t_0\sqrt{\pi}>(\xi_1-\xi_2)/\ [E_2(\xi_2)-E_1(\xi_1)]$, one will have, according to (4), (6) $\bar{\omega}_2>\bar{\omega}_1$. In this sense, a more favorable table will have a smaller longevity. . . . The reason for this paradox is very simple: the largest deviation to be expected for any distribution is a multiple of the standard deviation. But, for a mortality table unfavorable relative to another, the standard deviation will be larger. Being given this superiority of unfavorable tables, the data on men living at a lower level, who will nevertheless have attained high ages is not necessarily devoid of all sense. There are then two contrary points of view for judging a mortality table: the normal age and the limiting age. The usual criterion is preferable, even though the highest normal age is followed by a 'premature' death of the last survivor." (Compiler's translation, pp. 918-920).

**References:** von Mises (1923), Gumbel (1932, 1934g).

**Citations:** Gumbel (1935a,c, 1936c, 1937a, 1939f).

* Gumbel, E. J. (1934e). La distribution finale des valeurs voisines de la médiane. *Comptes Rendus de l'Académie des Sciences de Paris* **199,** 1174-1176.

**Summary:** "For an initial distribution $w(x)$ and for N observations the distributions of the $m^{th}$ values from above and from below are [Gumbel (1933b)] (1) $w_m(x) = \binom{N}{m} m W^{N-m}(1-W)^{m-1} w(x)$, ${}_m w(x) = \binom{N}{m} m (1-W)^{N-m} W^{m-1} w(x)$. Let us introduce $m = Np$, $N - m = Nq$. The modes $\tilde{u}_m$ and ${}_m\tilde{u}$ of the $m^{th}$ values are the roots of (2) $(Np/W)w - [(Np-1)/(1-W)]w + w'/w = 0$, $[Nq/(1-W)]w - [(Np-1)/W]w - w'/w = 0$. For the [double] exponential distribution $w(x) = 2e^{-1} e^{-|x|}$, these equations lead to (3) $W(\tilde{u}_m) = q = 1 - W({}_m\tilde{u})$. The median $\tilde{x}$ of an initial distribution being defined by $W = 1/2$, one calls the $m^{th}$ values such that $p \approx 1/2 \approx q$ *neighbors of the median*. For an initial distribution such that the median is not too far from the mode, the third terms in (2) may be neglected with respect to the other two. Then the modes of $m^{th}$ values near the median, given by (4) $W(\tilde{u}_m) = qN/(N-1) = 1 - W({}_m\tilde{u})$, arise, for sufficiently large values of N, from equations (3) valid for the extreme $m^{th}$ values. The probability density of the modes will be related for N sufficiently large by (5) $w_m(\tilde{u}_m)/w(\tilde{u}_m) = \sqrt{N/2\pi pq} = {}_m w({}_m\tilde{u})/w({}_m\tilde{u})$. For the Gaussian initial distribution, Haag [(1924)] has proved that the final distribution of $m^{th}$ values near the median is Gaussian. One can easily extend this theorem. For the $m^{th}$ values we set (6) $W(x) = q + y_m$; $W(x) = p - {}_m y$, and we obtain, according to (1), $w_m(x)dx = [w_m(\tilde{u}_m)/w(\tilde{u}_m)](1 \pm y/q)^{Nq}(1 \pm y/p)^{Np-1} dy = {}_m w(x)dx$. The upper (lower) signs hold for the left (right) equation with the index m of the variable y to the right (left). The usual expansion of the product in powers of y leads, according to (5), to distributions (7) $w_m(x)dx = \sqrt{N/2\pi pq}\, e^{-Ny^2/2qp}\, dy = {}_m w(x)dx$. The condition stated above allows us to set (8) $W(x) \approx 1/2 + (x - \tilde{x})w(\tilde{x})$. The transformed variables will be $y_m = w(\tilde{x})[x - \tilde{x} + (q-p)/2w(\tilde{x})] = {}_m y$. It follows that: The final distributions of $m^{th}$ values near the median of an initial distribution satisfying (8) are Gaussian, having in accord with (3) the expected value and standard deviation (9) $\bar{u}_m = \tilde{x} \pm (q-p)/2w(\tilde{x}) = {}_m\bar{u}$; $\sigma_m \approx [1/w(\tilde{x})]\sqrt{pq/N} = {}_m\sigma$. The difference between the $m^{th}$ values from above and from below will be a Gaussian distribution with mean and standard deviation $\bar{u}_m - {}_m\bar{u} = (q-p)/w(\tilde{x})$; $\sigma_\triangle = [1/w(\tilde{x})]\sqrt{2pq/N}$. For the median, which, in the ordinary sense, is defined only for N odd, it is necessary to choose a value $\tilde{m}$ such that $N = 2\tilde{m} - 1$, that is, $p = 1/2 + 1/N$. For sufficiently large values, one obtains a Gaussian distribution having, according to (9), the standard deviation (10) $\sigma_{\tilde{m}} = 1/2w(\tilde{x})\sqrt{N} < \sigma_m$. The condition (8) holds for the Gaussian initial distribution. Thus it reproduces itself for the $m^{th}$ value near the median. But one cannot extend these results beyond $m \approx N/2$. Because, like the Bernoulli distribution, the distributions of $m^{th}$ values lead for $N \to \infty$ to two limits. For extreme values, where $p \to 0$ or $q \to 0$ and $W \to 1$ or $W \to 0$, the previous procedure [Gumbel (1933c,d)], parallel to the deduction of Poisson, leads to a doubly exponential distribution. For values near the median, where $p \approx q$ and $W \to q$ or $W \to p$, one returns to the distribution of Gauss." (Compiler's translation, pp. 1174-1176).

**Comments:** In obtaining his expression for the standard deviation $\sigma_\triangle$ of the difference between the $m^{th}$ values from above and from below the author makes the unwarranted assumption that those values are uncorrelated.

**References:** Haag (1924), Gumbel (1933b,c,d).

**Citations:** Gumbel (1935a, 1937c, 1939b,c,e).

* Gumbel, E. J. (1934f). Les plus grands âges en Suisse. *Zeitschrift für Schweizerische Statistik und Volkswirtschaft* **70** (4), 602-614.

**Summary:** From a study of the greatest ages attained by Swiss people, the author draws the following conclusions (p. 614): "In our previous work we have calculated the limiting age by analogy with the mathematical expectation of the largest value for an unlimited distribution and for a large number of observations by treating the distribution of deaths according to age as Gaussian. We have assumed that the final distribution of the largest value was also Gaussian. In this article knowledge of the final form of

this distribution has allowed us to free our theory from those assumptions and nevertheless to apply it to our observations. For the three criteria considered, the dependence of the greatest age on the time duration considered, the cumulative frequencies of the greatest ages and their extreme values, our theory is well verified. This agreement is possible because these apparently individual cases rest upon numerous observations. This ties our researches to the works of Bortkiewicz. When that author proved that the number of soldiers killed each year by a horse followed this [Poisson] law, he hardly wanted to add to our equine knowledge. Likewise we have followed the fate of the woman who attained 106 years in order to show that extreme values, considered in general as arbitrary, follow well the rules of the calculus of probabilities, and that one can even verify that—provided the observations are good." (Compiler's translation).

**References:** Gumbel (1932, 1933a,b,c,d,f,g, 1934a,b).

**Citations:** Gumbel (1935a,d), Insolera (1935), Gumbel (1936c), Coutagne (1937), Gumbel (1937a), Coutagne (1939), Gumbel (1939f), Barricelli (1943).

* Gumbel, E. J. (1934g). L'età limite. *Giornale dell'Istituto Italiano degli Attuari* **5**, 52-79.

**Summary:** "We define the limiting age as the age for which it is expected that, among a finite number of individual observations, a single individual will surpass it. We apply a survival formula of Lexis to find a satisfactory agreement with American observations." (Compiler's translation of author's summary, p. 52). In section 1, the author gives the new definition of limiting age mentioned in his summary. In section 2, he discusses the median and the mode of the largest value. In section 3, he applies the results to a mortality table, and in section 4, he gives a numerical example. He tabulates the largest value which can be expected (as a function of the sample size N) under the hypothesis of a Gaussian distribution.

**References:** Czuber (1903) [1924], von Bortkiewicz (1922a,b), von Mises (1923), Tippett (1925), Fisher [& Tippett] (1928), Steffensen (1930), de Finetti (1932), Gumbel (1932, 1933c,e).

**Citations:** Gumbel (1933f), de Finetti (1934), Gumbel (1934d, 1935a,c,d, 1935-36), Insolera (1935), Gumbel (1936c, 1936-37, 1937a, 1937-38, 1939f).

* Insolera, F. (1934). Un premio di scienza attuariale e il giudizio di un'Accademia. *Giornale di Matematica Finanziaria* **16** (4-5), 134-147.

**Summary:** In Section 5 (pp. 143-147), the author discusses the extreme (oldest) age, $\omega$. He lists the titles of several of his earlier papers on the subject [including Insolera (1931)]. Let $p(x,y)$ be the probability that a person of age x survives to age y, continuous for $0 \leq x \leq y \leq \omega$, where the oldest age $\omega$ is the smallest value such that $p(x,\omega)=0$. Then the following relations hold: $\lim_{y\to\omega-0} p(x,y)=\lim_{y\to\omega-0}\exp[-\int_x^y \mu(z)\,dz] > 0$, where $\mu$ is the intensity of mortality, and $\lim_{y\to\omega+0} p(x,y)=0$. The author asserts that $\mu(y)$ has a discontinuity at $\omega$ such that $\lim_{y\to\omega-0}\mu(y)$ = a finite positive quantity (which he takes to be 1) and $\lim_{y\to\omega+0}\mu(y) = +\infty$.

**Comments:** As de Finetti (1934) and de Franchis (1934) have pointed out, the author is incorrect in stating that the upper limit of the intensity of mortality is + 1 or, for that matter, any finite value. The intensity of mortality is not a probability, which has an upper limit of + 1, but a probability density, which has no finite upper limit.

**Reference:** Insolera (1931), de Franchis (1934).

**Citations:** de Finetti (1934), de Franchis (1934).

* Jeffreys, Harold (1934). Probability and scientific method. *Proceedings of the Royal Society of London* (A) **146,** 9-16.

**Summary:** The author makes a further contribution, not directly relevant to our study, to his controversy with Fisher concerning probability and statistical inference. Of greater interest are the following statements (p. 14): "I have little to add here to the analysis already given by Bartlett [(1933)]. There was an unfortunate obscurity in my paper, which is partly removed by a note I had an opportunity of communicating to Bartlett. It is quite true that if we have no previous observations the probability of the third observation lying between the first two is 1/3, whatever the law of error and the distribution of probability of a and h [mean and precision constant]. But when we have made two observations we have relevant information; the mean gives a clue to the true value, and the difference gives a clue to the standard error. My postulate is that with this extra knowledge the probability that the third observation lies between the first two is still 1/3, and this is equivalent to the postulate that the standard error is completely unknown until there are two observations. This is a common case, but obviously not universal. If the standard error is already known, the postulate is untrue.

**References:** Fisher (1922), Bartlett (1933), Jeffreys (1933).

**Citation:** Thompson (1936).

* Mortara, G. (1934a). Misure ed indici delle disuguaglianze statistiche. *Rendiconti del Seminario Matematico e Fisico di Milano* **8**, 83-103. (JFM 60, 1172-1173).

**Summary:** After an introduction dealing with measures and indices in general, the author discusses the problem of measurement of statistical variability and the criteria applied to its solution. He makes a comparative study of two principal criteria, the mean deviation (usually taken from the arithmetic mean) and the mean difference. Calculation of the latter is facilitated by first arranging the n observations in order of magnitude: $u_1 \leq u_2 \leq \ldots \leq u_n$. Next comes a study of the application of the two criteria, followed by consideration of a modification of the second criterion and its consequences. This modification is the inclusion of the n zero differences (between each observation and itself), making a total of $n^2$ differences instead of $n(n-1)$. The author gives reasons for preferring the first criterion, and insists that the single apparent advantage of the second criterion (lack of dependence on the mean) is illusory, since the concept of the mean, thrown out the door, comes back in through the window. He discusses similarities and differences between the two criteria in the application of the first with reference to the arithmetic mean. The remainder of the paper deals with the concentration of a statistical series and its indices, the distinction between the concepts of variability and of concentration, the graphical interpretation of these concepts, and further discussion of measures and indices.

**References:** Dominedò (1934), Fréchet (1935), Pietra (1935a).

**Citations:** de Vergottini (1939, 1940), Boldrini (1942).

* Mortara, Giorgio (1934b). Sulle disuguaglianze statistiche. (French summary). *Revue de l'Institut International de Statistique* **2** (1), 72-77; *Bulletin de l'Institut International de Statistique* **28** (1935), (2), 165-170.

**Summary:** The author discusses measures and indices of inequality and of concentration. Measures of inequality can be based on one or the other of two definitions: A) one calls "inequality" the difference between each datum and a supposed level of equality corresponding to a mean of the data (usually the arithmetic mean); B) one calls "inequality" the difference between each datum and each of the others. Criterion A leads to an indirect comparison, and Criterion B to a direct comparision, among the data of which one wishes to measure the inequality. If there are n data points, Criterion A gives n inequalities

and Criterion B gives $n(n-1)$. The sum of the absolute values of the inequalities measures total inequality, and the quotient of this sum by n in the first case, or by $n(n-1)$ in the second, measures the arithmetic mean inequality [the mean deviation in Case A or the mean difference in Case B]. The ratio between the total inequality and the sum (or between the arithmetic mean inequality and the arithmetic mean) of n data points is a measure of the relative inequality. The ratio of the observed value of the total inequality (or the mean inequality) to the maximum value which it can attain (given the number of data points and the sum of the data) gives a measure of the degree of inequality. The choice between the two criteria depends upon theoretical and practical considerations. If $n = 2$, criterion B is simpler in application and more intuitive in interpretation, but as n increases, criterion A becomes more advantageous. Criterion B appears to have the advantage of being independent of the choice of a mean, but this advantage is illusory. For large n, one employs in practice an arithmetic mean of the particular inequalities. Moreover, one also uses a mean in computing measures of relative inequality. Criterion A also has the advantage of corresponding better with the habits of practice and of science. Since one can obtain measures of inequality very simply, one can see no reason to use indices of inequality. The use of several such indices can lead to extreme confusion, as in the case of the various indices employed in researches on the inequality of incomes [see, e.g., Yntema (1933)]. Another useless complication has been caused by the introduction of measures or indices of concentration, since the idea of concentration can easily be reduced to that of inequality, defined by one or the other of the criteria which the author has discussed.

**Reference:** Yntema (1933).

**Citations:** Dominedò (1934), Pietra (1935b), d'Addario (1936), Bresciani-Turroni (1939), Boldrini (1942), Kendall (1946).

* Münzner, Hans (1934). Über die Bewertung der Potenzmomente. *Monatshefte für Mathematik und Physik* **41**, 375-383. (JFM 60, 1165).

**Summary:** The author studies the precision of the $\alpha^{th}$ absolute moment ($\alpha>0$) for the generalized Gaussian distribution function $\phi_\chi(\epsilon) = [h_\chi/2\Gamma(1/\chi)]\, e^{-h^\chi|\epsilon|^\chi}$, $\chi \geq 1$. He shows that the maximum precision is attained when $\alpha=\chi$. He points out that in the special case $\alpha=\chi=2$, this reduces to the statement that the standard deviation is the most precise measure of dispersion for the Gaussian distribution, as shown by Gauss himself.

**Note:** Another special case, not emphasized by the author, is $\alpha=\chi=1$, in which the result reduces to the statement that the mean deviation is the most precise measure of dispersion for Laplace's first distribution. We have already seen [Glaisher (1872) and Fechner (1874)] that the mean deviation is a minimum when taken about the median rather than about the arithmetic mean.

**References:** Gauss (1816) [1880], Jordan (1869), Helmert (1876b), Bertrand (1889).

* Pollard, Harry S. (1934). On the relative stability of the median and arithmetic mean, with particular reference to certain frequency distributions which can be dissected into normal distributions. *Annals of Mathematical Statistics* **5**, 227-262. (JFM 60, 1175); abstract, *Bulletin of the American Mathematical Society* **40**, 31. (JFM 60, 480).

**Summary:** Most of the results in this paper deal with the relative stability of median and mean in samples from non-homogeneous populations (mixtures of two or three populations). Interesting by-products of the study include: (1) an exact expression $\sigma_M = 1/(2\sqrt{2n+3})$ for the standard deviation of the median of samples of $(2n+1)$ items (n an integer) from a rectangular population with probability density function $f(x)=1$ over a unit interval, which compares with the classical (large-sample) approximation $\sigma_M = 1/[2f(0)\sqrt{s}]$ where s is the sample size and $f(0)$ is the value of the p.d.f. at the population median; (2)

upper and lower limits for the standard deviation of the median for samples from any population; and (3) a method of determining the probable error (or any percentile) of the distribution of the median for samples of size (2n+1), for which Dodd (1922) has given the p.d.f.

**Reference:** Dodd (1922).

**Citations:** Kendall (1946), Eisenhart, Deming & Martin (1948a).

* Rosin, P.; Rammler, E. (1934). Die Kornzusammensetzung des Mahlgutes im Lichte der Wahrscheinlichkeitslehre. *Kolloid-Zeitschrift* **67**, 16-26.

**Summary:** The authors discuss the distribution of grain sizes in ground materials. They define the median as the value that divides the distribution into two equal parts; they find the median by taking the abscissa of the point on the cumulative distribution curve whose ordinate is 1/2. They note that the median usually lies between the arithmetic mean and the mode, except for symmetric distributions (e.g., the Gaussian or normal distribution), for which all three coincide. In connection with their study of typical distribution curves for grain sizes, they discuss the Pearson system of frequency curves.

**References:** Pearson (1895), Fechner (1897), Czuber (1903 [1914], 1921 [1927]), von Mises (1931).

**Citation:** Roller (1941).

* Slade, J. J., Jr. (1934). An asymmetric probability function. *Proceedings of the American Society of Civil Engineers* **60**, 1097-1123; reprint, *Transactions of the American Society of Civil Engineers* **101** (1936), 35-61; discussion, 62-104.

**Summary:** ". . . The paper is subdivided into the following parts: Section I contains a critical discussion of the various methods in use at present in the analysis of frequency distributions. . . . The requirements of a function that is to be one degree more general than the Gaussian are discussed in Section II. This function is introduced in Section II and is subjected to detailed mathematical analysis, and the constants of the curve are expressed in terms of the moments of the distribution which it is to fit. Finally, it is proved that the curve is a true generalization of the Gaussian, becoming identical with it, in fact, when the skewness parameter vanishes. In Section III the formulas derived in Section II are collected and a procedure for their use is outlined. A number of examples are given which serve both to illustrate the manipulation of the new curve and to compare the results obtained by its use with those obtained by the use of other functions [notably the Pearson Type III]. In Section IV, finally, the most general homograde function is discussed. Its parameters are expressed in terms of the bounds and standard deviation of the statistics, and a procedure for its application is outlined." (From the author's synopsis, pp. 1097-1098). The function developed in Sections II and III is bounded at one end and depends upon three parameters; the one developed in Section IV is bounded at both ends and depends upon four parameters. The reprint is followed by discussion by Gordon R. Williams, H. Alden Foster, R. D. Goodrich, F. T. Mavis, L. Standish Hall, Arne Fisher, Arthur W. Kempert, and the author. Williams points out (p. 62) that "the accurate determination of extremely high flows is more difficult than that of lower flows." Foster (pp. 63-64) mentions certain difficulties in the use of Slade's method and reiterates his preference for methods based on Pearson's Type I and Type III curves, for which the distance from the mean to any point on the duration curve is directly proportional to the coefficient of variation. Goodrich (pp. 64-67) regards Slade's work as a notable contribution, but disagrees with the latter's criticism of graphical and semi-graphical methods [Goodrich (1927)], as does Mavis (pp. 67-69). Hall (pp. 69-72) warns of difficulties in applying statistical methods to short records, and opines that "the distribution of data cannot be determined with accuracy for a 100-yr. period until at least a 200-yr. record is available." Fisher (pp. 72-88) reviews the history of the problem of fitting frequency curves, defends his use of the logarithmic transformation, which Slade had criticized, and gives numerical examples. Kempert (pp. 88-91) emphasizes certain pitfalls in the use of probability paper and other graphical devices, especially when drawn to logarithmic or semilogarithmic scales, and in the estimation of higher moments from small numbers of observations.

The author (pp. 91-104) answers some of the criticisms of his method contained in the discussion, and gives a 5-page table useful in applying his method.

**References:** Kapteyn (1903) [Kapteyn & van Uven (1916)], Foster (1924), Rietz (1924), Goodrich (1927), Hazen (1930), Foster (1934).

**Citations:** Foster (1936), Saville (1936), Kimball (1938), Beard (1942), Lane & Lei (1949).

* Smith, T. (1934). The mid-course method of fitting a parabolic formula of any order to a set of abservations. *Proceedings of the Physical Society (London)* **46,** 560-569; discussion, 569-573.

**Summary:** The author proposes minimizing the maximum residual instead of the sum of the squares of the residuals in fitting a curve whose equation is of the form $y=a+bx+cx^2+\ldots+kx^{n-1}$ to physical data. On page 561 he states: "The problem to be considered here is the determination of the coefficients a,b,c, . . . so that none of the larger discrepancies exceed a minimum [sic] value; the smaller discrepancies will not trouble us. In other words, we completely neglect all the observations that fall in mean positions and determine our coefficients from the outlying observations alone. If the extreme errors prove to be too great the order of the polynomial must be increased. The outlying observations can usually be selected without difficulty by plotting y against x, but it may sometimes be necessary to plot $y-\phi(x)$ against x, where $\phi(x)$ is an approximate expression for y. The number of outlying points should be one more than the number of terms in the polynomial, and therefore two more than the order of the polynomial. As x increases from its lowest to its highest value they must be chosen to lie on opposite sides of the mean locus of points." The author gives a method of fitting a curve which satisfies these conditions, and illustrates its use in three numerical examples. In each case the results are compared with those given by the method of least squares, and in one case with those given by the so-called zero-sum [Boscovich-Laplace (?)] method. The discussion by Lord Rayleigh, J. H. Awbery and others (with a reply by the author) is generally favorable, especially with regard to the simplicity of the calculations. However, Lord Rayleigh offers the opinion that the mid-course method proposed by the author hardly takes sufficient account of the difference between an error and a mistake, and Mr. Awbery questions whether it will often be suitable in pure physics. The latter points out that, for the extreme case of a formula of degree zero in the independent variable, in which we have a number of estimates of a single quantity, the criterion leads to the use of the midrange as a representative value, to which he objects.

**Citation:** Bruen (1938).

* Thomson, Godfrey H. (1934). Hotelling's method modified to give Spearman's g. *Journal of Educational Psychology* **25,** 366-374.

**Summary:** "A modification of Hotelling's process [for extracting the largest principal component; see Hotelling (1933)] is described which when applied to a hierarchical matrix of correlations gives Spearman's g, after which the unmodified process gives the specifics; and the interpretation of the two processes is discussed." (Author's summary, p. 374).

**Citation:** Lawley (1940).

* Anderson, Oskar N. (1935). *Einführung in die Mathematische Statistik*. Julius Springer, Wien. (JFM 61, 566).

**Summary:** In Chapter 2, Section 2 (pp. 135-145), the author discusses the arithmetic mean. In Section 3 (pp. 145-149), he discusses other averages which can be used in place of the arithmetic mean, including the geometric, harmonic and antiharmonic means, the median and the mode. He notes that, since the logarithm of the geometric mean of a number of quantities is the arithmetic mean of their logarithms, the geometric mean is also called the logarithmic mean. He states and proves the following theorem about the median: The sum of the absolute values of the deviations of all members of a series from their median is not greater than the sum of the like deviations from any other positive or negative number A

whatever. Separate proofs are given for odd and for even values of N, the number of members of the series, $x_1 \leq x_2 \leq \ldots \leq x_N$. For $N = 2k + 1$ (k an integer), the author shows that $\Sigma_{i=1}^{2k+1} |x_i - A| > \Sigma_{i=1}^{2k+1} |x_i - x_{k+1}|$ for $A \neq x_{k+1}$ (the median). For $N = 2k$, he shows that $\Sigma_{i=1}^{2k} |x_i - A| = \Sigma_{i=1}^{2k} |x_i - (x_k + x_{k+1})/2|$ if $x_k \leq A \leq x_{k+1}$ and $\Sigma_{i=1}^{2k} |x_i - A| > \Sigma_{i=1}^{2k} |x_i - (x_k + x_{k+1})/2|$ otherwise. In Section 4 (pp. 149-160), he discusses measures of dispersion, including the mean deviation, the standard deviation, and Gini's mean difference. He points out that, in view of the theorem proved in the preceding section, the mean deviation $\delta = \Sigma_{i=1}^{N} (x_i - M)/N$, where M is an average, is smallest when M is taken to be the median rather than the arithmetic mean or some other average. To those who object that the mean deviation from the arithmetic mean is easier to calculate, he replies that the mean deviation from the median can be easily calculated from the formulas $\delta = (\Sigma_{i=k+2}^{2k+1} x_i - \Sigma_{i=1}^{k} x_i)/N$ for odd $N = 2k+1$ and $\delta = (\Sigma_{i=k+1}^{2k} x_i - \Sigma_{i=1}^{k} x_i)/N$ for even $N = 2k$. He gives various methods of calculating Gini's mean difference $\Delta$, discusses its relation to the standard deviation $\sigma$, the mean deviation $\delta$ and the concentration ratio, and reviews the controversy between von Bortkiewicz and the Italian statisticians Gini, Savorgnan and Pietra concerning credit for priority with regard to certain results in this area [see von Bortkiewicz (1931), discussion].

**References:** Bowley (1901), Gini (1912), Czuber (1914), Pietra (1915), Julin (1921), March (1930), von Bortkiewicz (1931), de Finetti (1931a).

**Citations:** Czuber (1921) [1938], Castellano (1935), Nair (1940b), Kendall (1946).

* Baidaff, B. I.; Barral Souto, Jose (1935). Estudio de la derivada de una media general. *Boletin Matemático* (Buenos Aires) **8,** 81-83.

**Summary:** "The authors continue their study [Baidaff & Barral Souto (1934)] of means, dealing with the derivative with respect to x of the general mean $(\Sigma\, pa^x)^{1/x}$ with $\Sigma\, p = 1$, the a's and p's non-negative and the values of the a's not all equal. They show that it [the derivative] is not negative and thus arrive at the conclusion that the general mean is a monotone increasing function [of x]." (Compiler's translation of authors' French summary, page 83).

**Reference:** Baidaff & Barral Souto (1934).

**Citations:** Barral Souto (1935, 1938).

* Barral Souto, Jose (1935). Alrededor del signo de una derivada. *Boletin Matemático* (Buenos Aires) **8,** 149-150.

**Summary:** The author gives another proof, using logarithmic derivatives, of the proposition [Baidaff & Barral Souto (1935)] that the general mean $y = (\Sigma\, pa^x)^{1/x}$ is a monotone increasing function of x under the conditions stated in the earlier paper.

**Reference:** Baidaff & Barral Souto (1935).

**Citation:** Barral Souto (1938).

* Baticle, Edgar (1935). Le problème de la répartition. *Comptes Rendus de l'Académie des Sciences de Paris* **201,** 862-864. (JFM 61, 1299).

**Summary:** The author considers the following modification of the problem discussed in two previous papers [Baticle (1933a,b)]: What is the probability $\omega_0^p$ of a grouping of m objects into n groups such that no group contains more than p objects? The exact solution is given by $\omega_0^p = \{\binom{n-1}{m-1} - \binom{n}{1}\binom{n-1}{m-p-1} + \binom{n}{2}\binom{n-1}{m-2p-1} - \binom{n}{3}\binom{n-1}{m-3p-1} + \ldots\} / \binom{n-1}{m-1}$. If m is very large, one has, upon setting $\alpha = p/m$, $\omega_0^p = 1 - \binom{n}{1}(1-\alpha)^{n-1} + \binom{n}{2}(1-2\alpha)^{n-1} - \ldots$. If n is also large, a good approximation is given by $\omega_0^p = [1-(1-\alpha)^{n-1}]^p$.

**Note:** The result for m large is the probability that if a line segment of unit length is divided into n segments by $(n-1)$ points chosen at random, the length of the longest segment will not exceed $\alpha$.

**References:** Baticle (1933a,b).

**Citation:** Garwood (1940).

* Birkeland, B. J.; Frogner, E.(1935). Die extreme Variabilität der Lufttemperatur. *Meteorologische Zeitschrift* **52,** 349-352.

**Summary:** "It will be reported in the following work how the range of a meteorological element increases with duration of observation and that this dependence can be computed from the Gaussian law of error." (Compiler's translation of authors' summary, p. 349).

**References:** Fechner (1897), Bartels (1928), Köppen (1928).

**Citation:** Barricelli (1943).

* Bond, W. N. (1935). *Probability and Random Errors.* Edward Arnold & Co., London.

**Summary:** Chapter III (pp. 34-35) contains a section on averages (arithmetic mean, median, and mode) in which the arithmetic mean is associated with the normal law of error, the median with the "median law" [Laplace's first] and the mode (vaguely) with a distribution approaching a "spike." Chapter IV (pp. 46-65) contains sections on the normal or Gaussian law (pp. 46-48), typical errors (pp. 48-49), estimation of the error by quartile deviations (pp. 56-57), the median law of errors (pp. 57-63) and rejection of data (pp. 64-65). In the section on the normal law, the author mentions the "proof" by Hagen (1837) based on a large number of elementary errors of equal magnitude, each equally likely to be positive or negative, and its generalization by Sir Arthur Eddingfield (no reference given) in which the assumption of equal magnitude of the elementary errors is dropped. As measures of the typical error (dispersion), the author mentions the root-mean-square error, the mean absolute error, and the probable error; the quartile deviation occupies a section of its own. The author states that if a large number of measurements are made, positive and negative errors assumed to be equally likely, the normal law results if all the observations are equally liable to error, being carried out under similar circumstances and with equal care, but the median law (Laplace's first) results if the precision is not constant throughout, but varies haphazardly from observation to observation. In the section on rejection of data, the author gives Chauvenet's criterion and no other. Appendix I (pp. 132-136) is a summary of the chief formulae, and includes a section on normal equations and their solution (by the method of least squares).

**References:** Hagen (1837), Whittaker & Robinson (1924), Jeffreys (1932b).

**Citations:** Bruen (1938), Jeffreys (1938), Roller (1941).

* Castellano, V. (1935). Recente letteratura sugli di variabilità. *Metron* **12** (3), 100-131.

**Summary:** The author notes that several recent works [Yntema (1933), d'Addario (1934b), Dominedò (1934), Mortara (1934b), Pietra (1935a)] have recalled to the attention of statisticians the question of the preference that should be given to various proposed indices of variability, the contrasts between them, and the interrelationships among them. He proceeds to review the literature on the subject, going back as far as the work of Pareto (1897) and the early work of Gini (1909, 1910). He divides his discussion into the following parts: 1. Relation between the indices $\alpha$ of Pareto and $\delta$ of Gini. 2. Relations between the simple mean deviation S and the ratio of concentration R or the relative mean difference, and specifically: (a) whether the indices S and R are concordant or discordant; (b) whether the indices S and R

depend on the arithmetic mean; (c) whether R can furnish an approximation comparable with that furnished by S when the data are listed only by classes. 3. Whether a "measure" of variability exists, and whether one of the two indices S or R is preferable to the other.

**References:** Pareto (1897), Bowley (1901) [1920], Gini (1909, 1910), Furlan (1911), Mortara (1911), Dettori (1912), Gini (1912), Porru (1912), Czuber (1914), Gini (1914a), Pietra (1915), Savorgnan (1915), Ricci (1916), Weinberg (1916), Gini (1918), Niceforo (1919), Dalton (1920), de Pietri-Tonelli (1920), Julin (1921), Savorgnan (1921), Amoroso (1925), Pietra (1925), Gini (1926), Saibante (1928), d'Addario (1930), Gibrat (1930), March (1930), von Bortkiewicz (1931), de Finetti (1931a), d'Addario (1932), Gini (1932), Castellano (1933b), Watkins (1933), Yntema (1933), d'Addario (1934a,b), Dominedò (1934), Anderson (1935), Pietra (1935a,b).

**Citations:** de Vergottini (1940), Kendall (1946).

* Eyraud, H. (1935). Valori osservati di una variabile casuale e loro perequazione. *Giornale dell'Istituto Italiano degli Attuari* **6** (3), 243-255. (JFM 61, 1294).

**Summary:** "Attempts that have been made to arrive at an objective definition of probability have led to deepening the idea of a statistical distribution. The present work studies, from the point of view of the law of large numbers, the relation which exists between a statistical distribution and the random variable from which the data arose. This research has permitted placing in evidence some rules useful in the treatment of two problems, fundamental and inverse, one pertaining to the calculus of probabilities, the other to statistics. They concern observed values of a random variable and their standardization." (Compiler's translation of author's summary, p. 243). The author considers n ordered observations $x_1 \leq x_2 \leq \ldots \leq x_r \leq x_{r+1} \leq \ldots \leq x_n$ from a theoretical distribution with c.d.f. indicated by u(x), assuming for greater simplicity that u(x) is continuous and has continuous first and second derivatives for x varying from $-\infty$ to $+\infty$, with $u(-\infty)=0$, $u(+\infty)=1$. The sample c.d.f. has k/n as the probability of a value not greater than $x_k$. The author shows that the c.d.f. of $x_{r+1}$ is given by $u_{r+1}(x)=(n!/r!s!)\int_0^{u(x)} u^r v^s du = B[r,s;u(x)]$, where B is the incomplete Beta function, $s=n-r+1$, and $v(x)=1-u(x)$. He proceeds to study various properties of this distribution and a normal approximation to it, and uses Bayes' theorem to draw certain inferences.

**Reference:** Coolidge (1925).

**Citations:** Gumbel (1936b, 1937b), Brenet & Armand (1938).

* Fisher, Ronald A. (1935). *The Design of Experiments*. Oliver & Boyd Ltd., Edinburgh. (JFM 61, 566-567). Sixth edition, 1951.

**Summary:** Among the wealth of material in this important book, apparently the only part relevant to our study is the following passage (pp. 57-58 of the sixth edition) dealing with the problem of multiple comparisons: "When the z-test [or the F-test, where $F=e^{2z}$] does not demonstrate sufficient differentiation, much caution should be used before claiming significance for special comparisons. Comparisons, which the experiment was designed to make, may, of course, be made without hesitation. It is comparisons suggested subsequently, by a scrutiny of the results themselves, that are open to suspicion; for if the variants are numerous, a comparison of the highest with the lowest observed value, picked out from the results, will often appear to be significant, even from undifferentiated material. Properly, such unforeseen effects should be regarded only as suggestions for future experimentation, in which they can be deliberately tested. To form a preliminary opinion as to the strength of the evidence, it is sometimes useful to consider how many similar comparisons would have been from the start equally plausible. Thus, in comparing the best with the worst of ten tested varieties, we have chosen the pair with the largest apparent difference out of 45 pairs, which might equally have been chosen. We might therefore, require the probability of the observed difference to be as small as 1 in 900, instead of 1 in 20, before attaching statistical significance [at the 5% level] to the contrast."

**Citations:** Madow (1937), Savur (1937b), Snedecor (1937), Fisher & Yates (1938), Wishart (1938), Newman (1939), Nair (1940a), Chandra Sekar & Francis (1941), Scheffé (1943), Wilks (1943), Cramér (1946), Grant (1946), Kendall (1946), Duncan (1947), Tukey (1948c), Mather (1949), Tukey (1949a,b), Walsh (1949a), Wolfowitz (1949).

* Fréchet, Maurice (1935). Sur les précisions comparées de la moyenne et de la médiane. *Aktuárské Vědy* **5** (1), 29-34. (JFM 61, 1296). Abstract, *Časopis* **64,** 210-211. (JFM 61, 1330).

**Summary:** The author admits that the mean is to be preferred to the median in the majority of cases, but insists that it is not always so—median life or median income provides a more representative value than the corresponding mean. Certain statisticians object to the use of the median on the ground that its precision is less than that of the mean, which is true for the normal law of error (Laplace's second law), but not, as the author shows, for certain other probability laws. He studies two laws for which the median is more precise than the mean, at least for samples of size three. Let $\mu'$ be the standard error of the mean and $\mu''$ be that of the median. For samples of size three from Laplace's first law of error with c.d.f. $F(x)=e^{x}/2$ for $x\leq 0$, $F(x)=1-e^{-x}/2$ for $x\geq 0$, the median is slightly more precise [$\mu'^{2}=\mu''^{2}+1/36$], while for samples of size three from the probability law $F(x)=0$ if $x\leq 1$, $F(x)=1-x^{-\alpha}$ if $x\geq 1$, with $1<\alpha\leq 2$, $\mu''$ is finite but $\mu'$ is infinite. The author presents supporting evidence from a paper by Wilson & Hilferty (1929).

**Reference:** Wilson & Hilferty (1929).

**Citations:** Mortara (1934a), Gumbel (1937c), Fréchet (1940a).

* Gumbel, E. J. (1935a). Les valeurs extrêmes des distributions statistiques. *Annales de l'Institut Henri Poincaré* **5,** 115-158.

**Summary:** The author summarizes the theory of extreme values as developed by himself in 17 earlier papers and by several other authors. He reaches the following conclusions (pp. 154-155): "For an initial distribution w(x) having the [cumulative] probability W(x) and for an increasing number of observations the distributions of the $m^{th}$ largest and $m^{th}$ smallest values [$w_m(x,N)=\binom{N}{m} m\,\{W(x)\}^{N-m}\{1-W(x)\}^{m-1} w(x)$ and ${}_m w(x,N)=\binom{N}{m} m\,\{W(x)\}^{m-1}\{1-W(x)\}^{N-m} w(x)$] converge to two limits corresponding to two limits of the Bernoulli distribution. Let us introduce p and q by [$m=Np, N-m=Nq$]. For the $m^{th}$ values near the median, where $p\sim q$, and $W\to q$ or $W\to p$, one obtains the Gaussian distribution provided the median of the initial distribution is not too far removed from the mode. For the extreme values, where $p\to 0$ and $W\to 1$ or $W\to 0$, one arrives at the doubly exponential distribution [$w=\{m^{m}/(m-1)!\}\,\alpha e^{\mp my-me^{\mp y}}$, where $\alpha=(N/m)w(\tilde{u})$ and $\tilde{u}$ is the mode of the final distribution] provided the initial distribution belongs to the exponential type. The calculation of the modes is the same for the two limiting distributions. For large values of m there exists a complete analogy between the extreme $m^{th}$ values and the $m^{th}$ values near the median, because the two distributions are of the Gaussian type. . . ." (Compiler's translation). The author finds the final distributions and their means, standard deviations, and other moments, and gives applications of the theory.

**References:** Bertrand (1889) [1907], von Bortkiewicz (1922a,b), Dodd (1923), von Mises (1923), Haag (1924), Tippett (1925), Fréchet (1927), Fisher & Tippett (1928), Fogelson (1930), de Finetti (1932), Gumbel (1932, 1933a,b,c,d,e,f,g), Tricomi (1933), Eyraud (1934b), Gumbel (1934a,b,c,d,e,f,g, 1935b).

**Citations:** Gumbel (1935d, 1936a,b), von Mises (1936), Coutagne (1937), Gumbel (1937a,b,c,d), Madow (1937), Brenet & Armand (1938), Gumbel (1938a), Coutagne (1939), Gumbel (1939a,c,e,f), Kendall (1940), Gumbel (1941a), Wilks (1941), Gumbel (1942a,b,c, 1943a,c), Kendall (1943), Scheffé (1943), Wilks (1943), Gumbel (1944), Cramér (1946), Gumbel (1946a), Kendall (1946), Mosteller (1946), Elfving (1947), Gumbel (1947), Epstein (1948a,b), Epstein & Brooks (1948), Gumbel (1948), Wilks (1948), Gumbel (1949), Press (1949).

* Gumbel, E. J. (1935b). Les m^ièmes valeurs extrêmes et le logarithme du nombre d'observations. *Comptes Rendus de l'Académie des Sciences de Paris* **200,** 509-511.

**Summary:** "E. L. Dodd [(1923)] has calculated the largest value for the distributions of Gauss, of Poisson, of Pearson, and of Makeham. It would be interesting to complete these results by a general rule which would give the order of magnitude of this value and to extend this result to $m^{th}$ extreme values. This rule could serve for practical problems of observation, because, among observers, opinions of the largest value vary considerably. For a distribution w(x) unlimited in both directions having mathematical expectation $\bar{x}$ and standard deviation $\sigma$ and for N observations the modes $\tilde{u}_m$ and ${}_m\tilde{u}$ of the $m^{th}$ extreme values from above and from below may be calculated by (1) $W(\tilde{u}_m) = 1 - m/N = 1 - W({}_m\tilde{u})$, provided [see Gumbel (1933b)] (2) $w(\tilde{u}_m)/[1 - W(\tilde{u}_m)] \rightarrow -w'(\tilde{u}_m)/w(\tilde{u}_m)$, $w({}_m\tilde{u})/W({}_m\tilde{u}) \rightarrow w'({}_m\tilde{u})/w({}_m\tilde{u}) \ldots$. [This condition] determines a class of distributions for which this method is valid. Set, for the absolute values of the logarithmic derivatives of the initial distribution of modal $m^{th}$ extreme values, (3) $Nw(\tilde{u}_m)/m = \alpha_m$, $Nw({}_m\tilde{u})/m = {}_m\alpha$, values which have the reciprocal dimension of the variable. Then the increment of the modes with the number of observations will be [after correction of an obvious typographical error] (4) $d\tilde{u}_m/dN = 1/N\alpha_m$, $d_m\tilde{u}/dN = -1/N_m\alpha$. If the quantities $\alpha$ tend toward a constant for increasing values of N, that is, for increasing values of the variable, the modal $m^{th}$ values will be linear functions of the logarithm of the number of observations, which one may write (5) $\tilde{u}_m = \bar{x} \pm k\sigma \log (N/m) = {}_m\tilde{u}$, where k is that constant. The upper (lower) sign holds for the left (right) equation. If, however, the values of $\alpha$ increase with N, the modal $m^{th}$ values increase more slowly than the logarithm of the number of observations (5') $\tilde{u}_m = \bar{x} \pm k\sigma f[\log (N/m)] = {}_m\tilde{u}$. The treatment of $m^{th}$ values is thus legitimate only for initial distributions such that the first two moments do not diverge. The two cases also occur in final distributions [see Gumbel (1934a,c)] of $m^{th}$ values (6) $w_m(x) = [m^m/(m-1)!]\alpha e^{\mp my - me^{\mp y}} = {}_m w(x)$, where (7) $y = \alpha(x - \tilde{u})$. For the left (right) equation, it is necessary to add to the values $\alpha$, $\tilde{u}$ and y the subscript m to the right (left). If the values $\alpha$ tend toward a constant, this distribution no longer changes in form with N, and is only displaced to the right: one cannot increase the precision by increasing the number of observations. In the case where the $\alpha$'s increase, the distributions tighten for increasing numbers of observations and the precisions increase so that in reality there exists but one $m^{th}$ value. Finally Fréchet [(1927)] has constructed an initial distribution of a positive variable for which the value $\alpha$ decreases. Condition (2) is no longer satisfied; the mode increases faster than the logarithm of N and the distribution of the $m^{th}$ value from above, distinct from (6), is displayed, a possibility which an inexperienced observer might be tempted to eliminate a priori. Otherwise, like the final distribution of Fréchet, the distribution (6) of the last value is stable." (Compiler's translation, pp. 509-511).

**References:** Dodd (1923), Fréchet (1927), Gumbel (1933b, 1934a,c).

**Citations:** Gumbel (1935a,d, 1936c, 1939f), Kendall (1946).

* Gumbel, E. J. (1935c). Le plus grand âge, distribution et série. *Comptes Rendus de l'Académie des Sciences de Paris* **201,** 318-320.

**Summary:** "By treating the mortality table as the distribution of ages x at death one can verify by observations the theories about the largest value. One assumes this distribution to be unlimited to the right so that the greatest age becomes a random variable. Its mode $\tilde{\omega}$, a monotone increasing function of the numbers of observations N, that is, of the deaths in a year or in a period, will be calculated with the aid of the survival function $\ell(x)$ by (1) $\ell(\tilde{\omega}) = 1/N$. The probabilities that the greatest ages observed during a series of years are less than x must be the doubly exponential probabilities of largest values (2) $W(x) = e^{-e^{-y}}$, where, $\mu(\tilde{\omega})$ being the mortality rate of the modal largest age, (3) $y = \mu(\tilde{\omega})(x - \tilde{\omega})$ is a reduced variable. One determines the two constants $\mu(\tilde{\omega})$ and $\tilde{\omega}$ from the arithmetic mean $\bar{\omega}$ and the standard deviation $\sigma$ of the observed distribution with the aid of (4) $\tilde{\omega} = \bar{\omega} - 0.45005\,\sigma$; $1/\mu(\tilde{\omega}) = 0.77970\,\sigma$, and one can compare the theoretical probability (2) with the observed cumulative frequencies without needing an analytic formula for the biometric functions. For the observed ages of the last survivors in Sweden

1881-1932 and the two sexes one obtains with the aid of formulas (4) (4′) $\tilde{\omega}=101.28$; $1/\mu(\tilde{\omega})=1.5$. Figure 1 shows the agreement between the observed and theoretical numbers of years in which the age of the last decedent was less than or equal to x. In order to verify how the greatest age attained up to a certain time increases with time, it is necessary to introduce a hypothesis concerning the trend of biometric functions with age. Let us choose the formula of Gompertz (5) $\mu(x) = be^{\gamma x}$, which completes the necessary conditions for the calculation of the largest value and one obtains from (1) the modal value of the greatest age, as a function of the number of deaths N considered, by $e^{b/\gamma-(b/\gamma)e^{\gamma\tilde{\omega}}} = 1/N$, from which one draws the first approximation $\tilde{\omega} = (1/\gamma)\log(\gamma/b) + (\log\log N)/\gamma$. By introducing the most probable age at death $\xi$ and its life expectation $E(\xi)$ one obtains (7) $\tilde{\omega}=\xi+1.67688\,E(\xi)\log\log N$, a very simple formula and entirely analogous to that previously deduced with the aid of the hypothesis that the distribution of deaths about the age $\xi$ is Gaussian. Neither requires any knowledge of greatest ages. By choosing for $\xi$ and $E(\xi)$ the values from the Swedish mortality table for 1901-1910 situated in the middle of the interval studied, one obtains (7′) $\tilde{\omega}=78.667+10.035\log\log N$. Figure 2 compares these theoretical values with the observed trend of greatest age attained up to a certain year. The two graphs show that our theory on the greatest age is verified to the extent that one can expect for a quantity so difficult to determine." (Compiler's translation, pp. 318-320).

**References:** Dodd (1923), Gumbel (1932, 1933b,d, 1934a,c,d,g).

**Citations:** Gumbel (1937a,e, 1939f), Kendall (1946).

* Gumbel, E. J. (1935d). Le più alte età in Svezia. *Giornale dell'Istituto Italiano degli Attuari* **6** (4), 335-377.

**Summary:** "The author considers again the theory of the limiting age expounded in a preceding work [Gumbel (1934g)], avoiding consideration of an approximate value of the standard deviation of the largest value by substituting for it the exact value. He then introduces the formula of Gompertz and applies the extended theory to the observed distribution of greatest ages in Sweden and proceeding with this distribution in time, the author finds a better agreement between the theory and the observations than in the preceding work." (Compiler's translation of author's summary, p. 335).

**References:** Dodd (1923), Fisher & Tippett (1928), Steffensen (1930, 1931), Gumbel (1933b,d,g, 1934a,b,f,g, 1935a,b).

**Citations:** Gumbel (1936c), Coutagne (1937), Gumbel (1937a,d, 1939f), Barricelli (1943).

* Gumbel, E. J. (1935-36). La plus grande valeur. *Aktuárské Vědy* **5**, 83-89, 133-143, 145-160.

**Summary:** "For an initial distribution unlimited to the right and for N observations there exists a distribution of the largest value. Its mode can be calculated with the aid of formula (7) [$W(u)=1-1/N$] provided the number of observations is large enough to allow the introduction of l'Hopital's rule in the form (6) [$\lim_{N\to\infty}\{w(x)/[1-W(x)]\} = -w'(x)/w(x)$]. The condition (13) [$w^{(\nu)}(u)/w(u)=(-1)^\nu\alpha^\nu$, where $w(x)/\{1-W(x)\}=\alpha$] identical with (6) allows us to construct the final form of the distribution of the largest value for a very large number of observations. This final distribution is doubly exponential. It is for it that we have calculated the mathematical expectation of the largest value (25) [$\bar{u}=u+\gamma/Nw(u)$, where $\gamma\doteq 0.577216$ is Euler's constant] which approaches the mode under certain conditions. In addition we have found a recurrence formula (37) [$\alpha^n\mu_n=(n-1)\,S_n+(n-1)!\sum_{\nu=2}^{n-2}\alpha^\nu\mu_\nu S_{n-\nu}/\nu!$] which allows the calculation of all the moments with the aid of sums $S_\nu$ [$=\lim_{N\to\infty}\sum_{\lambda=1}^{N}1/\lambda^\nu$] known by the theory of Gamma functions. For the exponential distribution the final distribution is reached even for very small numbers of observations. On the other hand one needs 100,000 observations for the Gaussian distribution. For an exponential initial distribution and for the special case called the Pearson type III distribution, all the moments with the exception of the mathematical expectation will be independent of the number of observations. Thus the final distributions for increasing numbers of observations are dis-

placed to the right along the axis of the variable. For the Gaussian distribution the final distributions shrink for increasing numbers of observations. By choosing the same precision for the three distributions considered one obtains: The identification between the mathematical expectation of the largest value and the mode is legitimate if the number of observations exceeds the order of magnitude $(1/3)10^5$ for the exponential distribution, $10^5$ for the Gaussian distribution and $(1/3)10^7$ for the special case called the Pearson type III distribution. This high number comes from the fact that this final distribution does not shrink. The final distribution of the last value for every distribution which behaves at infinity like an exponential tends toward the final distribution of the exponential initial distribution. Its mathematical expectation tends toward the mode. The final distribution of the last value is propagated to the right for increasing numbers of observations. If the value $\alpha$, defined by (5) $[w(x)/\{1-W(x)\}=\alpha]$, tends toward a constant, the distribution no longer changes form. If the value $\alpha$ increases indefinitely, it shrinks. In the first case, the mode of the largest value increases as the logarithm of the number of observations; in the other case, it increases more slowly." (Compiler's translation of author's conclusion, p. 160).

**References:** Czuber (1903) [1924], Frank (1918), von Bortkiewicz (1922a,b), Dodd (1923), von Mises (1923), Splawa-Neyman (1923), Coolidge (1925) [1927], Tippett (1925), Fréchet (1927), Fisher [& Tippett] (1928), de Finetti (1932), Tricomi (1933), Gumbel (1934g).

**Citations:** Gumbel (1936-37), von Mises (1936), Gumbel (1937a,b,c,d, 1937-38, 1939f, 1941a).

* Insolera, Filadelfo (1935). A proposito di un ipotetico comportamento assintotico della mortalità. *Giornale di Matematica Finanziaria* **17** (2-3), 74-85; *Rendiconti del Circolo Matematico di Palermo* **59,** 303-306. (JFM 60, 1183).

**Summary:** The author responds to the criticisms by de Finetti (1934) and de Franchis (1934), especially the former, of his statement [Insolera (1934)] that the upper limit of the intensity of mortality is $\lim_{x\to\omega-0}\mu(x) = +1$. He admits that $\mu(x)$ is a probability density, not a probability, and hence does not necessarily have an upper limit of +1; rather, it is $\mu(x)dx$, the probability that an individual living at age x dies in the interval $(x, x + dx)$, that has an upper limit of +1. Nevertheless, he asserts that $\mu(x)$ has a finite, positive upper limit M. He defines the force of mortality $\mu(x, t, y_i)$, where x is the age, t is the time in which the age x is observed, and $y_i$ are specific biological parameters of the demographic group under consideration, and the force of vitality $\pi(x, t, y_i)$ and asserts that $\mu(x, t, y_i) + \pi(x, t, y_i) = M$, $\pi(\omega, t, y_i) = 0$ and $\mu(\omega, t, y_i) = M$, where $\omega$ is the oldest age and M is a positive constant independent of x. Then he suppresses the last two arguments of $\mu$, asserts that $\lim_{x\to\omega-0}\mu(x) = M$, and dismisses de Finetti's criticism as a short-lived product of nonserious meditation.

**References:** Bohlmann (1909), Steffensen (1930), Insolera (1931, 1933), de Finetti (1934), de Franchis (1934), Gumbel (1934f,g).

**Citation:** Insolera (1937-38).

* McKay, A. T. (1935). The distribution of the difference between the extreme observation and the sample mean in samples of n from a normal universe. *Biometrika* **27,** 466-471. (JFM 61, 572).

**Summary:** A study is made of the correlation of two linear combinations of the order statistics of a sample of size n from a normal population, and of the regression of one linear combination on another, and mathematical formulas are worked out for the general case. For the special case X = the highest observation, $\bar{x}$ = the mean, and $u = X - \bar{x}$, the correlation coefficient R for X and $\bar{x}$ is tabulated (3DP) for n=2,5,10,20,60,100,200,500,1000 along with the standard deviation (4DP) of u, which has previously been given by Pearson (1931), p. cxiv. The distribution of u for n=2 is shown to be half-normal; for n>2 it cannot be found in closed form, but an equation involving a single integral is given for the low percentage points needed in testing the significance of the departure of the highest observation from the sample mean. Examples are given of the use of this significance test in deciding whether or not to reject outlying observations, and this test is compared with a test suggested by Irwin (1925b) [see also K. Pearson

(1931), p. 160] and one based on the distribution of range tabulated by E. S. Pearson (1932).

**References:** Irwin (1925b), K. Pearson (1931), E. S. Pearson (1932).

**Citations:** Hartley (1942), Kendall (1946), Gumbel (1947), Nair (1948b), Wilks (1948).

* Olds, E. G. (1935). Distribution of greatest variates, least variates, and intervals of variation in samples from a rectangular universe. *Bulletin of the American Mathematical Society* **41,** 297-304. (JFM 61, 572). Abstract, *Bulletin of the American Mathematical Society* **40** (1934), 803-804. (JFM 60, 479—listing only).

**Summary:** Expressions are given for the distributions of greatest variates, least variates, and intervals of variation (ranges) in samples of size N drawn, without replacement, from the population characterized by the frequency distribution $f(x)=1$ for $x=0,1,2, \ldots, b$; $f(x)=0$ elsewhere. The author's results for discrete variates with a single variate in each class differ considerably from those of Neyman & Pearson (1928) for a continuous distribution and of Rider (1929) for discrete variates with an infinite number of variates in each class. The means and variances of the greatest variate G, the least variate L, and the range R for the author's population are found to be $M_G=[N(b+1)-1]/(N+1)$, $M_L=(b-N+1)/(N+1)$, $\sigma^2_G=\sigma^2_L=\{N(b+1)^2-[(N^2-N)b+(2N^2-N)]\}/[(N+2)(N+1)^2]$, $M_R=(N-1)(b+2)/(N+1)$, and $\sigma^2_R=\{2(N-1)(b+1)^2-2(N-1)[(N-1)(b+1)+N]\}/[(N+1)^2(N+2)]$.

**References:** Dodd (1922), Neyman & Pearson (1928), Rider (1929), Craig (1932a).

**Citation:** Kendall (1946).

* Pearson, E. S. (1935). *The Applications of Statistical Methods to Industrial Standardisation and Quality Control.* Publication No. 600, British Standards Association, London. Review by S. S. Wilks, *Journal of the American Statistical Association* **31** (1936), 421-422.

**Summary:** The portions of this publication which are relevant to our study are those which deal with methods based on the maximum and minimum observations in a sample and especially with their difference (the range). The following statements are of particular interest: "The larger the number of observations or tests, the greater the range of variation to be expected among them, since exceptional individual cases are more likely to be met." (p. 24). ". . . The minimum criterion might be applied to *the weakest strip in a sample of n pieces* instead of to the *mean*. If the variation in strength can be represented by a Normal curve with a mean $\bar{x}$, and standard deviation $\sigma$, then Tables are available from which such limits can be determined; they will be fixed at $\ell=\bar{x}-k\sigma$ where the value of k will depend upon (1) the number, n, of units in the sample; (2) the chance of rejection or acceptance which it is decided to adopt." (p. 64). The author discusses the estimation of $\sigma$ from the mean value of the range by means of the relation $\sigma_e=(1/d_n)$ (mean value of range) on p. 84, where he states: ". . . Experience suggests that if N be greater than 50 and the observations have been broken up into equal sub-groupings each containing not more than 10 units, the estimate from range will be adequate for control chart purposes." On pp. 89-90 the author discusses control charts for range. On p. 89 he states: "The range chart is clearly much quicker to prepare than that for standard deviation, and theoretical investigations suggest that its use in this form of control analysis is justifiable." He warns, however, that the use of range should be restricted to cases in which there are a number of small samples (not more than 10 units each) and that methods based on the range are affected more by non-normality than those based on the standard deviation. In Table 14 (p. 90) he gives, for $n=2(1)12$, factors for calculation of control limits for range $D_{0.005}$, $D_{0.050}$, $D_{0.950}$, $D_{0.995}$ for limits, $d_n$ and $1/d_n$ for central value, where outer limits are at $D_{0.005}\sigma$ and $D_{0.995}\sigma$, inner limits are at $D_{0.050}\sigma$ and $D_{0.950}\sigma$, and the mean value of $w=d_n\sigma$.

**References:** Yule (1911) [1932], Brunt (1917), K. Pearson (1931), Shewhart (1931), Davies & Pearson (1934).

**Citations:** Dodge (1933) [1935], Eisenhart (1939), Simon (1941), Dudding & Jennett (1942), Pearson & Hartley (1942), Knudsen (1943), Pearson & Hartley (1943), Grant (1946), Lord (1947), Shone (1949).

* Pearson, E. S.; Haines, Joan (1935). The use of range in place of standard deviation in small samples. *Journal of the Royal Statistical Society, Supplement* **2,** 83-98. (JFM 63, 1098).

**Summary:** A study is made of the use of range instead of standard deviation in small samples, with special emphasis on application to quality control charts. It is shown that estimates of the population standard deviation $\sigma$ based on the square root of average subgroup variance are affected more by lack of control than those based on either average subgroup standard deviation or average subgroup range. Since the range and the standard deviation are highly correlated for small samples, charts based on them tend to give similar indications of lack of control—in most cases a point out of (in) control on one is also out of (in) control on the other. The mean subgroup standard deviation is slightly more powerful in detecting the presence of assignable causes than the mean range, but the difference is small for samples (subgroups) of size $n \leq 10$. The authors therefore conclude that the range chart, which is much simpler and only slightly less efficient, may be used instead of the standard deviation chart for $n \leq 10$.

**References:** Tippett (1925), "Student" (1927), Shewhart (1931), Pearson (1932), McKay & Pearson (1933), Davies & Pearson (1934).

**Citations:** Newman (1939), Dudding & Jennett (1942), Hartley (1942), Kendall (1943, 1946), Lord (1947).

* Pietra, Gaetano (1935a). Intorno alla discordanza fra gli indici di variabilità e di concentrazione. (French summary). *Bulletin de l'Institut International de Statistique* **28** (2), 171-191.

**Summary:** Given a series of values $a_1 < a_2 < \ldots < a_n$, set $x_n = r/n$, $y_r = \Sigma_{i=1}^{r} a_i / \Sigma_{i=1}^{n} a_i$ and represent $(x_r, y_r)$ in a system of Cartesian coordinates. The curve in the unit square passing through the points $(x_r, y_r)$ $(r = 0, 1, \ldots, n)$ is the curve of concentration [Lorenz (1905)], increasing from left to right, convex toward the x-axis, and extending from (0, 0) to (1, 1). Twice the area between the curve of concentration and the line of equidistribution $y = x$ is the ratio of concentration [Gini (1914a)]. Denote by $\phi(x)$ the curve of concentration and by $f(x)$ the curve which connects the points $(x_i, a_i)$; then $\phi(x) = \int_0^x f(x)\,dx / \int_0^1 f(x)\,dx$. If $\Delta$ is the mean difference of the values $a_i$ and $A$ is their arithmetic mean, the ratio $\Delta/2A = 1 - 2\int_0^1 \phi(x)\,dx$ holds, where 2A is the maximum value which $\Delta$ can attain. Gini (1914a) has shown that the ratio $\Delta/2A$ is equal to the ratio of concentration. Denote by $S_A$ the simple mean deviation from the arithmetic mean of the values $a_i$; then the relation $S_A/2A = x_1 - \phi(x_1)$ holds, where $x_1$ is the value which maximizes the difference $x - \phi(x)$, which is the value (found by interpolation) such that $f(x_1) = A$. If the simple mean deviation from the median M of the values $a_i$ is denoted by $S_M$, the relation $S_M/A = 1 - 2\,\phi(1/2)$ holds between $S_M$ and its maximum value A. The author states that one of the most interesting aspects of the methodology concerning the statistical measure of concentration is the discordance of the results which follow from the adoption of one or another of the indices proposed by statisticians for this measure. He cites the case of two series (a) and (b) for which the concentration, if measured by the simple mean deviation relative to the arithmetic mean, is stronger for (a) than for (b), but if measured by the ratio of concentration, is weaker for (a) than for (b). The author reviews the literature on the problem of these discordances, and fixes the limits between which they can be confined and the probabilities of verifying them. The general results are illustrated by several applications.

**References:** Lorenz (1905), Gini (1909), Persons (1909), Dettori (1912), Gini (1912, 1914a), Pietra (1915), Ricci (1916), Amoroso (1925), Gini (1930), von Bortkiewicz (1931), Galvani (1931), Gini (1932), Pietra (1932a,b), Yntema (1933).

**Citations:** Mortara (1934a), Castellano (1935), Boldrini (1942).

* Pietra, Gaetano (1935b). A proposito della misura della variabilità e della concentrazione dei caratteri. *Supplemento Statistico ai Nuovi Problemi di Politica, Storia ed Economia* (Ferrara) **1** (1), 3-12.

**Summary:** The author notes that Mortara (1934b) discussed measures and indices of variability and concentration, and concluded that a single index of inequality-concentration (preferably one based on deviations from the arithmetic mean of the data) would suffice. The author asserts that the problem is not as simple as one might infer from the assumption and conclusions of Mortara, and undertakes a new study in which he considers a greater variety of indices than Mortara. He classifies the index $\alpha$ of Pareto and the index $\delta$ of Gini, both of which, like the standard deviation for the Gaussian distribution, correspond to parameters of the respective curves, as "descriptive", but the simple mean deviation from the arithmetic mean, the quadratic mean deviation, the simple and quadratic mean differences, etc., which are pure numbers, as "synthetic". He also points out that, contrary to what d'Addario (1934b) tried laboriously to show, an agreement does not exist between the ratio of concentration R and the simple mean deviation relative to the arithmetic mean S [see Pietra (1935a)]. Finally, he points out that the relation $\delta = \alpha/(\alpha-1)$ between the index $\alpha$ of Pareto and the index $\delta$ of Gini holds when the data obey the Pareto distribution law, but not otherwise [see Furlan (1911)].

**References:** Gini (1912, 1918, 1932), Pietra (1932a), Castellano (1933a), d'Addario (1934b), Dominedò (1934), Mortara (1934b).

**Citations:** Castellano (1935), Bresciani-Turroni (1939), Boldrini (1942).

* Sastry, N. Sundara Rama (1935). The range of samples taken from a rectangular population. *Journal of the Indian Mathematical Society* (New Series) **1** (7), 228-234. (JFM 61, 1311).

**Summary:** The author concludes (p. 234): ". . . The $m^{th}$ moment $\mu_m$ about the arithmetic mean of the range of samples of size n from an infinite rectangular population of range w is given by $\mu_m = [2w/(n+1)]^m F[-m,2; n+1; (n+1)/2]$, [where the confluent hypergeometic function is given by $F(a,b;c,z) = 1+\{a\cdot b/1\cdot c\}z + [a(a+1)b(b+1)/1\cdot 2\cdot c(c+1)]z^2 + \ldots$], and . . . the frequency distribution of the range of samples follows the type I curve of Karl Pearson." On p. 229 he gives the mean range $\bar{w} = [n-1)/(n+1)]w$, and on p. 231 he works out the following: $\mu_2 = [2(n-1)/(n+1)^2(n+2)]w^2$, $\mu_3 = -[4(n-1)(n-3)/(n+1)^3(n+2)(n+3)]w^3$, $\mu_4 = 24[(n-1)(n^2-2n+3)/(n+1)^4(n+2)(n+3)(n+4)]w^4$, $\beta_1 = 2(n+2)(n-3)^2/(n+3)^2(n-1)$, $\beta_2 = 6(n+2)(n^2-2n+3)/(n-1)(n+3)(n+4)$.

**References:** K. Pearson (1902), Irwin (1925a), Tippett (1925), Neyman & E. S. Pearson (1928).

**Citations:** Rajalakshman (1941, 1943).

* Smirnoff, N. [Smirnov, N. V.] (1935). Ueber die Verteilung des allgemeinen Gliedes in der Variationsreihe. *Metron* **12** (2), 59-81. (JFM 61, 1296).

**Summary:** The author finds the distribution of the $k^{th}$ (smallest) order statistic of a sample of size n from a distribution with p.d.f. f(x) and c.d.f. $\Phi(x)$. The c.d.f. of the $k^{th}$ order statistic is given by $W[x_k \leq x] = V_k(x) = \{n!/[(k-1)!(n-k)!]\} \int_{-\infty}^{x} \Phi^{k-1}(\xi)\,[1-\Phi(\xi)]^{n-k} f(\xi)d\xi$. If the variable X follows a uniform probability distribution in the interval (0,1), then $f(\xi) = 1$, $\Phi(\xi) = \xi$, and therefore $V_k(x) = [n!/(k-1)!(n-k)!]\int_0^x \xi^{k-1}(1-\xi)^{n-k}d\xi$. The distribution of the $(n-k+1)^{th}$ smallest ($k^{th}$ largest) order statistic is given by $W[x_{n-k+1} \leq x] = V_{n-k+1}(x) = [n!/(k-1)!(n-k)!] \int_{-\infty}^{x} \Phi^{n-k}(\xi)\,[1-\Phi(\xi)]^{k-1} f(\xi)d\xi$. If $X' = -X$, then $W[x_{n-k+1} \leq x] = W[x'_k \geq -x] = W[-x'_k \leq x]$, and we have $E(x_{n-k+1})^s = E(-x'_k)^s = (-1)^s E(x'_k)^s$, so that $E(x_{n-k+1}) = -E(x'_k)$, $E(x_{n-k+1})^2 = E(x'_k)^2$. The remainder of the paper deals with limiting properties of the distributions of order statistics, and includes a derivation of the doubly exponential distribution of extreme values.

**Reference:** de Finetti (1932).

**Citations:** Gumbel (1937c), Smirnoff (1937b), Scheffé (1943), Smirnov (1944), Kendall (1946), Mosteller (1946), Wilks (1948).

* Thompson, William R. (1935). On a criterion for the rejection of observations and the distribution of the ratio of the deviation to the sample standard deviation. *Annals of Mathematical Statistics* **6**, 214-219. (JFM 61, 1310-1311). Abstract, *Bulletin of the American Mathematical Society* **41**, 200. (JFM 61, 583 — listing only).

**Summary:** The author derives the distribution of $\tau = \delta/s$, where s is the sample standard deviation and $\delta$ is the deviation of an arbitrary observation of the sample from the sample mean. He determines $\tau_0$ so that by rejecting all observations deviating from the sample mean by more than $s\tau_0$ we shall have an average relative frequency of rejections per sample which is fixed at $\phi$. By contrast, Irwin (1925b) has proposed a method in which the relative frequency of rejection of observations is fixed, and hence the relative frequency of rejections per sample is proportional to the sample size. A table is given of the critical values $\tau_0$ and $t_0$, where $\tau_0 = t_0[(n+1)/(n+t_0^2)]^{1/2}$ and $t_0$ is the critical value for a Student t test with $n = N-2$ degrees of freedom, for $N = 3(1)22, 32, 42, 202, 502, 1002$ and $\phi = 0.2, 0.1, 0.05$. A modified procedure in which the samples are divided into subsamples is also considered.

**References:** Irwin (1925a,b), Rider (1933).

**Citations:** Pearson & Chandra Sekar (1936), Arley (1940), Chandra Sekar & Francis (1941), Smirnoff (1941), Cramér (1946), Kendall (1946), Nair (1948b).

* Wold, Herman (1935). A study on the mean difference, concentration curves and concentration ratio. *Metron* **12**(2), 39-58.

**Summary:** The author points out that Gini [(1912)] introduced the mean difference of a set of quantities as an index of variability and two years later [Gini (1914a)] the concentration ratio, showing its relation to the mean difference. He notes that later contributions to these tools of statistical analysis have been made by Gini (1932), Galvani (1931, 1932), Castellano (1933a) and others, who have extended these concepts to the case of continuous frequency curves and investigated special distributions in detail. He states, however, that some circumstances render the study of the subject rather inconvenient. In particular, the definitions are different in the cases of continuous and discontinuous distributions, an inconvenience which the author removes by using Stieltjes integrals to express sums and integrals by a single formula. This is quite straightforward in the case of the concentration ratio, R, but in the case of the mean difference, g, it requires a slight and natural modification for discontinuous distributions. Also, concentration has been defined only for distributions of a non-negative variable, though there is no such restriction in the case of the mean difference. The author generalizes the concepts of concentration ratio and concentration curve to admit negative values of the variable. Finally, he simplifies the inconveniences heretofore encountered in the calculation of g and R for a grouped frequency distribution, and gives numerical examples. He points out the analogy between the concentration ratio, the coefficient of variation, and the ratios (quartile deviation)/(mean of quartiles) and (mean deviation)/median. He also discusses the Cramér-von Mises statistic, $\omega^2$, and the test based upon it [see Cramér (1928), von Mises (1931)], which he says seems to have been proposed first by Cramér in a 1925 Swedish textbook on the theory of probabilities.

**References:** Bowley (1901) [1926], Gini ([1912], 1914a, 1918), Cramér (1928), Gini (1930), de Finetti (1931a), Galvani (1931), von Mises (1931), Galvani (1932), Gini (1932), Castellano (1933a).

**Citations:** Gini (1936), Wold (1937), Davis (1941), Kendall (1946).

* d'Addario, Raffaele (1936). Sulla curva dei redditi di Amoroso. *Annali dell'Istituto di Statistica* (Bari) **10**, 1-57.

**Summary:** The author considers the distribution curves of income proposed by Pareto (1897) and by Amoroso (1925). The equation of the latter is $y\,dx = K \exp[-\gamma(x-h)^{1/s}](x-h)^{(p-s)/s}dx$, where $y\,dx$ is the number of individuals having income in the interval $(x, x+dx)$, $h < x < \infty$, and K, h, $\gamma$, p are essentially positive constants. Special cases of this distribution were considered even earlier by March [s = +1 (Pearson type III)] and by Vinci [s = −1(Pearson type V)]. If $p - s \leq 0$, the curve is monotone decreasing, but if $p - s > 0$, it has a single maximum at $\theta = h + [(p-s)/\gamma]^s$; in either case, it is asymptotic to the positive direction of the x-axis. The author gives a brief explanation of the concentration curve of Lorenz (1905) and the concentration ratio, R. He then works out the parametric equation of the concentration curve corresponding to Amoroso's law of distribution of income in the general case and in the special cases s = +1 and s = −1, and discusses other forms of the concentration curve. After a brief explanation of Mortara's curve of marginal density of income, he discusses the mean numerical deviation relative to its upper limit (S). Then he discusses the determination of R and S in the general case and in the special cases s = +1, h = 0 and s = −1, h = 0. When s = +1, h = 0, he finds that $R = \Gamma(2p)/2^{2p-1}\Gamma(p)\Gamma(p+1)$ and $S = e^{-p}p^p/\Gamma(p+1)$; when s = −1, h = 0, the corresponding values are $R = \Gamma(2p-1)/2^{2p-2}[\Gamma(p)]^2$ and $S = e^{-(p-1)}(p-1)^{p-1}/\Gamma(p)$. In addition to the form of the curve and the degree of concentration of income, he discusses the minimum income and the mean income.

**References:** Pareto (1897), Lorenz (1905), Amoroso (1925), d'Addario (1934a), Mortara (1934b).

**Citation:** Bresciani-Turroni (1939).

* Crowe, P. R. (1936). The rainfall regime of the western plains. *The Geographical Review* **26**, 463-484.

**Summary:** For the analysis of rainfall data, the author proposes a graphical method, based on medians and quartiles, which he states as follows (page 466): "The method employed here can be stated briefly. In essence it regards the series of rainfall values for each month over a long period of years as a frequency group that can be summarized by locating median and quartile values . . . . Three values instead of one are plotted against each month. The median gives a central point or mean expectation of rainfall, and the distance between the quartiles, shaded to make it effective, measures the mean range of variation about it. Obvious advantages over the arithmetic normal are: (1) amounts as large as the median occur, by definition, in precisely half the years—the position has some reality; (2) the shaded area is a constant warning against false accuracy. As the relative position of medians and quartiles for two months clearly give some indication of the rapidity of the change experienced, it is argued that at some stage or other in the relationship we must be able to regard the difference as 'significant'. A convenient working hypothesis is to regard contrast in the precipitation of two months as justified only when the median of each falls very near or beyond a quartile of the other."

**References:** Crowe (1933), Mathews (1936).

**Citations:** Savur (1937a, 1937b).

* Darmois, Georges (1936). L'emploi des observations statistiques. Méthodes d'estimation. *Actualités Scientifiques et Industrielles*, No. 356. Hermann et Cie., Paris. (JFM 62, 1345).

**Summary:** The author summarizes the theory of estimation as outlined by Fisher (1922) and by Fisher and others in later papers. He discusses the relative precision of the median and the arithmetic mean of various populations, and the related question of whether to use the method of least squares or the method of least absolute first powers. In particular, he shows that the ratio of the standard error of the median to that of the mean is asymptotically equal to $\sqrt{\pi/2}$ for samples from a normal distribution, $\sqrt{1/2}$ for samples from Laplace's first distribution, and zero for samples from a Cauchy distribution.

**Reference:** Fisher (1922).

* **Citations:** Gumbel (1937c), Neyman (1937b).

* Foster, H. Alden (1936). Methods for estimating floods. *U. S. Geological Survey Water-Supply Paper 771: Floods in the United States* (ed. Clarence S. Jarvis et al.), pp. 28-67. U. S. Government Printing Office, Washington.

**Summary:** The author begins as follows (page 28): "The purpose of the study embodied in this chapter is to make a compilation of various methods representing the principal bases of approach thus far proposed for estimating the rate of flood to be expected on any given stream, together with a brief analysis of the several methods and a comparison of their relative advantages and disadvantages. No particular attempt is made to select the best methods for use, except as advantages and limitations of particular methods are inherently apparent." On page 32, he gives the following classification of methods: "A. Extreme-flood formulas. B. Flood-frequency formulas. 1. Formulas with frequency relation implied . . . . 2. Formulas with frequency relation expressed. C. Statistical or probability methods. 1. Theoretical probability curves. 2. The duration-curve method. D. Methods dependent on relation of rainfall to run-off. 1. 'Rational' methods . . . . . 2. The unit-hydrograph method. E. Method for estimating peak flow from 24-hour average flow." Several of these methods are based on order statistics.

**References:** Kapteyn (1903) [Kapteyn & van Uven (1916)], Fuller (1914), Hall (1921), Foster (1924), Jarvis (1926), Goodrich (1927), Hazen (1930), Foster (1934), Slade (1934).

**Citation:** Beard (1942).

* Gini, Corrado (1936). On the measure of concentration with special reference to income and wealth. *Cowles Commission Research Conference on Economics and Statistics, Colorado College Publication*, General Series No. 208, pp. 73-80.

**Summary:** The author reviews the literature on measures of concentration. He starts with nineteenth century work on the distribution of individual abilities by Galton (1869) and of individual incomes by Pareto (1897) and others. Pareto presented the formula $\log N_x = K - \alpha \log x$, where K and $\alpha$ are constants and $N_x$ indicates the number of taxpayers with an individual income greater than x, which became known as Pareto's law. Gini (1909) made a substantial improvement by taking account not only of the distribution of the taxpayers, but also of the amount of their incomes. He noticed that if $A_x$ indicates the amount of the incomes above the limit x, the data may be fitted by the formula $\log A_x = r - \beta \log x$. From this and Pareto's formula, he derived the formula $\log N_x = p + \delta \log A_x$. Theoretically, if income obeys Pareto's law, $\beta = \alpha - 1$, $\delta = \alpha/\beta = \alpha/(\alpha - 1)$, but in practice these equalities do not correspond exactly to reality. The author notes that $\delta$ furnishes a direct index of concentration of incomes, while $\alpha$ provides an inverse index. Other advantages of Gini's $\delta$ over Pareto's $\alpha$ given by the author are: 1. Gini's formula takes account of the taxpayers, often very numerous, in the lowest class, for which generally only the upper limit is given. 2. The data are generally fitted better by Gini's formula. 3. Gini's formula satisfactorily fits also the distributions of labor incomes and of rents, while Pareto's does not. 4. The index $\delta$ has a precise meaning; since Gini's formula may be written $N_x/N_0 = (A_x/A_0)^\delta$, $\delta$ is the exponent of the power to which we must raise the fraction of the total amount of incomes represented by the incomes above a certain limit in order to obtain the corresponding fraction of the taxpayers who possessed it. The author points out, however, that Gini's formula, though more widely applicable than Pareto's, is limited to total incomes, labor incomes and rents. A general measure of concentration is given by the concentration curve proposed by Lorenz (1905). Gini (1914a) proposed to take as a measure of concentration (concentration ratio) the area of concentration located between the concentration curve and the line of equidistribution divided by its maximum possible value (1/2), the area of triangle OAB with vertices O(0,0), A(1,1) and B(1,0). He demonstrated that the concentration ratio corresponds to the mean difference [Gini (1912)] divided by its maximum possible value (twice the arithmetic mean). Pietra (1915) demonstrated that the ratio of the mean deviation from the arithmetic mean to its maximum

possible value corresponds to the ratio of the area of the largest triangle (ODA) inscribed in the area of concentration to the area of triangle OAB, while the ratio of the mean deviation from the median to its maximum possible value corresponds to the ratio, to the area of triangle OAB, of the area of triangle OCA with OA as its base and the intersection C of the concentration curve with the vertical line erected at the median point E as its vertex. Gini (1930) demonstrated that the area (1/2) of triangle OAB represents the maximum area of concentration only when the character admits no maximum (may be $+\infty$) and no minimum (may be 0), and gave new limits for cases in which these hypotheses are not satisfied. The author summarizes the contributions of various other authors to theory and applications.

**References:** Galton (1869), Pareto (1897), Lorenz (1905), Gini (1909, 1910, 1912), Porru (1912), Czuber (1914), Gini (1914a), Pietra (1915), Savorgnan (1915), Gini (1918), Vinci (1918), Amoroso (1925), Gini (1926), Saibante (1926), Gumbel (1928), Saibante (1928), Gini (1930), Galvani (1931), Gini (1932), Castellano (1933a), Yntema (1933), Wold (1935).

**Citation:** Davis (1941).

* Girshick, M. A. (1936). Principal components. *Journal of the American Statistical Association* **31**, 519-528. (JFM 62, 1352).

**Summary:** The author (pp. 520-521) outlines two approaches to the problem of finding the principal components of a set of statistical variables, as follows: "We shall . . . get the principal components as by-products of the solution to the following two problems: (1) Given a set of variates, to find a linear function of the variates which has least variance ascribable to errors of measurement. (2) To find a linear function of these variates such that the sum of squares of the correlation between the function and each variate shall be a maximum. The first approach will exhibit a property of the principal components which is of statistical importance, viz. that if all tests have equal reliability coefficients, and therefore equal variances of errors of measurement, then the first principal component is that linear function of the variates which has least variance resulting from errors of measurement, and among all linear functions of our variates which are uncorrelated with the first component, the second component has least variance resulting from such errors, and so on for the other components. The second approach will emphasize the fact (pointed out by Professor Hotelling) that of all linear functions of the variates, the first principal component has the greatest mean-square correlation with the tests; the second component the next greatest mean-square correlation, etc." The author shows that the first principal component is the largest root of a determinantal equation.

**References:** Hotelling (1933, 1936a).

**Citations:** Girshick (1939), Kendall (1946), Tintner (1946).

* Grassberger, Hans (1936). Die Anwendung der Wahrscheinlichkeitsrechnung auf Hochwasserfragen. *Deutsche Wasserwirtschaft* (Stuttgart) **31**, 200-203, 222-225.

**Summary:** The author points out that one of the most important results of modern hydrology is the knowledge that floods on a river do not have an unconditional upper limit, but are governed by the laws of chance. He salutes American hydrologists, especially Fuller (1914) and Hazen (1914, 1930) as pioneers in the application of the theory of probability, and in particular the use of probability paper, in the field of hydrology. He states the principle, which he attributes to Fechner, that for strongly variable physical quantities, the frequency of the logarithms follows the Gaussian law of error; in other words, the quantities themselves follow a lognormal distribution. He notes that this principle, like all statistical laws, is only approximately true. It follows from Fechner's principle that the relation between the logarithms of the ordered characteristics and the Gaussian error integral (normal probability integral) is linear, so that if this principle holds the data will plot as a straight line on lognormal probability paper.

The author plots on lognormal probability paper the maximum daily discharges ($m^3$/sec) for each year over periods ranging from 20 to 66 years for various streams on three continents. In each case the plotted points lie approximately on a straight line.

**Note:** The author, like Fuller and Hazen, uses the plotting position $(2m-1)/2n$ for the $m^{th}$ order statistic of a sample of size n.

**References:** Fuller (1914), Hazen (1914, 1930), Gibrat (1932b), Grassberger (1932, 1933).

**Citation:** Gumbel (1937-38).

* Gumbel, E. J. (1936a). Les inondations et la théorie de la plus grande valeur. *Comptes Rendus de l'Académie des Sciences* (Paris) **203**, 27-29.

**Summary:** The author states: "Floods are certainly the largest values of flows. Now several authors, for example H. Graszberger [Grassberger (1932], have shown that one can treat the flow of a great river in first approximation as a random variable following a Galton [lognormal] distribution, which one obtains from the Gaussian [normal] distribution by a logarithmic transformation. Since the final distribution of the largest value has the same form for all distributions of the exponential type, one may try to represent the distribution of floods by that of the largest value [Gumbel (1935a)] without introducing the interpolation formulas employed up to the present. Among n years, the number of years W(x) having a maximum flow less than x is thus (1) $n\,W(x) = ne^{-e^{-y}}$; $y = \alpha(x-\tilde{u})$. If the number n is increased, the calculation of the two constants $\alpha$ and $\tilde{u}$ by the aid of the moments of the observed distribution becomes tedious. That is why one obtains them from the median $u_0$ and the two quartiles $u_1$, $u_2$ of the observed c.d.f., though this procedure involves, in the case of frequency (1) a loss of precision. The definitions of $u_0$, $u_1$, $u_2$ lead to equations (2) $\tilde{u} = u_0 - 0.36651/\alpha$; $1/\alpha = 0.63591\,(u_2-u_1)$. A table of values W(x) having being calculated [Gumbel (1935a)], one obtains the maximum flows x with probability W(x) by determining x by the linear transformation (1). One can expect that a maximum flow between $x_1$ and $x_2$ will occur $n[W(x_2)-W(x_1)]$ times. For the Rhine at Basle, and for the n = 118 years from 1808 to 1925, the maximum flows are contained in a publication by C. Ghezzi. One draws from these observations $u_0$ = 2682.5; $u_1$ = 2251; $u_2$ = 3078, which lead, with the aid of (2), to the linear transformation x = 2489.28 + 527.17y. Since the table of values W(x) increases by intervals $\Delta y = 1/2$, which corresponds to $\Delta x$ = 263.58, and, since the observations have been classified by intervals of $\Delta x = 250$, the values $\Delta W(x)$ of the table have been multiplied by 111.92. The graph [p. 28] compares the observed distribution of flood flows of the Rhine with the theoretical distribution calculated from (1). The good agreement shows that the application of the largest value to floods is entirely legitimate." (Compiler's translation).

**References:** Grassberger (1932), Gumbel (1935a).

**Citations:** Gumbel (1937a,d, 1938a), Coutagne (1939), Gumbel (1939c,e,f).

* Gumbel, Emil J. (1936b). Les distances extrêmes entre les émissions radioactives. *Comptes Rendus de l'Académie des Sciences (Paris)* **203**, 354-357.

**Summary:** "The distances x between radioactive emissions follow an exponential distribution (1) $w(x) = \beta e^{-\beta x}$, where $\beta$ is of the nature of a decay constant and 1: $\beta$ is the mean distance. According to the theory of the greatest value [Gumbel (1935a)], the mode $\tilde{u}_N$ of the greatest among N distances is (2) $\tilde{u}_N = (1/\beta)\log N$. The final distribution $\mathbf{w}(x)$ of the greatest distances and the probability $\mathbf{W}(x)$ that the greatest distance is less than x are (3) $\mathbf{w}(x) = \beta e^{-y-e^{-y}}$, $\mathbf{W}(x) = e^{-e^{-y}}$, where (4) $y = \beta(x-\tilde{u})$. The mathematical expectation $\bar{u}$ of the greatest distance and the standard deviation $\sigma$, a value independent of the number of observations, are $\bar{u} = \tilde{u} + 0.57722/\beta$, $\sigma = \pi/\sqrt{6}\beta$. . . . [On pp. 355-357, the author gives a numerical example involving a large number of observations on polonium separated into k = 80 groups.] . . .

In order to judge the agreement between the theory and the observations, we have calculated the standard deviations of the theoretical values x. Let us consider x as the $m^{th}$ value from below with c.d.f. $\mathbf{W}(x)$. Then its standard deviation is, according to Eyraud [(1935)], (7) $s=[1-\mathbf{w}(x)]\sqrt{\{\mathbf{W}(x)[1-\mathbf{W}(x)]\}/k}$, a function of y, a method which is valid provided m is neither too small nor of the order of magnitude of k. We have added and subtracted these values to the values x, which leads to two curves between which the observed frequencies should lie. The good agreement between the frequencies and the probabilities proves that the extreme distances between radioactive emissions follow well the theory of the largest value." (Compiler's translation).

**References:** Eyraud (1935), Gumbel (1935a).

**Citations:** Gumbel (1937a,b,d, 1937-38), Brenet & Armand (1938), Coutagne (1939), Gumbel (1939d,f).

* Gumbel, E. J. (1936c). La table de mortalité traitée comme distribution. *Annales de l'Université de Lyon A* (3) **1,** 48-68.

**Summary:** The author treats a mortality table as a statistical distribution and obtains a number of interesting results. Among the averages which he discusses is the median. Except for that, nothing in this paper is based on order statistics until one reaches the fifth and last section, which deals with the limiting age. The author writes (pp. 64-65): "We have denied the existence of a fixed limiting age. Yet there should exist an age such that a given generation has died out. These two points of view meet as soon as one treats mortality tables as distributions. Then the question of the limiting age is reduced to the question of knowing what is the largest value of a distribution $\theta(x)$ unlimited on the right and for N observations. Since we have treated this problem in several earlier works, it will suffice here to give the results. According to them there exists a distribution (31) $w(x) = N[1-\ell(x)]^{N-1}\theta(x)$ of the greatest age, a new unlimited positive statistical variable which one can denote by x. One can easily calculate its mode $\tilde{\omega}$, the most probable greatest age, by (32) $\ell(\tilde{\omega}) = 1/N$, a value which increases with N. This quantity is the number of deaths used in the construction of the table and naturally differs from 100,000, chosen arbitrarily as the starting value. . . . One defines the limiting age as the mathematical expectation of the greatest age. von Mises [(1923)] has proved that this mean tends toward the mode $\tilde{\omega}$ provided that the distribution of deaths tends toward zero more rapidly than an exponential. On the other hand, if it is legitimate, for large values of x, to set (33) $\mu(x)\rightarrow -\theta'(x)/\theta(x)$, the probability that the greatest age at the moment of death is less than x tends to (34) $W(x) = e^{-e^{-\mu(\tilde{\omega})(x-\tilde{\omega})}}$. Formula (33) is an analytic condition imposed on $\theta(x)$. At the same time it is a numerical condition imposed on N. One cannot verify whether the former is fulfilled, since one does not know the natural law for biometric functions. But one can verify equation (34). The observations of the age of the last survivor made in Switzerland during the years 1879-1933 are in perfect accord with this probability [see Gumbel (1934f)]. If one limits himself to the treatment of ages at the moment of death around the normal age [mode, $\xi$], . . ., the difference between the limiting age and the normal age will be a multiple of the mean deviation calculated after that value [the life expectancy at age $\xi$, $E(\xi)$], which leads to (35) $\tilde{\omega} = \xi + E(\xi)f(N)$, where the function f(N) depends on the assumption that one makes about the distribution of deaths. One can prove that it will be in general of the order of magnitude of the natural logarithm of N [see Gumbel (1935b)]. . . ."

**References:** Laplace (1812), von Mises (1923), Steffensen (1930), Gumbel (1933f), de Finetti (1934), Gumbel (1934d,f,g, 1935b,d).

**Citations:** Gumbel (1936-37, 1937a,e, 1939f).

* Gumbel, E. J. (1936-37). L'âge limite. *Aktuárské Vědy (Prague)* **6** (1), 28-38; (2), 79-86; (3), 114-122.

**Summary:** The author reaches the following conclusions (pp. 121-122): "We reject the usual theories of the limiting age and the attempts made up to the present to determine this age in an appropriate manner, in particular the methods which determine the limiting age with the aid of a few observations of advanced ages. On the contrary our theory employs only data corresponding to the neighborhood of the

normal age, for which the numbers observed are numerous and quite certain. The probability of the survival of a single person seems — but only seems — to fall outside the conditions on which the theory of probability rests. However, our solution is based only on it. We assume that the distribution of deaths extends to infinity and approaches zero only asymptotically. Consequently we calculate the last age as the age after which one would expect that, among a given number of deaths, one and only one individual should die. This leads to the extremely simple determination (3) [$\ell(\tilde{\omega}) = 1/N$] from which one obtains the limiting age with the aid of (10) [$\bar{\omega} = \tilde{\omega} + \gamma/\mu(\tilde{\omega})$, where $\mu$ is the intensity of mortality and $\gamma$ is Euler's constant]. In order that a biometric formula be usable for this calculation, it suffices that the product of the age and the intensity of mortality does not diminish with age. The Lexis formula satisfies this condition. Then the last age is given by (18) [$\tilde{\omega} = \xi + \tau E(\xi) \sqrt{\pi}$, where $\xi$ is the normal age and $\tau$, the reduced last age, will be calculated as a function of N by numerical tables] and its standard deviation by (20) [$\sigma = \pi^{3/2} E(\xi)/\sqrt{6} \cdot 2\tau = 1.13663\ E(\xi)/\tau$]. It follows that an increase in the normal age, leading, as a rule, to a decrease in the expectation of life at that age, has an unfavorable influence from the point of view of longevity. Despite the simplicity of the analytic calculations and despite the very gross approximation used, the agreement between the theory and the experience shown by the American tables is satisfactory. It will be quite necessary to distinguish between our general theory of the limiting age and the application which we make by applying the Lexis formula. The observations made on the highest age in Switzerland during 55 years show that the distribution of these values is very much what should be expected. This comparison necessitates no choice of a biometric formula. The theory is again verified by the extreme values of these distributions. This agreement is possible because these apparently individual cases rest upon numerous observations. In this sense and also from the analytic point of view our reasoning is parallel to the research of Bortkiewicz which led to the law of rare events. In summary the theory, which seems paradoxical, according to which there is no fixed limiting age, is in accord with the observations, and thus all the arguments which rest upon the existence of such an age are contradicted". (Compiler's translation).

**References:** Splawa-Neyman (1923), Steffensen (1930), Insolera (1931), Steffensen (1931), Insolera (1933), Gumbel (1933g), de Finetti (1934), de Franchis (1934), Gumbel (1934g, 1935-36, 1936c).

**Citations:** Gumbel (1937a,e, 1937-38), Insolera (1937-38), Gumbel (1939f).

* Horton, Robert E. (1936). Hydrological conditions as affecting the results of the application of methods of frequency analysis to flood records. *U. S. Geological Survey Water-Supply Paper 771: Floods in the United States* (ed. Clarence S. Jarvis et al.), pp. 433-450. U.S. Government Printing Office, Washington.

**Summary:** After some introductory material, the author states (pp. 436-437): "As results of the same general character would be obtained by the application of any one of several existing frequency formulas or methods of frequency analysis applied to the flood data, an analysis by a single formula or method will serve for purposes of illustration. A formula developed by the author [Horton (1924)] will be utilized: $q = q_g - (q_g - q_0)\, e^{-k(t-t_0)^n}$ (1) in which q = magnitude of a flood having . . . an average recurrence interval t . . . . Recurrence interval may be defined as the average interval of occurrence of floods or other hydrologic events equaling or exceeding a given magnitude. Recurrence intervals are given by the formula $t = T/(N - n + 1)$ (2) in which T = total period, in years, covered by the record, N = total number of events in the record, n = the number of a given event or flood in order of magnitude, beginning with the lowest. . . . The recurrence intervals for the different observed events are computed by formula (2), and the magnitudes of the events are plotted in terms of their recurrence intervals. In formula (1), $q_0$ is the minimum value or magnitude of a flood or event of a given type, and $t_0$ is its recurrence interval. . . . The form of equation (1) is such that as $t \to \infty$, the flood magnitude q approaches a limiting or maximum value $q_g$ asymptotically. Formula (1) generally resembles other frequency formulas applied to flood data except in this respect. Some formulas indicate that flood magnitudes increase without limit—in other words, the calculated flood magnitude approaches infinity as the average recurrence interval approaches infinity. This is true, for example, with the normal or Gaussian law of error. In accordance with

formula (1), flood magnitudes always continue to increase as the recurrence interval increases, but they increase toward a definite limit and not toward infinity. This is believed to be the more rational form of expression. . . ." The author goes on to describe the method of determining the constants in equation (1), using three pairs of values of q and t selected from the curve plotted from the observed data. The method is illustrated by an analysis of data on Tennessee River floods at Chattanooga, Tenn. for the period 1874-1933. A mathematical derivation of the constants in equation (1) is given in an appendix.

**Reference:** Horton (1924).

**Citations:** Kimball (1938), Gumbel (1941a, 1942b, 1943b).

* Hotelling, Harold (1936a). Simplified calculation of principal components. *Psychometrika* **1**, 27-35. (JFM 62, 1352).

**Summary:** The author writes (p. 27): "The resolution of a set of n tests or other variates into principal components $\gamma_n$, each of which accounts for the greatest possible portion $\gamma_1, \gamma_2, \ldots$, of the total variance of the tests unaccounted for by the previous components, has been dealt with by the author in a previous paper [Hotelling (1933)]. Such 'factors', on account of their analogy with the principal axes of a quadric, have been called *principal* components. The present paper describes a modification of the iterative scheme of calculating principal components there presented, in a fashion that materially accelerates convergence. . . . The advantage of the present method is enhanced when, as will often be the case in dealing with numerous variates, not all the characteristic roots but only a few of the largest are required."

**References:** Whittaker & Robinson (1924), Hotelling (1933).

**Citations:** Girshick (1936), Aitken (1937), Bartlett (1938), Tintner (1945, 1946), Kincaid (1947).

* Hotelling, Harold (1936b). Relations between two sets of variates. *Biometrika* **28**, 321-377. (JFM 62, 618-619). Abstract, *Bulletin of the American Mathematical Society* **41** (1935), 632-633. (JFM 61, 583).

**Summary:** In the abstract, the author states: "Between two sets of normally correlated variates $x_1, \ldots, x_s, y_1, \ldots, y_t$ the correlations invariant under internal linear transformations are roots $\rho_1, \ldots, \rho_s$ (where $s \leq t$) of a certain determinantal equation. The number of non-vanishing roots is the number of independent common components of the two sets. The primary statistical problem here is equivalent to the algebraic problems of finding invariants and standard forms for two quadratic forms in the x's and y's respectively and a bilinear form in the x's and y's, under transformations of the two sets separately. The solution of the algebraic problem gives the canonical forms $\Sigma_1^s x_i^2$, $\Sigma_1^t y_j^2$, $\Sigma_1^s \rho_i x_i y_i$. For statistical purposes it is also necessary to determine sampling distributions of these roots and of functions of them. Some of these sampling problems are solved completely in this paper, and approximations and partial solutions are given for others. . . ." The author finds, among other things, the joint distribution of the squares of the canonical correlations, arranged in descending order of magnitude, of $(x_1, x_2)$ and $(x_3, x_4)$, two mutually independent pairs of variables of which the first pair is normally distributed.

**Reference:** Hotelling (1933).

**Citations:** Aitken (1937), Bartlett (1938), Fisher (1938, 1939), Girshick (1939), Hsu (1939), Wilks (1943), Anderson (1945), Tintner (1946), Bartlett (1947a,b, 1948), Rao (1948).

* Logan, K. H. (1936). Soil-corrosion studies, 1934. Rates of loss of weight and pitting of ferrous specimens. *Journal of Research of the National Bureau of Standards* **16**, 431-466. (Research Paper RP883).

**Summary:** The following quotations summarize the results relevant to our study: "The soil-corrosion investigation which was started in 1922 was originally planned to cover 12 years, plus the time required to write the final report. In 1934 the last of the original specimens were removed from 23 soils . . . . In general, the results of the examination of the specimens removed in 1934 confirm earlier conclusions. . . . The average maximum penetrations for the wrought-iron, Bessemer, and open-hearth specimens do not differ by more than a few percent." (p. 431). "Early in this soil-corrosion investigation [see Logan & Grodsky (1931)] it was observed that the average maximum pit for the 1-1/2-inch specimens was less than the corresponding average maximum pit on the 3-inch specimens of the same material." (p.455). "Generally speaking, the larger the area from which the deepest pit is chosen the deeper the pit. This fact has an important bearing on the determination of the conditions of a pipe line by means of local inspections. In order to make pit-depth measurements comparable, similar methods of inspecting pipes must be used." (p.465).

**Reference:** Logan & Grodsky (1931).

* Mathews, H. A. (1936). A new view of some familiar Indian rainfalls. *Scottish Geographical Magazine* **52**, 84-97; note by P. R. Crowe, 187-188.

**Summary:** After pointing out the fact that the arithmetic mean is totally inadequate and often misleading in summarizing monthly rainfall data in India, the author uses the method proposed by Crowe (1933), plotting medians and quartiles. He extends this plot in months of high rainfall to include upper and lower octiles, and mentions the possibility of indicating the highest and lowest rainfalls experienced. He espouses the median as a measure of central tendency and the semi-interquartile range (quartile deviation) as a measure of variability. He proposes that the latter be expressed as a percentage of the median. The note by Crowe deals with a slight difference between the method used by Crowe (1933) and that used by Mathews in this paper to test whether the rainfalls of two months differ significantly over a period of years. Crowe shows that the two methods give essentially the same results when applied to Mathews' data on Indian rainfalls.

**Reference:** Crowe (1933).

**Citations:** Crowe (1936), Savur (1937a,b).

* de Mises, R. [von Mises, R.] (1936). La distribution de la plus grande de n valeurs. *Revue Mathématique de l'Union Interbalkanique* (Athens) **1**, 141-160. (JFM 63, 495).

**Summary:** The author begins (p. 141) as follows: "It is in connection with the following problem that one speaks of the 'largest of n values.' A random variable is subject to any distribution law V(x) whatever; in other words V(x) is the probability that the distinctive character of a collective (in a single dimension) does not exceed the value x. One considers a group of repeated trials, that is to say the composition of n collectives equal among themselves and independent of each other. The probability $W_n(x)$ that all n results of these trials fail to exceed x is equal to the $n^{th}$ power $V^n(x)$ of V(x). One can call $W_n(x)$ *the distribution of the largest of n values* since this function gives the probability that the largest of the n results is less than or equal to x. What retains our attention above all is the study of $W_n(x)$ for n very large." (Compiler's translation). The author derives the three asymptotic distributions of largest values and gives sufficient conditions on the initial distribution for the validity of each.

**References:** von Mises (1923), Fisher & Tippett (1928), Gumbel (1935a, 1935-36).

**Citations:** Gumbel (1939e,f), Gnedenko (1941), Gumbel (1941a), Kimball (1942), Gnedenko (1943), Gumbel (1943c, 1944), Cramér (1946), Wilks (1948), Juncosa (1949).

* Nair, U. S. (1936). The standard error of Gini's mean difference. *Biometrika* **28**, 428-436. (JFM 62, 599).

**Summary:** The author begins (p.428) with the following introduction: "The Mean Difference has been introduced by Professor Corrado Gini (1914a) [see comments below]. It is defined as the mean value of all the possible differences between individual observations. Thus, if $x_1, x_2, \ldots, x_n$ are n observed values of a variate x, the mean difference is defined as $g = [2/n(n-1)] \Sigma_{i \neq j} |x_i - x_j|$. In the present paper, expressions will be derived for the standard error of g when the variate follows any continuous distribution law, and, in particular, explicit formulae will be given for the normal, the exponential and the rectangular distributions. In these three cases, it is possible to use the value of g calculated from a sample to estimate the value of the parameter measuring the scale of variation in the population. The reliability of this form of estimate will be compared with that of other alternatives." The author (p.436) reaches the following conclusion: "In the case of the exponential and rectangular distributions the Mean Difference as an estimate of the scale of variation in the population seems to have no advantage over the simpler estimates with which it has been compared. For the normal distribution, $(1/2)\sqrt{\pi} \times$ Mean Difference may be used to estimate the population standard deviation; it is less reliable than estimates based on squares of deviations from the mean, but somewhat more reliable than that based on the Mean Deviation and considerably more reliable (for $n > 10$, say) than that based on the Range. . . ."

**Comments:** The mean difference is also discussed in an earlier paper by Gini (1912), who notes that it was used much earlier by von Andrae (1869, 1872), Jordan (1869, 1872), and Helmert (1876a). The mean difference is defined in terms of the unordered observations, but ordering the observations facilitates its computation.

**References:** Gini (1914a), Neyman & Pearson (1928), Hojo (1931), Gini (1932), Davies & Pearson (1934).

**Citations:** Wold (1937), Kendall (1943, 1946), Hoeffding (1948), Nair (1949b).

* Neyman, J.; Pearson, E. S. (1936). Contributions to the theory of testing statistical hypotheses. *Statistical Research Memoirs* **1**, 1-37. (JFM 62, 609).

**Summary:** After a brief review of the approach to the problem of testing statistical hypotheses developed by them in a number of earlier publications, the authors consider methods which may be followed in choosing a critical region for testing a hypothesis $H_0$ when no uniformly most powerful test exists. They suggest that the practical value of any critical region, w, depends on the properties of its power function throughout the range of admissible values of $\theta$, the parameter of interest, and that a comparison of alternative regions of the same size may be readily effected by a comparison of their power functions. They define unbiased critical regions of types A and $A_1$ and show how to determine such regions. They give a number of examples, including one in which the power of the unbiased region based on the range r is compared with that of the unbiased type A region based on $v = \Sigma(x_i^2)$ for testing the hypothesis $H_0$: $\sigma^2 = 1.0$ for the case $n = 3$, $\alpha = .02$. The greater power of the type A region is due almost entirely to the fact that it makes use of knowledge of the population mean, while its competitor does not. If the population mean is not known, v must be replaced by $v' = \Sigma(x_i - \bar{x})^2/\sigma_0^2$, where $\bar{x}$ is the sample mean. The v'-test is only slightly more powerful than the r-test. The authors note that for $n = 2$ the tests are identical, but as the sample size n increases, the standard deviation test becomes steadily more and more powerful than the range test. They prove three fundamental propositions, two of which deal with the relations among unbiased type A critical regions, sufficient statistics, and regular estimation.

**Reference:** McKay & Pearson (1933).

**Citations:** Pearson & Chandra Sekar (1936), Neyman (1937a,b), Sukhatme (1937), Pitman (1939), Wald (1939), Paulson (1941), Wilks (1943), Cramér (1946), Kendall (1946).

* Pearson, E. S.; Chandra Sekar, C. (1936). The efficiency of statistical tools and a criterion for the rejection of outlying observations. *Biometrika* **28**, 308-319; appendix by J. M. C. Scott, 319-320. (JFM 62, 610).

**Summary:** The authors study the criterion for rejection of outlying observations proposed by Thompson (1935). They point out that it provides complete control over the probability of Type I error (rejecting the hypothesis that all the observations have been drawn from a single normal population, with unspecified mean and standard deviation, when that hypothesis is true). This is true because it is based on a studentized statistic and does not assume, as many tests for rejecting outlying observations do, that the population standard deviation $\sigma$ is known. Nevertheless, the authors show that the criterion is quite inefficient in the presence of two or more outliers unless the sample is quite large. The values of $\tau_i = (x_i - \bar{x})/s$ $(i = 1, 2, \ldots, N)$ are not independent. In the appendix, Scott determines max $|\tau_i|$ $(i = 1, 2, \ldots, N)$ subject to the restriction $\tau_1^2 \geq \tau_2^2 \geq \ldots \geq \tau_n^2$. The authors use the results to show that for significance level $\phi$ of 0.10, Thompson's criterion cannot reject more than 1 observation unless $N > 11$; more than 2 unless $N \geq 22$; or more than 3 unless $N \geq 32$.

**References:** Student (1927), Neyman & Pearson (1928), K. Pearson (1931), Rider (1933), Thompson (1935), Neyman & Pearson (1936).

**Citations:** Arley (1940), Chandra Sekar & Francis (1941), Kendall (1946), Eisenhart & Solomon (1947), Nair (1948b).

* Pitman, E. J. G. (1936). Sufficient statistics and intrinsic accuracy. *Proceedings of the Cambridge Philosophical Society* **32**, 567-579. (JFM 62, 611).

**Summary:** The author proves that if there exists a sufficient statistic for the estimation of an unknown parameter of a population, the frequency function of the population must be of a certain type. He shows that some modification of previous theory of the intrinsic accuracy of statistics is necessary when the range of the population sampled is a function of the paramenter to be estimated. Consider a population distributed over the range (a,b) according to the frequency function $f(x,\theta)$, where $\theta$ is the parameter to be estimated. The author shows that if a is a function of $\theta$ and b is fixed, then a sufficient statistic for $\theta$, if one exists, must be a function only of $x_0$, the smallest member of the sample. Similarly, if b is a function of $\theta$, and a is fixed, any sufficient statistic for $\theta$ must be a function only of $y_0$, the largest member of the sample. If $\theta = a$, $x_0$ is a sufficient statistic for $\theta$, while if $\theta = b$, $y_0$ is a sufficient statistic for $\theta$. The author extends the theory to sufficient sets of statistics, i.e. sets of statistics which together contain all the information provided by a sample about the unknown parameter. When a and b, the end-points of the distribution, are both functions of $\theta$, the greatest and least of a sample constitute a pair of sufficient statistics for $\theta$ if the frequency function of the distribution is of the type $f(x)/F(\theta)$, where $F(\theta) = \int_a^b f(x)\,dx$ or $\Sigma_a^b f(x)$.

**Reference:** Fisher (1922).

**Citations:** Pitman (1937, 1939), Kendall (1946), Lehmann (1947), Huzurbazar (1948).

* Ruark, Arthur; Devol, Lee (1936). The general theory of fluctuations in radioactive disintegration. *Physical Review* (2) **49**, 355-367; preliminary communication, **48** (1935), 772.

**Summary:** "The chance, $W_n(0,t)$ that n atoms of a radioactive source will disintegrate in the interval (0,t) may be found if we know $f_r(t)dt$, the chance that a disintegration will occur in dt at t after r have occurred in (0,t). Bateman's differential equation is generalized to cover the case in which $f_r$ depends on t and r. The solution is given for the case in which it depends only on t. A detector of efficiency g subtends a solid angle $4\pi A$ at a source (decay constant $\lambda$) containing N atoms at time zero. The probability of n counts in the interval $(T_1, T_1 + T_2)$ is $P(T_1, T_1 + T_2) = C_n^N (e^{\lambda T_2} - 1)^n e^{-N\lambda(T_1 + T_2)} (ga)^n [1 + e^{\lambda T_2}(e^{\lambda T_1} - 1) +$

$(1-gA)(e^{\lambda T_2}-1)]^{N-n}$. Putting g and A equal to 1 we get the probability $W_n(T_1, T_1+T_2)$ of n disintegrations in this interval. Bortkiewicz's formula $W_n(0,t) = C_n^N (e^{\lambda t}-1)^n e^{-N\lambda t}$, is a special case. If we measure a great number, r, of intervals between the disintegrations of atoms in a single source, the fraction of the intervals that exceed t has the 'expected' value $(e^{-N\lambda t}-e^{-(N-r)\lambda t})/r(1-e^{\lambda t})$. The problem of determining fluctuations in the disintegration of a single substance from fluctuations in counting is solved for the case in which each disintegration gives a single ray capable of actuating the counter. Fluctuations in counts produced by two or more independent sources are considered. . . ." (From the authors' summary, p. 355).

**Reference:** Czuber (1903).

**Citation:** Ruark (1939).

* Saville, Thorndike (1936). A study of methods of estimating flood flows applied to the Tennessee River. *U. S. Geological Survey Water-Supply Paper 771: Floods in the United States* (ed. Clarence S. Jarvis et al.), pp. 398-420. U. S. Government Printing Office, Washington.

**Summary:** The author begins with the following introduction (p. 398): "This paper is an attempt to apply all the more generally used statistical methods of estimating flood flows to the 57-year record (1875-1931) of the Tennessee River at Chattanooga . . . . The investigation was designed to afford an impartial and unprejudiced analysis of the applicability of various methods to a long-term series of observations on a single stream and to compare results obtained by different methods. As the study is confined to a single river, and in general to a single record on that river, too broad generalizations from the results are probably not warranted. However, a comprehensive comparative study of this kind has not appeared in the literature of flood flows, and it is believed that the results here presented will have some general significance and may stimulate similar investigations of records on streams having different characteristics. This paper presents in a series of sections the application of generally similar methods advanced by different investigators." The methods considered are those of Fuller (1914), Foster (1924), Hazen (1930), Goodrich (1927), and Slade (1934). The Fuller method gives results that represent the estimated average annual flood, or most probable magnitude of annual flood, to be expected in a period of T years, while the other four methods give results that represent the estimated magnitude of annual flood likely to be equaled or exceeded in a given period of years. The differences between Fuller's method and the others have been pointed out by W. E. Fuller and E. W. Lane in their contributions to the discussion of a paper by Jarvis (1926).

**References:** Fuller (1914), Foster (1924), Jarvis (1926), Goodrich (1927), Hazen (1930), Slade (1934).

**Citations:** Gumbel (1938b), Kimball (1938), Gumbel (1939c, 1941a,b).

* Slade, J. J., Jr. (1936). The reliability of statistical methods in the determination of flood frequencies. *U.S. Geological Survey Water-Supply Paper 771: Floods in the United States* (ed. Clarence S. Jarvis et al.), pp. 421-432. U. S. Government Printing Office, Washington.

**Summary:** The author makes the following statement (p. 430) directly relevant to our study: "It appears to be generally conceded not only that there is a definite upper limit to the magnitude of floods in a stream, but also that this limit is not enormously in excess of the greatest flood observed in a long record." He reaches the following conclusions (pp. 431-432) about the application of statistical methods in hydrology: "To represent adequately a stream-flow distribution curve with four arbitrary constants is necessary; this is the required degree of flexibility in order that the curve correspond to the population in mean, standard deviation, skewness, and kurtosis. Because of the errors of sampling only [the first] two of these constants can be determined with any degree of reliability from the items of a small sample. No method of curve fitting can circumvent the indeterminacies introduced by these errors, because the

errors are inherent in the sample. Two more constants may be determined from a knowledge of the limits of the variation. Zero flow may be taken as the lower limit of floods. A modification of the Sherman-Bernard unit-graph method, designed to give the peak flow from a possible storm, the probability of whose occurrence, however, is negligible, can be used to give a reasonable estimate of the upper limit of floods. In the writer's opinion the statistical method, in whatever form employed (graphic or analytical), is an entirely inadequate tool in the determination of flood frequencies. When used in conjunction with nonstatistically inferred data, however, it may attain a high degree of precision."

**Citations:** Kimball (1938), Gumbel (1941a, 1942c).

* Smirnoff, N. [Smirnov, N. V.] (1936). Sur la distribution de $\omega^2$ (Criterium de M. R. v. Mises). *Comptes Rendus de l'Académie des Sciences* (Paris) **202**, 449-452. (JFM 62, 592).

**Summary:** By a slight modification of the Cramér-von Mises statistic [von Mises (1931); the author does not mention Cramér], the author writes it in the form $\omega_n^2 = n\int_{-\infty}^{\infty} g[F(x)]\,[S_n(x)-F(x)]^2\,dF(x)$, where $F(x)$ is the population cumulative distribution function (c.d.f.) and $S_n(x)$ is the sample c.d.f. of a sample of size n: $S_n(x) = k/n (x_k \leq x \leq x_{k+1}; k = 1, 2, \ldots, n-1)$; $S_n(x) = 0 (x<x_1)$ and $S_n(x) = 1 (x>x_n)$, where $x_1 \leq x_2 \leq \ldots \leq x_n$ are the ordered sample values. He points out that the c.d.f. $\Phi(\omega)$ of $\omega_n^2$ is independent of the form of the function $F(x)$, and derives an asymptotic expression for $\Phi(\omega)$ which reduces to the form $\Phi(\omega) = 1-(1/\pi)\sum_{k=1}^{\infty}\int_{(2k-1)\pi}^{2k\pi}(e^{-z^2\omega^2/}\sqrt{-z\sin z})dz$ for $g(x) = 1$.

**Reference:** von Mises (1931?).

**Citations:** Smirnoff (1937a), Cramér (1946), Kendall (1946), von Mises (1947).

* Sukhatme, P. V. (1936). On the analysis of k samples from exponential populations with especial reference to the problem of random intervals. *Statistical Research Memoirs* **1**, 94-112. (JFM 62, 613).

**Summary:** The author states at the outset (p. 94): "In applying statistical technique to determine whether a series of events occur randomly in time (or space), it has been a common practice to make use of the fact that if this hypothesis is true the frequency of these events occurring in a fixed interval of time (or space) will follow a Poisson distribution. . . . Another method of attack which has been suggested by various writers, including [von] Bortkiewicz [(1915)], Morant [(1921)], and Neyman & Pearson [(1928)], is based on the analysis of the intervals between the events which, if they be random, should be distributed according to the exponential law. The first method may be described as that of count analysis, the second as that of interval analysis." He proceeds to develop a statistical technique for the exponential population analogous to that given by Neyman and Pearson (the L-test) for the normal distribution. His technique makes use of order statistics, especially the smallest value in a sample. He points out that problems where there is necessarily a 'closed period' [see Morant (1921)] must be treated by the interval method of analysis.

**References:** von Bortkiewicz (1915), Morant (1921), Neyman & Pearson (1928), Pearson (1932).

**Citations:** Pitman (1937), Paulson (1941), Kendall (1946).

* Thompson, William R. (1936). On confidence ranges for the median and other expectation distributions for populations of unknown distribution form. *Annals of Mathematical Statistics* **7**, 122-128. (JFM 62, 1346). Abstract, *Bulletin of the American Mathematical Society* **41** (1935), 786-787.

**Summary:** The author obtains confidence intervals, based on order statistics of a random sample of size n, for the median and other quantiles of a continuous population of unknown distribution form. Let U be an infinite population and let $p_k$ be the probability that random sample values from U will be less than

the $k^{th}$ order statistic of a given random sample (of size n) from U. Then the expected value and the variance of $p_k$ are given by $\bar{p}_k = k/(n+1)$ and $E[(p_k - \bar{p}_k)^2] = \bar{\sigma}^2_{p_k} = k(n-k+1)/(n+1)^2(n+2)$. Let x be a value chosen at random from U. Then $\overline{P}(x_k<x<x_{n-k+1}) = (n+1-2k)/(n+1)$ for $2k<n+1$ and $\overline{P}(x<x_k) = \overline{P}(x>x_{n-k+1}) = k/(n+1)$. Let p' represent the p-quantile and M the median of the population U and let $q = 1-p$. Then $\overline{P}(p'<p_k) = I_q(n-k+1,k)$ and $\overline{P}(M<x_k) = I_{0.5}(n-k+1,k)$, where I is the incomplete Beta-function ratio, and $\overline{P}(x_k<M<x_{n-k+1}) = 1-2I_{0.5}(n-k+1,k)$ for $k<n+1$ gives a central confidence interval. Consider a second infinite population U', and draw a sample of size n' from it. Let $x'_m$ be the $m^{th}$ order statistic of this sample and let $k_m$ be the number of observations in the sample of size n from U that are less than $x'_m$, so that $x_{k_m}<x'_m<x_{k_{m+1}}$. For x and x' drawn at random from U and U', respectively, $0 < (n+1)(n'+1)\cdot\overline{P}(x<x') - \Sigma_{m=1}^{n'} k_m < n+n'+1$, provided that the expectations for U and U' may be treated as independent, and, under the same conditions, $\Sigma_{m=1}^{n'} \binom{n'}{m} I_{0.5}(n-k_m+1, k_m) < 2^{n'}Z\overline{P}(M<M') < 1+\Sigma_{m=1}^{n'} \binom{n'}{m-1} I_{0.5}(n-k_m, k+1)$ [the author does not define Z in this inequality]. Similar expressions are given for the case of sampling without replacement from a finite population $U_N$ of size N.

**References:** Laplace (1812), Todhunter (1865), Jeffreys (1932b), Bartlett (1933), Fisher (1933), Jeffreys (1933), Fisher (1934), Jeffreys (1934).

**Citations:** Thompson (1938), Nair (1940a,b), Paulson (1940), von Schelling (1941), Scheffé (1943), Kendall (1946), Bickerstaff (1947), Noether (1948), Wilks (1948), Wolfowitz (1949).

* Ville, Jean-André (1936). Sur la convergence de la médiane des n premiers résultats d'une suite infinie d'épreuves indépendantes. *Comptes Rendus de l'Académie des Sciences* (Paris) **203**, 1309-1310. (JFM 62, 614).

**Summary:** Let X be a random variable with c.d.f. F(x). Suppose that one draws a denumerably infinite sample of values of X, and let $X_1, X_2, \ldots, X_n, \ldots$ be the series of results obtained. It is known that if the variable X has a well determined mean value $\overline{X}$, the series of means of the results $U_n = (X_1 + X_2 + \ldots + X_n)/n$ converges almost surely to $\overline{X}$. The object of this note is to show that an analogous result is true for the median. Specifically, it is shown that if the random variable X has a unique median m, the series of median values of the first $2n+1$ values $X_1, X_2, \ldots, X_{2n+1}$ of X converges almost surely to the value m as n increases without limit.

**Citation:** Kendall (1946).

* Aitken, A. C. (1937). Studies in practical mathematics. II. The evaluation of the latent roots and latent vectors of a matrix. *Proceedings of the Royal Society of Edinburgh* **57**, 269-304. (JFM 63, 1133).

**Summary:** The author begins with the following introduction (pp. 269-270): "In many branches of applied mathematics problems arise which require for their solution a knowledge of the latent roots of a matrix A, sometimes only the root of greatest modulus but often the second and other roots as well, and the corresponding latent vectors. A few examples, among many that might be cited, are problems in the dynamical theory of oscillations, problems of conditioned maxima and minima, problems of correlation between statistical variables, the determination of the principal axes of quadrics, and the solution of differential or other operational equations. It is important, therefore, to have a choice of methods for obtaining latent roots and latent vectors without undue labour, and the object of the present paper is to augment the existing store of such methods. . . . We take the opportunity of acknowledging the stimulus received from some related work of Hotelling (1933, 1936a,b) on finding latent roots and latent vectors of a symmetric matrix by successive operations on an arbitrary vector. When the matrix A is not symmetric there are two kinds of latent vectors to be considered, row vectors u, satisfying $uA = \lambda u$ (or, in transposed form, $A'u' = \lambda u'$), and column vectors x, satisfying $Ax = \lambda x$. In addition there may be complex roots, or multiple roots real or complex which may also be associated with non-linear elementary divisors of the characteristic determinant. We extend the method of the arbitrary vector to meet these

various complications; and we wish to draw special attention to the processes for obtaining a greatly enhanced convergence of the sequences, not only towards latent roots, but also towards latent vectors. These processes are so effective that it is usually better not to spend time in carrying out a first sequence to a long series of terms, but to stop at a comparatively early stage and apply the accelerating methods. (Other procedures, such as preliminary matrix-squaring, or alteration of diagonal elements, will be suggested as occasion arises.) Under appropriate treatment a few crudely convergent first terms may be made to yield valuable information not merely with respect to the greatest root, but also with respect to the smaller roots and the vectors associated with them. . . ." Among other results, the author evaluates the product of the k greatest roots.

**References:** Whittaker & Robinson (1924) [1929], Hotelling (1933, 1936a,b).

**Citations:** Lawley (1940), Kendall (1946), Kincaid (1947).

* Bowley, A. L. (1937). Standard deviation of Gini's mean difference. *Comptes Rendus du Congrès International des Mathématiciens* (Oslo, 1936), Vol. 2, pp. 182-185. A. W. Brøggers Boktrykkeri A/S, Oslo. (JFM 63, 523).

**Summary:** The mean difference, $g_N$, of N quantities, $x_1, \ldots, x_s, \ldots, x_r, \ldots, x_N$, arranged in ascending order, is defined by $[N(N-1)/2]g_N = \Sigma|x_r - x_s| = \Sigma_{r=1}^{N} (2r-N-1)x_r$. For n of these quantities selected at random, say $x_1, \ldots, x_r, \ldots, x_n$, the mean difference, $g_n$, satisfies the relation $G = [n(n-1)/2]g_n = \Sigma_{r=1}^{n} (2r-n-1)x_r$. The author finds the variance of $g_n$ to be $\sigma_g^2 = \{2/n(n-1)\}^2 (S+2P-M^2)$, where $M = \text{Mean } G = \Sigma_{k=1}^{N} 2\kappa x_k n(n-1)/N(N-1)$, $S = \Sigma_{k=1}^{N} [n(n-1)/N(N-2)] \{4[(n-2)/(N-1)]\kappa^2 + N-n\}x_k^2$, $2P = [2n(n-1)/N(N-1)(N-2)(N-3)] \Sigma_{\ell=2}^{N} \Sigma_{k=1}^{\ell-1} [4\kappa\lambda(n-2)(n-3) + (N-n)\{4(\kappa-\lambda)(n-2) + nN-3N-n+1\}] x_k x_\ell$, $2k = N+1+2\kappa$, and $2\ell = N+1+2\lambda$. By setting $2\kappa = N\alpha$, $2\lambda = N\beta$, $\alpha<\beta$, one obtains $\sigma_g^2 = 4[(N-n)/n(n-1)(N-2)] \Sigma[(Nn-2N-2n+2)\alpha^2/(N-1)^2 + 1/N] x_k^2 - 8[(N-n)/n(n-1)(N-1)(N-2)(N-3)] \Sigma\Sigma\{[2Nn-3N-3n+3)/(N-1)] \cdot 2\alpha\beta - [2N(\alpha-\beta)(n-2) + nN-3N+n+1]/N\} x_k x_\ell$. The expression for large samples from an infinite universe becomes $\sigma_g^2 = (4/\lambda N) \Sigma_{\alpha=-1}^{+1} \alpha^2 x_k^2 - (32/nN^2) \Sigma_{\beta=-1}^{+1} (\beta-1/2)x_\ell \Sigma_{\alpha=-1}^{\beta} (\alpha+1/2)x_k\}$, since one can ignore 3/n and n/N. For the normal distribution $f(x) = e^{-x^2/2\sigma^2}/\sigma\sqrt{2\pi}$, one finds $\sigma_g^2 = \{4/3 - 8(2-\sqrt{3})/\pi\} \sigma^2/n$, while $g = 2\sigma/\sqrt{\pi}$, so that $\sigma_g \doteq 0.807\sigma/\sqrt{n} \doteq 0.715g/\sqrt{n}$. For the uniform distribution, $f(x) = 1/N$, one finds $n\sigma_g^2 = N^2/45$, $\sigma^2 = N^2/12$, $g = N/3$, so that $\sigma_g \doteq 0.516\sigma/\sqrt{n} \doteq 0.447g/\sqrt{n}$. The author compares the precision of g not only with that of the standard deviation $\sigma$ but also with that of the probable error $\rho$ and the mean deviation $\eta$. For the normal distribution he finds approximately $\sqrt{n}\,\sigma_\sigma = 0.79\sigma = 1.2\rho$, $\sqrt{n}\sigma_\eta = 0.603\sigma = 0.756\eta$, $\sqrt{n}\sigma_\rho = 0.707\sigma$, $\sqrt{n}\sigma_g = 0.807\sigma = 0.715g$; for the uniform distribution, $\sqrt{n}\sigma_\sigma = 0.866\sigma = \rho$, $\sqrt{n}\sigma_\eta = 0.707\sigma = 0.577\eta$, $\sqrt{n}\sigma_\rho = 0.707\sigma$, $\sqrt{n}\sigma_g = 0.516\sigma = 0.447g$. Thus the absolute precision of g is less than that of the other three measurements for the normal distribution but greater for the uniform distribution, while the relative precision of g is very nearly as great as that of $\sigma$ for the normal distribution and considerably greater for the uniform distribution.

**Citation:** Wold (1937).

* Bresciani-Turroni, C. (1937). On Pareto's law. *Journal of the Royal Statistical Society* **100**, 421-432.

**Summary:** The author notes that a number of later writers have declared that Pareto (1897) was wrong when he stated that as his index $\alpha$ increases, so does the inequality of incomes. He points out that whether Pareto was right or wrong depends upon the definition of inequality, which is essentially arbitrary. According to Pareto's definition, based on the notion that inequality diminishes when the number of "rich" increases compared with the number of "poor", he was right. Let $N_x$ be the number of individuals with incomes of x or more and $N_h$ the number of individuals with incomes of h or more ($h<x$). According to Pareto's definition, the inequality of incomes diminishes when $U_x = N_x/N_h$ increases. If we substitute for $N_x$ and $N_h$ the values given by Pareto's first law, we have $U_x = (h/x)^\alpha$, so that as $\alpha$ increases,

$U_x$ decreases and therefore the inequality of incomes increases. An alternate definition is that the inequality of incomes increases as the ratio of the average income of the subgroup $N_x$ to the average income of the whole group $N_h$ increases. If $M_x/M_h$ is the ratio of the average incomes and if Pareto's first law holds, $M_x/M_h = x/h = \sqrt[\alpha]{N_h/N_x}$, so that, as $\alpha$ diminishes, the inequality of incomes increases—a conclusion opposite to Pareto's. If $N_x/N_h = p$ is the relative number of persons with incomes greater than or equal to x (x might be the upper centile, decile, or quartile) and v is the median ($v < x$), then, according to a modified form of the last definition, the distribution of incomes becomes more unequal when the ratio x/v increases. If Pareto's first law holds, we have $x/v = \sqrt[\alpha]{0.50/p}$ and, again, as $\alpha$ diminishes, the inequality of incomes increases. The author also notes that Bowley's quartile measure of dispersion $(Q_2 - Q_1)/(Q_2 + Q_1)$, where $Q_2$ and $Q_1$ are the upper and lower quartiles, is another measure of inequality. If Pareto's first law holds and if the minimum income is h, $Q_2 = h/\sqrt[\alpha]{0.25}$, $Q_1 = h/\sqrt[\alpha]{0.75}$ and $(Q_2 - Q_1)/(Q_2 + Q_1) = (\sqrt[\alpha]{3}-1)/(\sqrt[\alpha]{3}+1)$ increases as $\alpha$ decreases. The author modifies Pareto's definition slightly by assuming that inequality diminishes when the number of people $N_m$ with incomes above the mean increases compared with the total number of people $N_h$, and shows that this leads to a conclusion which is the reverse of Pareto's. If Pareto's first law holds and if $\alpha > 1$ and the highest income is unlimited, we have $N_m/N_h = [(\alpha-1)/\alpha]^\alpha$; as $\alpha$ increases, so does $N_m/N_h$, and hence the inequality of incomes decreases. In a footnote (p. 430), the author mentions the concentration index $\delta$ of Gini (1914a) and its relation to $\alpha$ when Pareto's first law holds.

**References:** Pareto (1897), Persons (1909), Gini (1914a), Ricci (1916), Dalton (1920 [1925], von Bortkiewicz (1931).

**Citations:** Bresciani-Turroni (1939), Davis (1941).

* Calichiopulo, Antonio (1937). Méthode de sélection des erreurs d'observation. *Comptes Rendus de l'Académie des Sciences* (Paris) **204,** 642-644. (JFM 63, 509).

**Summary:** The author discusses the question of treatment of outlying observations and the choice of measures of central tendency and dispersion when outliers may be present. He notes that rejection criteria have been proposed which are based on the subjective judgment of the calculator and are therefore very questionable. He proposes an objective and systematic method, to be applied in all cases without previously examining the measurements. Given a set of n measurements, he proposes discarding the s of them which deviate farthest from the mean, where the ratio s/n may vary from 0.10 to 0.50 depending on the care and skill of the observers, then taking the mean E of the absolute values of the extremes $+E_1$ and $-E_2$ of the remaining errors (deviations from the mean), and dividing it by the proper factor (2.44 for s/n=0.10, 1 for s/n=0.50), determined from a table of the normal distribution, to find the probable error. As the effective value of the measure he advocates neither the mean of the n original measures nor that of the n−s measures, but a weighted mean of the two with weights inversely proportional to their mean deviations. He gives several numerical examples.

**Reference:** Bertrand (1889) [1907].

* Camp, Burton H. (1937). Methods of obtaining probability distributions. *Annals of Mathematical Statistics* **8,** 90-102. (JFM 63, 1080).

**Summary:** The author's emphasis is on methods, with special results cited in order to illustrate the methods. The only portion relevant to our study is the following illustration (pp. 100-101): "Let a sample of N be drawn from the universe $\phi = Ae^{-at^{2\lambda}}$ if $t>0$, $\phi=0$ if $t\leq 0$. It is readily proved . . . that the distribution f(x) of the parameter, $x=(t_1^{2\lambda}+ \ldots +t_N^{2\lambda})^{1/N}$ is a curve of the form $f(x)=Bx^{N-1}e^{-x^{2\lambda}}$ where $x>0$, $f(x)=0$ elsewhere. Now let $\lambda$ become infinite. The universe approaches as a limit the rectangle: $\Phi=A$ where $0\leq t<1$, $\Phi=0$ elsewhere. The parameter x approaches as a limit X, where X=maximum $t_i$. The distribution f(x) approaches as a limit the new distribution $F(X)= NX^{N-1}$ where $0<X<1$, $F(X)=0$ elsewhere. Hence we have proved in a new way, what was already known: that the distribution of the greatest variate obtained by sampling from a rectangular universe is of the form F(X)."

**Citation:** Kendall (1946).

* Coutagne, A. (1937). Étude statistique des débits de crue. *Revue Générale de l'Hydraulique* **3** (18), 317-321.

**Summary:** The following summary (by the editor?) appears on p. 317: "Our readers have found in the transactions of the Societé Hydrotechnique de France an exposition by Gumbel of his theory of the largest value and a first application to the determination of flood flows. Coutagne, recognizing a priori the interest in these studies, has examined in ten different cases the results obtained with the new formulas and established a comparison with the old laws of Galton, Gauss, and Fuller. The contrast of the theories of Galton [lognormal] and Fuller-Gumbel [extreme-value] cannot be defined in a most exact absolute manner; their difference is moreover often very slight." (Compiler's translation.)

**References:** Gibrat (1932a), Grassberger (1932, 1933), Gumbel (1934f, 1935a,d).

**Citations:** Gumbel (1939c, 1941a, 1943c).

* Friedrich, Konrad (1937). Allgemeine für die Rechenpraxis geeignete Lösung für die Aufgaben der kleinsten Absolutsummen und der günstigsten Gewichtsverteilung. *Zeitschrift für Vermessungswesen* **66,** 305-320, 337-358. (JFM 63, 1104-1105).

**Summary:** Let $F_i = a_i x_1 + b_i x_2 + \ldots + f_i$ $(i = 1,2,3,\ldots,n)$ be n linear functions in m unknowns $x_1, x_2, \ldots, x_m$ $(m<n)$. The unknowns are to be determined so that the sum of the absolute values of the $F_i$, namely $\Sigma|F_i| = \Sigma|a_i x_1 + b_i x_2 + \ldots + f_i|$, shall be as small as possible. The preceding problem was first posed by Boscovich in 1755 [actually 1757] for the adjustment of errors of observation. Laplace (1799) accepted this adjustment principle and gave an elegant solution for the case of a single unknown. Gauss (1809) knew that for m equations in m unknowns the value of the F's must be equal to zero. From a purely theoretical standpoint the problem is thereby solved, since one can solve the proper systems of linear equations for all $\binom{n}{m}$ possible combinations of m out of n linear expressions in m unknowns and form the corresponding absolute sums. Then the smallest of the sums so obtained gives a solution [not necessarily unique] of the problem, as shown on pp. 309-312. Of course such a solution is not useful in practical computation on account of its tediousness. Since the beginning of the nineteenth century, however, a simple computational solution has not been necessary, since in the meantime the general adjustment method of least squares was introduced. The author points out, however, that several German authors, including Schreiber (1882), Bruns (1882), Jordan (1888?), and Runge (1890) returned to the earlier method late in the nineteenth century. He discusses their contributions to the solution of the problem, as well as those of several twentieth century writers, including himself [Friedrich (1927)]. He works out the details of the theory and gives several numerical examples.

**References:** Laplace (1799), Gauss (1809), Helmert (1872) [1907], Bruns (1882), Schreiber (1882), Jordan (1888?), Runge (1890), Friedrich (1927).

* Gumbel, E. J. (1937a). La durée extrême de la vie humaine. *Actualités Scientifique et Industrielles*, No. 520. Hermann et Cie., Paris. (JFM 63, 1116).

**Summary:** The author writes (pp. 3-5): "If one considers biometric functions as capable of being discontinuous, a well determined limit of human life can exist. This age would be defined by the fact that the probability of dying in an interval of one year is equal to one. But according to the usual opinion, these functions are continuous. Even in this case, a fixed age limit would be possible, the probability of survival being zero for this age and not defined for greater ages. The extreme length of human life would be one of the constants which exist in biometric functions, independent of the number of observations. But this determination would be tied to a special analytic hypothesis about the increase of mortality with

age. Such hypotheses have been set forth for example by A. De Moivre and Th. Wittstein. Most books on actuarial theory consider the mortality table as discontinuous at a certain point. But this theory leads, as Steffensen has proved, to very great difficulties from the point of view of logic. Because for a sufficient increase in the number of observations, one could make the discontinuity in the distribution of the number living according to ages as large as one wishes. In order to avoid this difficulty we will introduce the simpler idea that the survival function is everywhere continuous and approaches zero only asymptotically. Starting from this hypothesis, we are going to expound a method based on the theory of probabilities for defining the last age corresponding to a definite number of observations. . . . Our point of departure is the study of the distribution of deaths according to age [Gumbel (1936c)], assumed unlimited to the right. Then the question of the highest age takes a new aspect. To seek a fixed age limit is devoid of sense. Its existence is not known and its determination with the aid of analytic conditions or from values determined from biometric functions can be regarded as artificial. However, this idea does not disappear. It should simply be regarded in a new way. It is necessary to distinguish between the last age and the greatest age. The latter becomes a statistical variable, relative to a definite number of observations, whose distribution depends on that number. We will introduce the form called final, toward which it tends as the number of observations increases. One could give an idea of the greatest age by the means of this distribution, which one distinguishes from each other by chosen designations. The mathematical expectation of this distribution, a function of the number of observations, will be considered as the limiting age, while the mode is considered as the last age. Since these values increase with the (number of) observations, the difficulty pointed out by Steffensen is set aside. . . . Moreover, this method allows many other applications." (Compiler's translation).

**References:** von Bortkiewicz (1922a,b), Dodd (1923), von Mises (1923), Fisher & Tippett (1928), Steffensen (1930), Insolera (1931), von Mises (1931), Steffensen (1931), Gumbel (1932, 1933c,e,f,g), Insolera (1933), de Finetti (1934), de Franchis (1934), Gumbel (1934b,d,f,g, 1935a,c,d, 1935-36, 1936a,b,c, 1936-37).

**Citations:** Gumbel (1938a, 1939f, 1941a), Kendall (1946), Epstein (1948a,b), Epstein & Brooks (1948).

* Gumbel, E. J. (1937b). Les intervalles extrêmes entre les émissions radioactives. *Le Journal de Physique et le Radium* (7) **8** (8), 321-329; (11), 446-452.

**Summary:** Part I begins with the following summary (p. 321): "The purpose of the following lines is to apply the theories concerning the extreme values of statistical distributions in the domain of radioactivity. 1. The initial distribution [exponential]. 2. Theory of $m^{th}$ intervals [distribution of $m^{th}$ largest value among N observations]. 3. The final distribution of extreme $m^{th}$ intervals [Type I asymptotic distribution of largest values]. 4. Method of comparisons between theory and observations" (Compiler's translation). It closes with the following conclusion (p. 329): "After having calculated the mean interval between consecutive emissions, we have established the distribution of the $m^{th}$ interval and its characteristics. For the extreme intervals, this distribution tends, even for moderate numbers of observations, to the final distribution. We have established the necessary methods for comparison between theory and observations." (Compiler's translation). Part II deals with the comparison between theory and observations, which are generally in good agreement.

**References:** Fisher & Tippett (1928), von Mises (1931), Eyraud (1935), Gumbel (1935a, 1935-36, 1936b).

**Citations:** Brenet & Armand (1938), Gumbel (1938a, 1939b,f), Lévy (1939), Gumbel (1941a, 1942a,b, 1943c, 1945a), Wilks (1948).

* Gumbel, E. J. (1937c). La précision de la moyenne arithmétique et de la médiane. *Aktuárské Vědy* **6** (4), 145-154. (JFM 63, 1089).

**Summary:** The author begins with the following introduction (p. 145): "In order to characterize an observed distribution with the aid of a single value, one chooses one of the means. Among them, the arithmetic mean and the median play the most important role. It is a question of knowing which to choose. The classical problem is solved with the aid of many criteria—which contradict each other. We are not going to take up all of these arguments. We limit ourselves rather to contributing some details on the special question of comparative precision of the arithmetic mean and of the median." (Compiler's translation). He closes with the following conclusions (p. 154): "The median of the distribution of the median is equal to the median of the initial distribution. The mode of the distribution of the median . . . tends toward the same value. In the limit, the distribution of the median tends toward a Gaussian distribution. The distributions considered prove that the relations between the two means are multiform. It may happen that the mathematical expectations of the two means coincide and their precisions differ. For another distribution, the two means differ and their precisions coincide. Finally—and this will be the general case—the two means and their precisions differ. We have verified that the arithmetic mean is more precise than the median for the uniform distribution, for the Gaussian distribution, and for the distribution of the largest value; that it is less precise for the symmetric [double] exponential distribution, while the relation of the precisions depends on the magnitude of the standard deviation for the Galton [lognormal] distribution. In a special case of this distribution and for the asymmetric [single] exponential distribution the two precisions are equal." (Compiler's translation).

**References:** Wilson (1923), Haag (1924), Wilson & Hilferty (1929), Fogelson (1930), Kolmogoroff (1931), Eyraud (1934b), Gumbel (1934e), Fréchet (1935), Gumbel (1935a, 1935-36), Smirnoff (1935), Darmois (1936).

**Citations:** Gumbel (1937-38), Fréchet (1940a), Gumbel (1942a).

* Gumbel E. J. (1937d). Der grösste Wert einer statistischen Veränderlichen. *Comptes Rendus du Congres International des Mathématiciens* (Oslo, 1936), Vol. 2, pp. 200-203. A. W. Brøggers Boktrykkeri A/S, Oslo. (JFM 63, 1129 — listing only).

**Summary:** The author writes (pp. 200-203): "To a continuous initial distribution $w(x)$ unlimited to the right, with cumulative probability $W(x)$ there belongs, for N observations, a distribution $w_N(x)$ of the largest value (1) $w_N(x) = N[W(x)]^{N-1}w(x)$ which constitutes a special case of the general distribution of the $m^{th}$ value. The distribution (1) possesses in turn a most probable value $\tilde{u}$, which will be called the last value, a mean value $\bar{u}$, which will be called the limiting value, and a root-mean-square error s. The last value for sufficiently large N is given by (2) $N = -w'(\tilde{u})/w^2(\tilde{u})$ or whenever l'Hopital's rule (3) $w(\tilde{u})/[1-W(\tilde{u})] = -w'(\tilde{u})/w(\tilde{u})$ can be considered as satisfied with a prescribed accuracy, by (2') $W(\tilde{u}) = 1-1/N$. For a single exponential initial distribution with mean value $1/\alpha$ the distribution (1) tends with increasing N to the limiting form (4) $w(x) = \alpha e^{-y-e^{-y}}$, so that the transformed variable and the cumulative probability read $y = \alpha(x-\tilde{u})$ and $\mathfrak{W}(x) = e^{-e^{-y}}$. The last value then has the magnitude (5) $\tilde{u} = (1/\alpha)\lg N$. . . . The two constants $\tilde{u}$ and $\alpha$ can be estimated on the basis of an observed distribution of the largest value either from the mean $\bar{u}$ and the standard deviation s or from the median $u_0$ and the two quartiles $u_1$ and $u_2$. The first method leads to (6) $\tilde{u} = \bar{u} - 0.57722/\alpha$; $1/\alpha = 1.28255/s$; the last, somewhat less accurate, but on the other hand simpler, yields (6') $\tilde{u} = u_0 - 0.23307(u_2-u_1)$; $1/\alpha = 0.63592(u_2-u_1)$. In the general case, one obtains for the reduced last value (7) $\tau = (\tilde{u}-\xi)/\theta$ where $\xi$ is the mode and $\theta$ the mean error of the initial distribution, with the help of (2) or (2'), (8) $\tau = f(\lg N)$, where f depends on the nature of the initial distribution, so for the last value itself (8') $\tilde{u}_N = \xi + \theta f(\lg N)$. The largest observed value up to a certain sample size will correspondingly always increase, other things being equal, with the sample size, and thus also with the time, which can be compared with the observations. The constant (9) $\alpha = -w'(\tilde{u})/w(\tilde{u})$ and with it the standard error of the distribution of the largest value is in general itself a function of N. This allows, among the initial distributions which lead to the limiting distribution (4), three categories to be distinguished according to their behavior at infinity: (1) such as behave like the exponential function itself [one-sided and two-sided exponential and Pearson Type III]. One obtains $f(\lg N) = \lg N$

and a standard error independent of N. . . . (2) Still worse: the last value increases faster than lgN. This is the case for the Galton [lognormal] distribution, for which lg f(lgN) = $\sqrt{N}$. (3) Conversely, the last value increases more slowly than lgN [Gauss and Gompertz distributions] . . . . This account of the distribution of the largest value as the last value itself as a function of N allows the question to be decided whether a certain extreme value is still compatible with a chosen initial distribution or whether essentially, for example, secular variations have occurred." (Compiler's translation).

**References:** Gumbel (1935a,d, 1935-36, 1936a,b).

**Citation:** Gumbel (1939f).

* Gumbel, E. J. (1937e). Das Grenzalter. *Comptes Rendus du Congrès International des Mathématiciens* (Oslo, 1936), Vol. 2, pp. 203-205. A. W. Brøggers Boktrykkeri A/S, Oslo. (JFM 63, 1130 — listing only).

**Summary:** The author writes (pp. 203-205): "Numerous authors have occupied themselves with the general viewpoint of the interesting problem of the longest duration of human life and endeavored to determine this from the observations through complicated interpolation formulas. This problem acquires a new look as soon as one considers the series of deaths as a distribution of deaths according to age and especially assumes that this initial distribution is unlimited. Then the limiting age becomes identically simply a special case of the largest value of an unlimited distribution for a given large number of observations. The correctness of this analogy follows without incisive analytical assumptions by equating the distribution of age at death of the oldest of a certain population with the limiting distribution of the largest value. The mode of this distribution will be the last age $\tilde{\omega}$, the mean limiting age is called $\bar{\omega}$, the standard error is in turn proportional to the intensity of mortality at the last age. Comparisons carried out for the last 50 years in Sweden and in Switzerland have proved the correctness of the method. Therefore it is admissible to calculate the last age as a function of the observation time. If one confines himself to the distribution of deaths around the normal age $\xi$ and if one denotes the life expectancy at this age by $E(\xi)$, the probability that it is attained by $\ell(\xi)$, then the last age amounts to (1) $\tilde{\omega} = \xi + E(\xi) f(\lg N \ell(\xi) k)$, where N is the number of deaths and k is a constant which serves only to normalize the distribution of deaths above the normal age, and f depends on the assumption concerning the course of mortality at the highest ages. For the Gompertz formula the mortality function $\ell(x)$ and the intensity of mortality $\mu(x)$ reduce for the three normal values $\xi$, $\ell(\xi)$ and $E(\xi)$ to (2) $\ell(x) = \ell(\xi) e^{1-e^{pt}}$; $\mu(x) = [p/E(\xi)]e^{pt}$ where the reduced age $t = (x-\xi)/E(\xi)$ and $p = 0.59635$ is a universal constant. After introduction of the distribution of deaths $\theta(x) = \ell(x)\mu(x)$ one obtains . . . the last age, after slight neglects, from (1′) $\log N\ell(\xi) = 2 \log e \cdot \text{Sin hyp}\,(p\tau - 1/2)$ where (3′) $\tau = (\tilde{\omega}-\xi)/E(\xi)$ is the reduced last age. . . . The increase of the last age with time is so slow that the highest age of 106 years observed until now in Sweden would first be exceeded in a century. Since according to experience the normal age increases somewhat with hygienic conditions, the corresponding life expectancy drops at the same time, so that an improvement of mortality, as long as it remains within the bounds of this explanation, does not increase the limiting age." (Compiler's translation).

**References:** Gumbel (1935c, 1936c, 1936-37).

**Citation:** Gumbel (1939f).

* Gumbel, E. J. (1937-38). La distribution des inondations. *Aktuárské Vědy* (Prague) **7** (2), 85-93.

**Summary:** The author writes (pp. 85-89): "In a previous article [Gumbel (1935-36)] we have developed the theory of the largest value. Afterward, we have applied it to the problem of the limiting age [Gumbel (1936-37)] and to distances between radioactive emissions [Gumbel (1936b)]. But this general theory

allows many other practical applications, since the final distribution has the same form for all initial distributions of exponential type. In what follows we are going to study the problem of floods. . . . We are going to represent the distribution of floods by that of the largest value, being given that the peak discharges are certainly the largest values of the discharges. We base our work on the fact that the final distribution of the largest value is, in a certain measure, independent of the initial distribution that one considers. This simplifies our task greatly. . . . The necessary condition for the passage of the exact distribution of the largest value to its final form is verified. Because the discharges of large rivers are observed daily, and the floods are sometimes known even for a century. . . . Let x be the discharge, let n be the number of years observed. Then the number of years n W(x) having a maximum discharge less than x is n W(x) = $ne^{-e^{-y}}$ (1) where $y = \alpha(x-u)$ (2) is a variable reduced by the introduction of the mode u of the largest value, and of the quantity $\alpha$, of dimension $x^{-1}$. These two constants $\alpha$ and u can be determined from the observations with the aid of the method of moments, which leads to the equations $u = \bar{u} - 0.57722/\alpha$ (3) and $1/\alpha = (\sqrt{6}/\pi)\sigma$ (4) where $\bar{u}$ is the arithmetic mean and $\sigma$ the standard deviation of the distribution of the largest value, quantities which one draws from the observations. But this calculation becomes tedious when the number n is very large. That is why we propose a new determination of the two constants with the aid of the observed median $u_0$ and quartiles $u_1$, $u_2$. It is a question then of finding the relation between these three quantities, called values of position, and the two constants u and $\sigma$. The median of the distribution of the largest value is, according to the definition of that value, following (1) and (2), the solution of $e^{-e^{-\alpha(u_0-u)}} = e^{-\lg 2}$ which leads to $e^{-\alpha(u_0-u)} = \lg 2$ or $u_0 = u - \lg \lg 2/\alpha$. Thus one determines the constant u with the aid of the observed value $u_0$ by: $u = u_0 - 0.36651/\alpha$. (5) In order to determine $\alpha$, one makes use of the two quartiles $y_1$ and $y_2$ of the reduced variable, which are $y_2 = 1.24590$, $y_1 = -0.32663$. One obtains according to (2) $\alpha(u_2-u_1) = 1.57253$. Thus one can calculate the constant $1/\alpha$ from $1/\alpha = 0.63529\ (u_2-u_1)$ (6) and the constant u, following (5), from $u = u_0 - 0.23307(u_2-u_1)$. (7) This determination with the aid of values of position involves a certain loss in precision as compared with the determination with the aid of moments. Because the precision of the median [Gumbel (1937c)] is, for the determination of the largest value, 90% of the precision of the arithmetic mean. But this fault is amply compensated by the ease of the determination." The author applies these results to data for floods on the Rhine at Basel during the 118-year period 1808-1925. On page 92 he states: "In general, the distribution of peak discharges follows well the distribution of the largest value. . . . The introduction of special hypotheses concerning floods is not at all necessary. This shows once more that the domain of application of the theory of largest values is quite vast."

**References:** Hazen (1930), Gibrat (1932b), Grassberger (1932, 1933), Gumbel (1935-36), Grassberger (1936), Gumbel (1936b, 1936-37, 1937c).

**Citations:** Gumbel (1939e,f), Kendall (1946).

* Insolera, F. (1937-38). Au sujet de l'âge limite. *Aktuárské Vědy* (Prague) **7** (2), 49-55.

**Summary:** The author attempts to answer the objections raised by Gumbel (1934g, 1936-37) and by de Finetti (1934) to the definition of the limiting age which he stated earlier [Insolera (1933, 1935)], and which he here (p. 50) restates as follows: "For an age of $\omega$ years and only for this age, if $\varepsilon$ is a positive quantity as small as one wishes, there is an interval $(\omega-\Delta\omega,\omega)$, with $\Delta\omega>\epsilon$, in which the difference between the number of survivors and of deaths is smaller than $\epsilon$." (Compiler's translation). The author also advances certain other arguments to support his theory of the limiting age. To counterbalance the "force of mortality" [instantaneous rate of mortality] he proposes a "force of vitality", and asserts that, while the former is not zero at age 0, thus allowing for still-births, the latter is zero at the limiting age $\omega$.

**References:** Bohlmann (1909), Insolera (1933), de Finetti (1934), Gumbel (1934g), Insolera (1935), Gumbel (1936-37).

* Kozakiewicz, W. (1937). Sur un théorème de Glivenko. (Polish summary). *Comptes Rendus de la Société des Sciences et des Lettres de Varsovie* (Classe III) **30**, 83-87. (JFM 63, 513).

**Summary:** The author writes (pp. 83-84): "This note contains a very simple demonstration of a theorem of Glivenko [(1933)]. We denote by $P\{E\}$ the probability of an event E, by $\mathfrak{E}(z)$ the mathematical expectation of a random variable z. Given a sequence $\{z_n\}$, $n = 1, 2, \ldots$ of random variables, we denote—for t real—by $P_m\{z_n < t\}$ the probability of having simultaneously all the inequalities: (1) $z_n < t$; $n = m, m + 1, \ldots$. Let x be a random variable. For t real we set: (2) $F(t) = P\{x \leq t\}$. F(t) is the c.d.f. of the variable x. Let $x_1, x_2, \ldots, x_n$ be n values of the variable x, corresponding to n independent trials. We determine for each $x_k$ $(1 \leq k \leq n)$ a function $u_k(t)$ by the conditions $u_k(t) = 1$ if $x_k \leq t$ and $u_k(t) = 0$ if $x_k > t$. Set: (3) $V_n(t) = (1/n)\Sigma_{k=1}^{n} u_k(t)$. One calls $V_n(t)$ the empirical probability law, because for $x_1, x_2, \ldots, x_n$ fixed, $V_n(t)$ possesses all the properties of the c.d.f. The theorem of Glivenko can be expressed by the relation: (4) $P\{\lim_{h\to\infty}[\sup_t |V_n(t) - F(t)|] = 0\} = 1$, which means: the sequence of random variables $\sup_t |V_n(t) - F(t)|$ converges stochastically strongly to zero. Stated otherwise, to every pair of positive numbers $\epsilon, \eta$ one can find a corresponding number N such that $m \geq N$ implies: (5) $P_m\{\sup_t |V_n(t) - F(t)| < \epsilon\} > 1 - \eta$." (Compiler's translation). The proof, which is simple but too long to reproduce here, is given on pp. 84-87. The author points out that the theorem can be generalized to the case of several dimensions, and that the method of proof used in this paper can be applied, with suitable modifications, when the trials are not independent.

**References:** Glivenko (1933), Kolmogoroff (1933).

* Madow, William G. (1937). Contributions to the theory of comparative statistical analysis. I. Fundamental theorems of comparative analysis. *Annals of Mathematical Statistics* **8**, 159-176. (JFM 63, 1094). Abstract, *Bulletin of the American Mathematical Society* **43**, 176.

**Summary:** The author states (pp. 159-160): "This is the first of several papers in which there will be presented a general approach to the statistical examination of hypotheses which are false if any of several things are true. Phenomena requiring such a statistical theory are investigated quite frequently. . . . The theorems of this paper have one purpose: to permit the reduction of the distributions by which the hypotheses are to be tested to essentially the joint distribution of the statistics which contain the information offered by the data concerning the truth or falsity of the things which will negate the hypotheses. . . . The fundamental theorems of comparative analysis are now obtained in such a form that they are applicable to problems in the theory of probability no matter what the distributions may be. Some special cases of these theorems have been used in connection with the derivation of distributions of positional statistics such as the $k^{th}$ in order of N elements [Gumbel (1935a)], and others." The author proceeds to prove nine theorems which are too long to state here; these theorems do not deal directly with order statistics, but are useful in proving results such as the one mentioned by the author in the passage quoted above.

**References:** Poincaré (1896) [1912], Fisher (1935), Gumbel (1935a).

**Citation:** Kendall (1946).

* Neyman, J. (1937a). "Smooth test" for goodness of fit. *Skandinavisk Aktuarietidskrift* **20**, 149-199. (JFM 63, 1092).

**Summary:** The author writes (pp. 159-160): "In what follows, I shall be concerned only with the case when the hypothesis tested, say $H_0$, is simple, specifies a continuous probability law, common to all the x's, and when the admissible alternative hypotheses are smooth. This case has been treated by H. Cramér [(1928)] who suggested a special criterion and applied it systematically to test the goodness of fit of the Charlier series. Later on the same criterion was advanced by R. v[on] Mises [(1931)] who seemed to be unaware of the previous work by Cramér. In the work of [von] Mises the criterion was named 'the $\omega^2$

test' and under this name it is now generally known. However, the method of approach of the two authors mentioned, as well as the results, differ essentially from what follows." The test proposed by the author, unlike the Cramér-von Mises test, is not based on order statistics.

**References:** Cramér (1928), von Mises (1931), K. Pearson (1931), Neyman & E. S. Pearson (1936).

**Citations:** Gumbel (1939d, 1942a), Pearson (1942), Cramér (1946), Kendall (1946), Bickerstaff (1947), Kimball (1947).

* Neyman, J. (1937b). Outline of a theory of statistical estimation based on the classical theory of probability. *Philosophical Transactions of the Royal Society of London* (A) **236**, 333-380. (JFM 63, 515).

**Summary:** This paper deals primarily with the theory of interval estimation and only incidentally, in an example, with order statistics. Let $x_1$ and $x_2$ be random variables subject to the probability law $p(x_1, x_2|\theta) = \theta^{-2}$ for $0 < x_1, x_2 < \theta$, $p(x_1, x_2) = 0$ otherwise. The value of $\theta$ is unknown and interval estimates of $\theta$ are desired. Let $x'_1$ and $x'_2$ be values of $x_1$ and $x_2$, respectively, chosen at random and let $L = \max(x'_1, x'_2)$. The author shows that $L(1-\alpha)^{-1/2}$ is a $100\alpha$ percent upper confidence bound on $\theta$, or since we know that $\theta \geq L$, the interval $[L, L(1-\alpha)^{-1/2}]$ is a (one-sided) $100\alpha$ percent confidence interval on $\theta$. He finds that one-sided intervals based on L are always shorter than two-sided intervals with the same confidence level based on the sum $T = x'_1 + x'_2$. In fact, the former sometimes (but not always) lie entirely within the latter.

**References:** Lévy (1925), Darmois (1936), Neyman & Pearson (1936).

**Citations:** Wilks (1938), Eisenhart (1939), Pitman (1939), Wald (1939), Kullback & Frankel (1940), Kendall (1943), Wilks (1943), Cramér (1946), Kendall (1946), Lehmann (1947), Wolfowitz (1949).

* Pitman, E. J. G. (1937). The "closest" estimates of statistical parameters. *Proceedings of the Cambridge Philosophical Society* **33**, 212-222. (JFM 63, 515).

**Summary:** The author defines and studies "closest" estimates of the parameters of statistical populations. Of two estimates $X_1$ and $X_2$ of a parameter $\theta$, $X_1$ is a closer estimate than $X_2$ if the probability that $|X_1-\theta| < |X_2-\theta|$ is greater than 1/2. The closest estimate, if one exists, is the one which is closer than any other. The author shows that if X is a random variable with a unique median M, then M is the closest estimate of X, or, more precisely, if C is any fixed number not equal to M, the probability that $|X-M| < |X-C|$ is greater than 1/2. He relates this property of the median to the well-known fact that the mean absolute deviation is a minimum when deviations are taken about the median. He also shows that if $x_1, x_2, \ldots, x_n$ is an ordered sample from a two-parameter exponential distribution with p.d.f. $f(x) = (1/c)e^{-(x-a)/c}$, $x \geq a$, $c > 0$, then $x_1$ and $\Sigma(x_r - x_1)$ are a sufficient pair of statistics for the estimation of a and c. Moreover, these two statistics are independent of each other and of any function F of the x's which is both homogeneous of degree 0 and invariant with respect to change of origin. The author shows that $X = n(x_1-a)/c$ has a $\Gamma(1)$ distribution and $Y = \Sigma(x_r-x_1)/c$ has a $\Gamma(n-1)$ distribution; hence $Z = X/Y$ has a known distribution, and Z can be used as a statistic to test whether the smallest member, $x_1$, of a sample from an exponential distribution differs significantly from zero or any other specified value. There are corresponding tests for significance of differences between two values of $x_1$ and between two values of $\Sigma(x_r-x_1)$. The closest estimates of a and c are $A = x_1 - (2^{1/(n-1)}-1)\Sigma(x_r-x_1)/n$ and $C = \Sigma(x_r-x_1)/g_{n-1}$, where $g_{n-1} \doteq (n-1) - 1/3 + 0.09/(n-1)$ is the median of the $\Gamma(n-1)$ distribution. The author also gives the closest estimates of the location and scale parameters of a rectangular distribution, which are functions only of the smallest and largest sample values, which form a sufficient pair.

**References:** Pitman (1936), Sukhatme (1936).

**Citations:** Pitman (1939), Craig (1943), Geary (1944), Cramér (1946), Kendall (1946).

* Savur, S. R. (1937a). The median as a statistic (letter to the Editor). *Current Science* **5** (8), 419-421.

**Summary:** The author points out that impetus to the use of the median in statistical analysis has been given by Crowe (1933, 1936) and by Mathews (1936). Whereas Crowe has justified his methods either by appealing to intuition or by assuming normality of distribution, the author finds distribution-free lower and upper 95% confidence bounds for the population median based on the order statistics of a random sample of size n, $y_1 \leq y_2 \leq \ldots \leq y_{\ell_1} \leq \ldots \leq y_{n+1-\ell_1} \leq \ldots \leq y_n$. These bounds are given by $y'_{\ell_1}$ and $y'_{n+1-\ell_1}$, respectively, where $y'_{\ell_1}$ is $y_{\ell_1}$ increased by unity in the last significant digit and $y'_{n+1-\ell_1}$ is $y_{n+1-\ell_1}$ decreased by unity in the last significant digit, and $\ell_1$ is the largest value of $\ell$ for which the incomplete B-function ratio $I_{1/2}(n-\ell, \ell+1)$ is less than 0.05. The interval between these two bounds is an approximate 90% central confidence interval for the population median. If this interval does not include zero, the author states that the median is significantly different from zero. If such intervals for two random samples do not have a common part, he states that the sample medians are significantly different from each other. He applies these tests to an example. He states that the median has two advantages over the mean in tests of significance: (1) Tests based on the median are far simpler than those based on the mean; (2) It is necessary to know the distribution of the population in the case of tests based on the mean, but not in the case of tests based on the median.

**References:** Crowe (1933, 1936), Mathews (1936).

**Citation:** Savur (1937b).

* Savur, S. R. (1937b). The use of the median in tests of significance. *Proceedings of the Indian Academy of Science* (A) **5**, 564-576.

**Summary:** The author gives the following summary (p. 576): "In this paper a new test of significance, called the 干 [Sanskrit, pronounced like "me" in "calmer"] test, in which the median is used, is given. It is shown that this test is quite independent of the, to us unknown, frequency-distributions in the populations from which our samples are drawn. The application of this test is quite simple. Hence the test appears to be far better than those in use at present and which are based on some assumed frequency-distribution, generally normal distribution. The 干 test is applied to some illustrative examples and the results compared with those obtained after the application of the existing tests." The author contends that any advantages the sample mean may have over the sample median hold only in the case of the normal distribution, whereas "using the median we can devise tests which are applicable to samples irrespective of the nature of the frequency-distributions in the populations from which the samples were obtained." (p. 567).

**References:** Fisher (1922), Whittaker & Robinson (1924), Crowe (1933), Fisher (1935), Crowe (1936), Mathews (1936), Savur (1937a).

**Citations:** Nair (1940a,b), Scheffé (1943), Kendall (1946), Bickerstaff (1947), Wilks (1948).

* Smirnoff, N. V. [Smirnov, N. V.] (1937a). Sur la distribution de $\omega^2$ (criterium de M. v. Mises). (Russian. French summary). *Matematicheskii Sbornik (Recueil Mathématique)* **44** [N. S. **2**] (5), 973-993. (JFM 63, 513).

*Summary:* "Let $x_1 \leq x_2 \leq \ldots \leq x_n$ be a series of values of a random variable X obtained in n independent trials and numbered in increasing order. Let us assume that the distribution law F(x) of X is continuous and denote by $S_n(x)$ the empirical distribution function $S_n(x) = k/n$ $(x_k \leq x < x_{k+1}, k = 1, 2, \ldots, n-1)$, $S_n(x) = 0$ $(x<x_1)$, $S_n(x) = 1$ $(x \geq x_n)$. By modifying slightly the well-known expression of von Mises [(1931)], we set $\omega_n^2 = n\int_{-\infty}^{+\infty} g[F(x)]\,[S_n(x) - F(x)]^2 dF(x)$, assuming that the function $t(1-t)g(t)$ has a continuous derivative $(0<t<1)$. Then one can prove the following theorem: Whatever the function F(x), the distribu-

tion law $\Phi_n(\omega)$ of $\omega^2$ tends for $n\to\infty$ to $\Phi(\omega) = 1-(1/\pi)\sum_{k=1}^{\infty}\int_{\lambda_{2k-1}}^{\lambda_{2k}}[e^{-\omega\lambda/\lambda}\sqrt{-D(\lambda)}]\,d\lambda$, where $\lambda_k > 0$ are the characteristic numbers of the kernel $K(x,y) = \sqrt{g(x)g(y)}\,x\,(1-y)$ $(x\leq y)$, $K(x,y) = \sqrt{g(x)g(y)}y\,(1-x)$ $(x>y)$ and $D(\lambda)$ is the Fredholm determinant of the same kernel. See my note [Smirnoff (1936)]." (Compiler's translation of author's French summary, p. 993).

**References:** von Mises (1931), Kolmogoroff (1933), Smirnoff (1936).

**Citations:** Smirnov (1944), von Mises (1947).

* Smirnoff, N. [Smirnov, N. V.](1937b). Sur la dépendance des membres d'une série de variations. *Bulletin de l'Université d'État à Moscou, Serié International, Section A (Mathématiques et Mécanique)* **1**, Fascicule 4, 12 pp. (JFM 63, 1570—listing only).

**Summary:** Let $x_1 \leq x_2 \leq \ldots \leq x_n$ (1) be the observed values, in nondecreasing order, of a random variable X. Let $\Phi(x)$ be the distribution law (c.d.f.) of X, assumed to be differentiable so that $f(x) = \Phi'(x)$ is the p.d.f. of X. We call (1) the series of variations of x. The author examines the asymptotic form of the joint distribution of two members $x_k$ and $x_m (k<m)$ of this series, assuming the number of observations n to be large. He writes the joint p.d.f. of $x = x_k$ and $y = x_m$ in the form $\omega_{km}(x,y) = B_{km}^n\ \Phi_{k-1}(x)\,[\Phi(y)-\Phi(x)]^{m-k-1}\,[1-\Phi(y)]^{n-m}\,f(x)f(y)$, where $B_{km}^n = n!/\,[(k-1)!(m-k-1)!(n-m)!]$. If x is uniformly distributed on the interval (0, 1), then $\Phi(x) = x$ for $0 \leq x \leq 1$, $\Phi(x) = 0$ for $x < 0$ and $\Phi(x) = 1$ for $x \geq 1$, and the joint p.d.f. becomes $\omega_{km}^0(x,y) = B_{km}^n\ x^{k-1}(y-x)^{m-k-1}\,(1-y)^{n-m}$ $(0\leq x<y\leq 1)$. Let a,b,c,d be four members such that $a<b<c<d<1$; then the probability $P_{km}(a,b,c,d)$ of the inequalities $\Phi^{-1}(a) < x_k < \Phi^{-1}(b)$ and $\Phi^{-1}(c) < x_m < \Phi^{-1}(d)$ can be written as $P_{km}(a,b,c,d) = \int_{\Phi^{-1}(a)}^{\Phi^{-1}(b)}\int_{\Phi^{-1}(c)}^{\Phi^{-1}(d)}\omega_{km}(x,y)\,dx\,dy$, or, after making the change of variables $\Phi(x) = \xi, \Phi(y) = \eta$, by $P_{km}(a,b,c,d) = \int_a^b\int_c^d \omega_{km}^0\ (\xi,\eta)d\xi d\eta$. If $a_1 < a_2$ and $b_1 < b_2$ are four numbers whose absolute values are all less than some arbitrarily large given number L, then $P_{km}(a_1,a_2,b_1,b_2) = [(1+\delta'_n)/(2\pi\sqrt{1-r_{km}^2})]\int_{a_1}^{a_2}\int_{b_1}^{b_2} e^{[1/2(1-r_{km}^2)]\,[\xi^2-2r_{km}\,\xi\eta+\eta^2]}\,d\xi d\eta$, where $r_{km} = \sqrt{[k(1-m/n)]/\,[m(1-k/n)]}$ is the correlation coefficent of $x_k$ and $x_m$ and $\delta'_n\to 0$ as $n\to\infty$, is the probability that $\Phi^{-1}(\lambda_k+a_1\tau_k) < x_k < \Phi^{-1}(\lambda_k+a_2\tau_k)$ and $\Phi^{-1}(\lambda_m+b_1\tau_m) < x_m < \Phi^{-1}(\lambda_m+b_2\tau_m)$, where $\lambda_k = k/n$, $\nu_k = 1-\lambda_k$, and $\tau_k = \sqrt{\lambda_k\nu_k/n}$ $(k = 1, 2, \ldots, n)$.

**Reference:** Smirnoff (1935).

**Citations:** Scheffé (1943), Smirnov (1944), Mosteller (1946), Wilks (1948).

* Snedecor, George W. (1937). *Statistical Methods.* Iowa State College Press, Ames. (JFM 64, 1202). Fourth edition, 1946; fifth edition, 1956; sixth edition (with W. G. Cochran), 1967.

**Summary:** The fourth edition contains sections on the range (2.3), on the median (and the quartiles) (2.17), on the ratio of range to standard deviation (in the same sample) (5.7), and on comparisons not planned before the experiment is performed (15.5). Presumably much of this material is also in earlier editions, which the compiler has not seen. In the fifth edition, the material is rearranged and the treatment of the median and the quartiles is expanded to include their use in nonparametric tests. Sections are added on range analogues of the t-test (5.6) and on the problem of multiple comparisons (10.6), the latter including the use of the studentized range and of Hartley's method of sequential testing of pairs of adjusted means.

**References:** Gauss (1823), Galton (1888, 1889), Yule (1911), Rietz (1924), Tippett (1925), K. Pearson (1931), E. S. Pearson (1932), Fisher (1933, 1935), {Fisher & Yates (1938), Newman (1939), Deming (1943), Kendall (1943), Wilks (1943), Kendall (1946), Lord (1947)}.

**Citations:** Newman (1939), Cramér (1946), Duncan (1947), Tukey (1948c, 1949a,b).

* Sukhatme, P. V. (1937). Tests of significance for samples of the $\chi^2$ population with two degrees of freedom. *Annals of Eugenics* **8**, 52-56. (JFM 63, 1091).

**Summary:** Consider the two-parameter exponential population with p.d.f. $f(x) = (1/\alpha)\, e^{-(x-\beta)/\alpha}$. Let $x_1, x_2, \ldots, x_n$ be the order statistics of a random sample of size n from this population. The maximum-likelihood estimators are $\hat{\beta} = x_1$ and $\hat{\alpha} = \Sigma_{i=2}^{n} (x_i - x_1)/n$, the corresponding unbiased estimators being $\tilde{\beta} = x_1 - \Sigma_{i=2}^{n} (x_i - x_1)/[n(n-1)]$ and $\tilde{\alpha} = \Sigma_{i=2}^{n} (x_i - x_1)/(n-1)$ respectively. Let $\xi_1 = nx_1$, $\xi_2 = (n-1)(x_2 - x_1)$, $\xi_3 = (n-2)(x_3 - x_2)$, $\ldots$, $\xi_r = (n-r)(x_{r+1} - x_r)$. Then the $\xi$'s are distributed independently of each other, each in an exponential population with the same scale parameter as the original data. Let $\ell = \Sigma_{i=2}^{n} (x_i - x_1)/(n-1)$. Then $2(n-1)\ell/\alpha$ is distributed as $\chi^2$ with $2(n-1)$ d.f. (degrees of freedom), so that $(n-1)\ell$ is distributed as $\alpha\, \chi^2/2$ with $2(n-1)$ d.f. Also $n(x_1 - \beta)$ is distributed as $\alpha\, \chi^2/2$ with 2 d.f. Hence $z = 1/2 \log_e t$, where $t = (x_1 - \beta)/(\ell/n)$, is distributed as Fisher's z [or t as Snedecor's F] with 2 and $2(n-1)$ d.f. If $y_1, y_2, \ldots, y_{n_1}$ are the order statistics of a second ordered sample of size $n_1$ and if we let $\ell_1 = \Sigma_{i=2}^{n_1} (y_i - y_1)/(n_1 - 1)$, then $t = (x_1 - y_1)/\{[\ell(n-1) + \ell_1(n_1 - 1)][1/n + 1/n_1][n + n_1 - 2]\}$ is distributed as F with 2 and $2(n + n_1 - 2)$ d.f. and $\ell/\ell_1$ is distributed as F with $2(n-1)$ and $2(n_1 - 1)$ d.f. If we have k samples $x_{t_1}, x_{t_2}, \ldots, x_{t_{n_t}}$ $(t = 1, 2, \ldots, k)$ drawn at random from an exponential population with scale parameter $\alpha$, then under the null hypothesis that there is a common value of $\beta$, the ratio $[\Sigma_{t=1}^{k} \Sigma_{i=1}^{n_t} (x_{t_i} - x_{t_1})/\Sigma_{t=1}^{k} (n_t - 1)]/[\Sigma_{t=1}^{k} \{n_t(x_{t_1} - x_1)\}/(k-1)]$ is distributed as F with $\Sigma_{t=1}^{k} (n_t - 1)$ and $(k-1)$ d.f. This test is analogous to the analysis of variance test in normal theory.

**Note:** For convenience, the author's statements concerning Fisher's z distribution have been converted to the corresponding statements concerning Snedecor's F distribution, where $F = e^{2z}$, which is more familiar to present-day statisticians.

**References:** Neyman & Pearson (1936).

**Citations:** Paulson (1941), Kendall (1946), Lehmann (1947), Nair (1948a).

* Thibaud, Jean (1937). Étude experimentale de la distribution statistique des émissions de particules alpha. Confrontation avec la théorie. *Bulletin de la Société Francaise de Physique* **8** (406), 94 S-95 S.

**Summary:** The author compares extensive data on the emission of alpha particles by disintegrating atoms of polonium with results predicted by statistical theory, by means of five different criteria. These include fitting a Poisson distribution to the number of particles emitted in time periods of a given length, an exponential distribution to the intervals between emissions, and a Type 1 asymptotic distribution of largest values to the longest intervals in groups of 100 consecutive intervals. In each case, the observations agree quite well with the theory, except that there appears to be some oscillation about the theoretical values predicted by the exponential distribution. The author states that only further research can elucidate whether or not these oscillations are of a systematic character.

**Citations:** Thibaud & Ferber (1937), Brenet & Armand (1938).

* Thibaud, Jean; Ferber, Martin (1937). Sur l'indépendance des désintégrations des atomes de polonium. *Comptes Rendus de l'Académie des Sciences* (Paris) **205**, 1383-1385.

**Summary:** The authors study two sets of intervals between emissions of alpha particles by disintegrating atoms of polonium, one set of 7420 intervals between emissions from a strong source, with particles admitted to the counter only for a small solid angle (case a) and one set of 2160 intervals between emissions from a weak source, with particles admitted to the counter over the entire sphere (case b). In each case the series of intervals is divided into groups of 10 successive intervals, and the intervals in each group are arranged in decreasing order. The means and variances of the ordered intervals are compared with those of the order statistics of samples of size 10 from an exponential distribution, which holds if

successive intervals are independent. The agreement is quite good in case a, but in case b the long intervals are significant longer and the short ones significantly shorter than those predicted by theory under the hypothesis of independence, so this hypothesis is rejected in case b.

**Reference:** Thibaud (1937).

**Citation:** Brenet & Armand (1938).

* Wold, Herman (1937). On the mean difference at random samples — a note on Prof. Bowley's lecture. *Comptes Rendus du Congrès International des Mathématiciens* (Oslo, 1936), Vol. 2, pp. 212-213. A. W. Brøggers Boktrykkeri A/S, Oslo. (JFM 63, 523).

**Summary:** Let x be a statistical variable and F(x) its cumulative distribution function. The mean difference g of the distribution is $g = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|t-u|dF(t)dF(u) = 2\int_{-\infty}^{\infty} dF(t)\int_{-\infty}^{t} F(u)du$. Let $x_1, x_2, \ldots, x_n$ be a random sample of the variable x. The author shows that the empirical mean difference $\bar{g} = \sum_{i=1}^{n}\sum_{k=1}^{n}|x_i-x_k|/n(n-1)$ has expected value $E(\bar{g})=g$ (and thus is unbiased) and variance $\sigma_{\bar{g}}^2 = (4/n)\{\mu_2-[(2n-3)/(2n-2)]g^2 + 4[(n-2)/(n-1)]\int_{-\infty}^{\infty} dF(t)\int_{-\infty}^{t} F(u)du[m-t+\int_{-\infty}^{t} F(v)dv]\}$, where $m = E(x)$ and $\mu_2 = E[(x-m)^2]$. He gives results for two special cases: for the normal distribution, $F(x) = \int_{-\infty}^{x} e^{-(t-m)^2/2\sigma^2}dt$, $g = 2\sigma/\sqrt{\pi}$ and $\sigma(\bar{g}) = (g/\sqrt{n})\{[(n+1)/(n-1)]\ \pi/3-[4n-6-2(n-2)\sqrt{3}]/(n-1)\}^{1/2}$; for the rectangular distribution, $F(x) = 0$, $x\leq\alpha$, $F(x) = (x-\alpha)/(\beta-\alpha)$, $\alpha\leq x\leq\beta$, $F(x) = 1$, $\beta\leq x$, $g = (\beta-\alpha)/3$ and $\sigma(\bar{g}) = (g/\sqrt{5n})[(n+3)/(n-1)]^{1/2}$.

**References:** Wold (1935), Nair (1936), Bowley (1937).

* Barral Souto, Jose (1938). El modo y otras medias, casos particulares de una misma expresion matematica. *Boletin Matemático* (Buenos Aires) **11** (3), 29-41. (JFM 64, 1208).

**Summary:** The author shows that the mathematical expression $M_h = (\sum_{i=1}^{n} p_i a_i^h)^{1/h}$, where the $a_i$ are non-negative real numbers and the $p_i$ are positive weights whose sum is 1, leads for particular values of h to the following "means": $h = -\infty$ gives the smallest of the $a_i$, $h = -1$ gives the (weighted) harmonic mean, $h = 0$ gives the (weighted) geometric mean, $h = 1$ gives the (weighted) arithmetic mean, $h = 2$ gives the (weighted) quadratic mean, and $h = \infty$ gives the largest of the $a_i$. If the $a_i$ are replaced by their absolute deviations from a certain value of x, $M_h$ is transformed into $M_h(x) = (\sum_{i=1}^{n} p_i|x-a_i|^h)^{1/h}$. The author shows that the values of x which minimize $M_h(x)$ are the following for particular values of h: for $h\to 0$, the mode; for $h = 1$, the (weighted) median; for $h = 2$, the (weighted) arithmetic mean; and for $h = \infty$, the midrange. The corresponding minimum values of $M_h(x)$ are respectively the geometric mean deviation (zero), the mean (absolute) deviation, the root-mean-square deviation (standard deviation), and the semirange. These results are a generalization and extension of those given by Bruen (1938) and earlier authors.

**References:** Jackson (1921), Rietz (1924), C. Jordan (1927), de Finetti (1931b), Baidaff & Barral Souto (1934, 1935), Barral Souto (1935).

* Bartlett, M. S. (1938). Further aspects of the theory of multiple regression. *Proceedings of the Cambridge Philosophical Society* **34**, 33-40. (JFM 64, 547).

**Summary:** This paper, which is closely related to earlier work by Hotelling [(1933), (1936a)] on the method of principal components, considers the problem of testing the significance of the largest root of a matrix equation. The author writes (p. 38): "The effect of selection for the largest root $\lambda_1$ makes the problem of the exact distribution of $\lambda_1$ or the other roots exceedingly complex, and a comprehensive test of $\Lambda[\equiv \prod_{i=1}^{p}(1-\lambda_i)$, where p is the number of variates] might sometimes prove more useful." The author considers $\chi^2$ approximations to tests of significance for $\Lambda$ and for $\Lambda' = \prod_{i=2}^{p}(1-\lambda_i)$, given that $\lambda_1$ is significant.

**References:** Hotelling (1933, 1936a,b).

**Citations:** Kendall (1946), Bartlett (1947b), Rao (1948).

* Bose, S. S. (1938). Relative efficiencies of regression coefficients estimated by the method of finite differences. *Sankhyā* **3**, 339-346.

**Summary:** The author investigates the estimation of the regression coefficient of y on x by the use of differences of the independent variable y. He considers three methods: (1) the method of successive differences; (2) the method of differences at half range; and (3) the method based on the range. He calculates the relative efficiencies of these estimators in relation to the least squares estimator. The method of differences at half range has limiting efficiency (as n, the number of pairs of values of x and y, tends to infinity) of 75% relative to the method of least squares, while the other two methods have zero limiting relative efficiency. For small values of n, however, they may save a considerable amount of labor with only slight loss of efficiency.

**Reference:** Davies & Pearson (1934).

**Citations:** Nair & Shrivastava (1942), Kendall (1946).

* Brenet, J.; Armand, H. (1938). Distributions expérimentales des plus grands intervalles entre émissions de particules alpha. *Journal de Physique* (7) **9** (2), 73-80.

**Summary:** "Two series of automatic recordings of alpha particles of polonium, obtained by J. Thibaud, are studied from the point of view of the largest intervals between emissions taken in groups of n consecutive intervals among the set of $N_0$ intervals recorded. 1. The experimental material comprises two recordings: one of 7,000 intervals, the other of 14,000. The resolving power of the recording system was in the first case 1/1000 of a second and in the second 1/350. 2. The precision obtained on the mean interval was 0.8% in the first case, 2.5% in the second. Thus one can consider that the error introduced by the mean interval in the calculations is practically negligible, at least in the first case. 3. The legitimacy of applying to the grouping corrections arising from the number of intervals not registered is discussed. It appears that it is preferable to reduce the importance of this correction. 4. The agreement between the theory of largest values and the observations upon the intervals between emissions appears satisfactory. However some anomalies noted on the cumulative frequency curves and in the experimental values of dispersions and of arithmetic means lead to an interpretation which is discussed." (Compiler's translation of author's summary, p. 73).

**References:** Eyraud (1935), Gumbel (1935a, 1936b, 1937b), Thibaud (1937), Thibaud & Ferber (1937).

**Citation:** Gumbel (1939f).

* Bruen, Curtis (1938). Methods for the combination of observations: Modal point or most lesser-deviations, median loci or least deviations, mean loci or least squares, and midpoint of least range or least greatest-deviation. *Metron* **13** (2), 61-140.

**Summary:** The author considers various methods of combining observations based on the concept of power-means, as defined by Fechner (1874). The $p^{th}$-order power-mean of a set of observations, $x_i (i=1,2,3,\ldots,n)$, is that value, x, which makes the sum, $\Sigma|x_i-\mathbf{x}|^p$, a minimum. It is well known that the median is the first-order power-mean, the arithmetic mean is the second-order power-mean, and the midrange is the limiting value of the $p^{th}$-order power-mean as $p\to\infty$. Not so well known is the fact, which the author attributes to R. M. Foster [see Rietz (1924), p. 7], that the mode is the limiting value of the $p^{th}$-order power-mean as $p\to 0$. The author generalizes the concept of the power-mean from the case of direct observations to that of indirect observations or of implicit functional observations, for which it leads to the method of least power-sums of the absolute values of the deviations. Corresponding to mode,

median, mean, and midrange one has then the methods of least number (least sum of zero powers), least sum of first powers, least sum of squares, and least maximum (least sum of infinite powers) of the absolute deviations. Jackson (1924) has studied the existence and the uniqueness of solutions by the method of least $p^{th}$ powers ($0<p\leqslant\infty$) of sets of n simultaneous linear equations in m unknowns, when $m<n$. The author reviews the contributions to the theory of errors of Mayer, Boscovich, Laplace, Legendre, Gauss, Cauchy, Glaisher, Fechner, Edgeworth, Turner, Goedseels, de la Vallée Poussin, Rhodes and others. He closes with a discussion as to the choice of method, in which he points out that the choice depends on the presumed distribution of deviations, each method being best for a particular distribution — the mode in one variable or the modal point in two or more variables for a spike distribution (single isolated value), the median or median loci for a symmetric exponential (first Laplacean) distribution, the mean or mean loci for a normal (Gaussian or second Laplacean) distribution, and the midrange or midpoint of least range for a uniform (rectangular) distribution.

**References:** Mayer (1750), Maire & Boscovich (1755) [1770], Boscovich (1760), Laplace (1774, 1786, 1793, 1799), Legendre (1805), Gauss (1809), Laplace (1812), Cauchy (1837), Glaisher (1872), Fechner (1874), Edgeworth (1887b,c), Turner (1887), Edgeworth (1888), Goedseels (1909), Edgeworth (1911), Goedseels (1911), Keynes (1911), de la Vallée Poussin (1911), Yule (1911) [1932], Trachtenberg (1915), Brunt (1917), Jackson (1921), Jones (1921), Keynes (1921), Edgeworth (1923), Jackson (1924), Rietz (1924), Whittaker & Robinson (1924), Coolidge (1925), Bowley (1928), Dufton (1928), Rhodes (1930), Horst (1931), Smith (1934), Bond (1935).

**Citation:** Kendall (1946).

* Cisbani, Renzo (1938). Contributi alla teoria delle medie. *Metron* **13** (2), 23-59. (JFM 64, 161-162).

**Summary:** Part 1 deals with generalized means, which have been proposed by Dunkel (1909). Given n numbers $a_1, a_2, \ldots, a_n$, Dunkel defines their generalized mean as $y_n(x) = [\Sigma a_i^x/n]^{1/x}$ for $x\neq 0$, $y_n(x) = \sqrt[n]{a_1 a_2 \ldots a_n}$ for $x=0$. The values of $y_n(x)$ for $x = -1,0,1$, and 2 are the harmonic, geometric, arithmetic, and quadratic means, respectively, while $\lim_{x\to-\infty} y_n(x) = \min a_i$ and $\lim_{x\to+\infty} y_n(x) = \max a_i$. The author also discusses a further extension by Galvani of the concept of generalized means, but it is not relevant to our study. In Part 2, he extends the method of Jackson (1921) for determining uniquely the median of a set of $n = 2k$ numbers, where k is an integer, so as to obtain the median of a set of grouped data. Let $x_1, x_2, \ldots, x_n$ be the observed values (discrete values or class marks) and let $y_1, y_2, \ldots, y_n$ be their respective frequencies. Then, by analogy with Jackson's definition for ungrouped data, the median is defined as the value of x for which $\lim_{p\to 1} \Sigma_{i=1}^{n} |x-x_i|^p y_i$ is a minimum. The value which satisfies this condition, which is again analogous with Jackson's result, is the solution of the equation $(x-x_1)^{y_1} \ldots (x-x_r)^{y_r} - (x_{r+1}-x)^{y_{r+1}} \ldots (x_n-x)^{y_n} = 0$, where r is chosen so that the median lies between $x_r$ and $x_{r+1}$. Gini & Galvani (1929) have discussed various averages, including the median, of data which are cyclically ordered, e.g. wind directions. The usual definition of the median (the value with equal numbers of observations above and below it) is meaningless for such data, but Jackson's definition can be used to obtain a unique median. The author gives numerical examples, both for data which are ordered in the usual sense and for data which are cyclically ordered.

**References:** Dunkel (1909), Jackson [(1921)], Galvani (1927), Gini & Galvani (1929).

**Citations:** Zappa (1939), Dodd (1940), Zappa (1940), Boldrini (1942), Kendall (1946).

* Dodd, E. L. (1938). Definitions and properties of the median, quartiles, and other positional means. *American Mathematical Monthly* **45**, 302-306; note by E. J. M., 462. (JFM 64, 545).

**Summary:** The author reproduces some of the results of Jackson (1921) concerning the median and gives similar results concerning the quartiles and other positional means (quantiles). He enumerates

some of the properties of these measures. Specifically, he shows that these measures (as he defines them) are substitutive [i.e. $f(c,c,\ldots,c) = c$], internal [i.e. minimum $(x_i) \leq f(x_1, x_2, \ldots, x_n) \leq$ maximum $(x_i)$], simulvariant to all continuous monotone increasing transformations $\psi(x)$ [i.e. $f\{\psi(x_1), \psi(x_2), \ldots, \psi(x_n)\} = \psi\{f(x_1,x_2,\ldots,x_n)\}$], translative and homogeneous, but not transitive and not associative. The note by the editor, E. J. M[oulton], states that the author, too late to change the paper, discovered that the results on quartiles and other quantiles in his Sections 2 and 3 (pp. 302-304) had been published by Jackson (1923).

**References:** Jackson (1921, 1923).

* von Eberhard, O. (1938). Über das Fehlergesetz der grössten Fehlers einer Serie und das Gesetz der Salvenausdehnung. *Zeitschrift für Angewandte Mathematik und Mechanik* **18**, 128-135. (JFM 64, 1192).

**Summary:** "Since a certain relation exists between the expected largest error in a series of observations and the average spread of a salvo, both problems will be treated together here. One finds observations on the probability that the largest error in a series of n shots takes a certain value z already in Czuber [(1891a)], pp. 202 ff., also an expression for the average value of the smallest error, that is, the number which one obtains when he forms the arithmetic mean of the smallest errors from very many series of n shots each. On the other hand, one finds in Rohne [(1907)], pp. 232 ff., an empirical determination of the ratio of the salvo spread for salvos of n shots each to the probable deviation of a single shot. Thus Rohne applies a large number of test results at his disposal to state the mean salvo spread for many salvos of n shots each. An exact study of the probable error of all these means has not to my knowledge been published, though the answer to this question has a certain inherent practical importance." (Compiler's translation of author's summary, p. 128).

**References:** Chauvenet (1863), Helmert (1877b), Czuber (1891a), Cranz (1896) [1925], Rohne (1907).

* Fisher, R. A. (1938). The statistical utilization of multiple measurements. *Annals of Eugenics* **8**, 376-386. (JFM 64, 1216).

**Summary:** This is a fundamental paper on linear discriminant analysis. The following portion [p. 383 (errata)], dealing with $s-1$ orthogonal comparisons of s components, each with p degrees of freedom, is relevant to our study: "In choosing the component making the largest contribution it will be observed that $s-2$ coefficients have been adjusted, consequently the number of degrees of freedom to be ascribed to the first component is $p+s-2$, leaving $(s-2)(p-1)$ for the remainder. Similarly, if the next largest component be separated, it will contain $p+s-4$, and leave $(s-3)(p-2)$. It will be noticed that the sum of the arithmetic series $(p+s-2) + (p+s-4) + \ldots + (p-s+2)$ adds to the total of $p(s-1)$ degrees of freedom for the $(s-1)$ components, and represents the partition of degrees of freedom among them if they are chosen in order of magnitude."

**Reference:** Hotelling (1936b).

**Citations:** Fisher (1939), Hsu (1939, 1941a,b), Anderson (1945), Tintner (1945), Kendall (1946), Tintner (1946), Bartlett (1947b), Rao (1948).

* Fisher, Ronald A.; Yates, Frank (1938). *Statistical Tables for Biological, Agricultural and Medical Research*. Oliver and Boyd Ltd., Edinburgh. (JFM 64, 1202; MR 5, 207). Sixth edition, Hafner Publishing Company Inc., New York, 1963.

**Summary:** This volume includes more than thirty tables, of which two are relevant to our subject. Table XX is a two-decimal-place table of scores for ordinal (or ranked) data (expected values of normal order statistics) for samples of size $n = 2(1)50$. Table XXI gives the sums of squares of the values in Table XX for each value of n. Uses of these tables are discussed in the introduction (pp. 31-32 of the sixth edition). If

data can only be ranked but not scaled, or if there is uncertainty as to the proper scale, they can be transformed to standard normal scores by use of Table XX, and Table XXI is then useful in carrying out the analysis of variance. The scoring of tied ranks and the associated adjustment in the sum of squares are discussed. Other applications suggested for Table XX are determining the factor by which the range or a linear combination of quasi-ranges should be multiplied to obtain an unbiased estimate of the standard deviation. Use of the range instead of the standard deviation is advocated only in case a number of small samples (preferably all of the same size n, $n \leq 10$) are available, since the efficiency of the range estimator drops sharply for larger sample sizes.

**References:** Fisher (1922, 1929, 1935), {Thompson (1941)}.

**Citations:** Snedecor (1937), Arley (1940), Arley & Buch (1940), Finney (1941), Thompson (1941), Curtiss (1943), Wilks (1943), Cramér (1946), Kendall (1946), Mosteller (1946), Duncan (1947), Eisenhart, Hastay & Wallis (1947), Hastings, Mosteller, Tukey & Winsor (1947), Geary (1948), Tukey (1948c, 1949a).

* Gini, Corrado (1938). Die una formula comprensiva delle medie. *Metron* **13** (2), 3-22. (JFM 64, 160-161).

**Summary:** The author considers various types of averages or means, including the median, the mode, the midrange, the generalized geometric mean of Dunkel (1909), combinatorial means, combinatorial power means, and further generalizations. Dunkel's generalized geometric mean $M^p = (\Sigma_{i=1}^n a_i^p/n)^{1/p}$ has as special cases the harmonic mean ($p=-1$), the geometric mean (limit as $p \to 0$), the arithmetic mean ($p=1$), the quadratic mean ($p=2$), and the cubic mean ($p=3$). The author does not mention the special cases $p \to -\infty$ and $p \to +\infty$, which yield the minimum and maximum values, respectively, so the only averages specifically discussed which are based on order statistics are the median and the midrange.

**Reference:** Dunkel (1909).

**Citations:** Dodd (1940), Zappa (1940), Boldrini (1942), Kendall (1946).

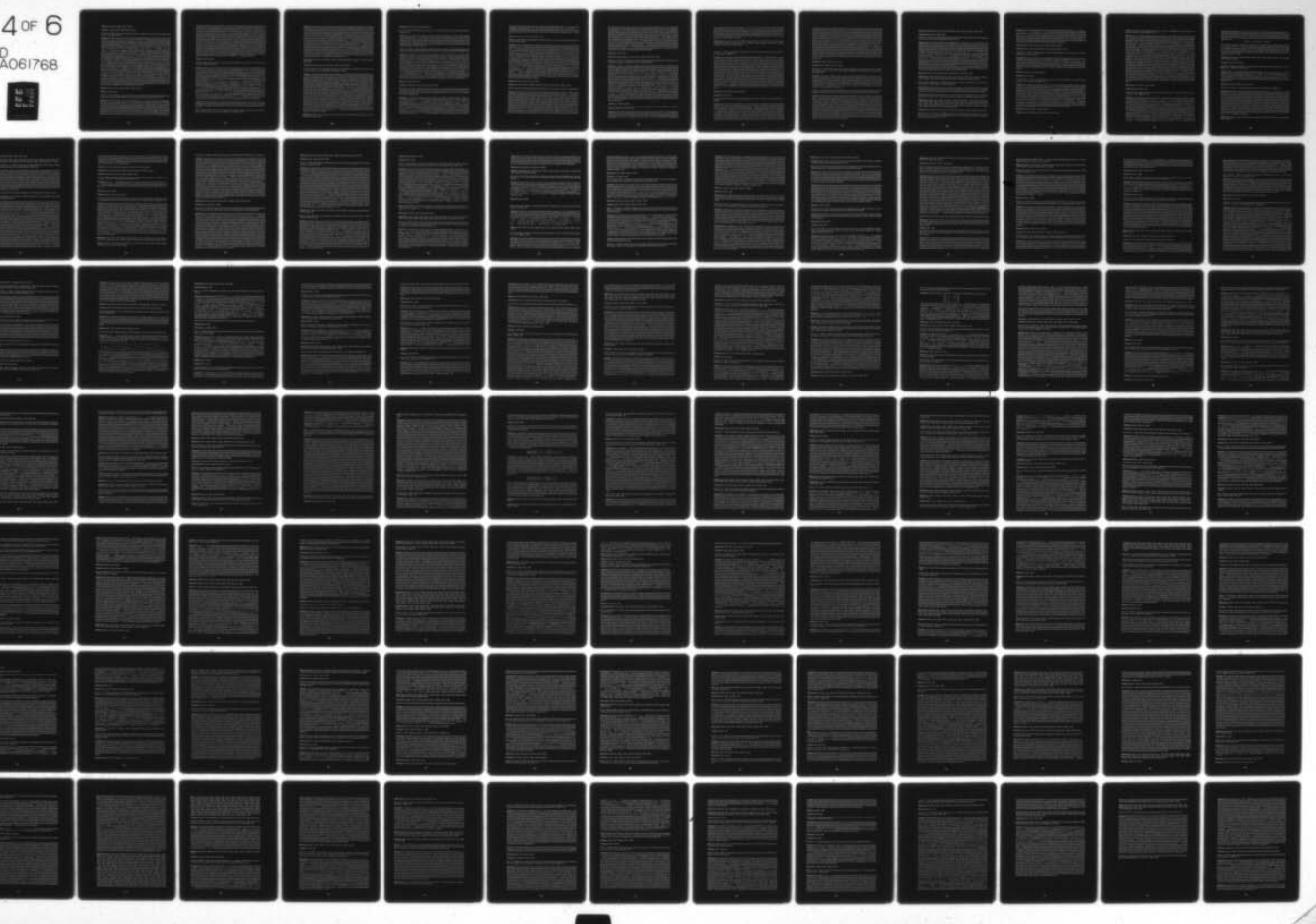
* Gumbel, E. J. (1938a). La prévision des inondations. *Comptes Rendus de l'Académie des Sciences* (Paris) **206,** 558-560.

**Summary:** The author writes (pp. 558-559): "Like the distribution of flood flows [Gumbel (1936a)], their growth over time can be interpreted by the theory of the largest value [Gumbel (1935a)]. If the initial distribution $\phi(x)$ of daily flows is (1) $\phi(x) = (a/\sqrt{\pi}\, x) \exp[-a^2 \lg^2(x/b)]$, b being the median and 1/a proportional to the difference of the two quartiles, the probability $W(x)$ of a flow less than x is (2) $2W(x) = 1 + \Phi(\lg t)$, where t is the reduced flow t related to the flow x by (3) $x = bt^{1/a}$. Then l'Hôpital's rule applied to the distribution (1) of Galton (4) $\phi(x):[1-W(x)] = -\phi'(x):\phi(x)$ is confirmed by the asymptotic value of the Gaussian integral $\Phi$, if (5) $2 \lg t >> 1$. The distribution of flood flows thus tends for an increasing number of daily observations N, very slowly, toward (6) $w(x) = \alpha \exp\{-\alpha(x-u) - \exp[-\alpha(x-u)]\}$, u being the most probable annual flood flow. Since $\alpha = -\phi'(u):\phi(u)$ decreases with x, this distribution spreads for an increasing number of observations, unlike that of extreme intervals between radioactive emissions [Gumbel (1937b)], which remains unchanged, and that of greatest ages [Gumbel (1937a)], which contracts. For the prediction of floods and their precision, we use the median $u_0$ and the two quartiles $u_1$ and $u_2$ of the largest value (7) $W(u_0) = \exp[-(\lg 2)/N]$, $W(u_1) = \exp[-(\lg 4)/N]$, $W(u_2) = \exp[-\lg(4/3)/N]$. Following (3) one obtains the three reduced flood flows $\tau_0, \tau_1, \tau_2$ corresponding to $u_0, u_1, u_2$, as a function of the number of years observed $T = N/365$, (8a,b,c) $T_0 = 3.79807 \times 10^{-3}/[1-\Phi(\lg \tau)]$, $T_1 = 2T_0$, $T_2 = 0.415T_0$, where $T_0, T_1, T_2$ are the years corresponding to $\tau_0, \tau_1, \tau_2$, calculated for the same value $\tau$. These formulas assume only that $N^2 >> 1$, a much less restrictive condition than (5). One determines the two constants a and b from (3) according to the observed trend of the floods themselves. . . . " (Compiler's translation). The remainder of the paper is devoted to a numerical example of the application of the theory developed by the author to the use of data on floods on the Rhine at Basel during the period 1808-1925 to predict future floods.

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**References:** Gumbel (1935a, 1936a, 1937a,b).

**Citations:** Gumbel (1938b, 1939f), Kendall (1946).

* Gumbel, E. J. (1938b). Le débit de crue médian au cours des années. *Comptes Rendus de l'Académie des Sciences* (Paris) **207,** 832-834.

**Summary:** The author writes (pp. 832-833): "The median flood discharge $u_0$ as a function of the number of years $T_0$ is (1) $u_0(T_0) = b \exp(\lg \tau : a)$, $[1-\Phi(\lg \tau)]T_0 = 3.79807 \cdot 10^{-5}$, $\Phi$ being the Gaussian integral, b the median of the daily discharges, and 1:a proportional to the logarithm of the quotient of the two quartiles of the daily discharges. One can easily obtain from this a linear relation between lg $\tau$ and lg $T_0$ valid for reduced median flood discharges $2.6 < \lg \tau < 3$, that is to say for a limited time interval, say $16 < T_0 < 170$. Because one obtains, by expanding in series about $\log \tau' = 2.8$, $d \lg \tau / d \lg T_0 = [1-\Phi(\lg \tau')] / [2 \exp(-\lg^2 \tau') : \sqrt{\pi}] + (\lg \tau - \lg \tau')\{-1 + 2 \lg \tau' [1-\Phi(\lg \tau')] / [2 \exp(-\lg^2 \tau') : \sqrt{\pi}]\} + \ldots = 0.16884 + (\lg \tau - 2.8)\, 0.0545 + \ldots$. The first factor is approximately equal to the quotient of the differences for the limits of the interval $\log_e[\Delta \lg \tau / \Delta \lg T_0] = 0.43429(3.0-2.6)/(2.23517-1.20676) = 0.16892$. Since the second factor has only a small influence in the interval considered, one can set approximately (2) $\lg \tau(T_0) = A + 0.16884 \lg T_0$, from which follows, according to (1),(3) $u_0(T_0) = b\, e^{A:a}\, T_0^{0.16884:a}$. In the interval considered, the increase of the logarithm of the median flood discharge is independent of time and proportional to 1:a. Because (4) $d \log u_0(T_0) / d \log T_0 = 0.16884/a$. Another approximation, valid for small values of 1:a and short durations $T_0$, is, according to (1) and (2), $u_0(T_0) = u_0(50) \exp[0.16884 (\log T_0 - \log 50) : a \log e]$. (5) $u_0(T_0) = u_0(50)[1 + (\log T_0 - \log 50) 0.38877 : a]$. The median flood discharge becomes, under the stated conditions, a linear function of the logarithm of the number of years. An analogous formula, but with different and purely empirical signification, is due to Fuller [(1914)]. For the prediction of floods, one sets, according to (1), (6) $\log u_0(T_0) = \log \tilde{u}(1) + [\lg \tau(T_0) - 2.0466] \log e : a$, $\tilde{u}(1)$ being the median annual flood discharge obtained from the observed distribution of flood discharges. One identifies the largest observed flood discharges $'u(T)$, up to an epoch T, called the base year, with median flood discharges $u_0(T)$. One introduces, according to (1), the corresponding values $\lg \tau(T)$. Finally, n being the number of base years, one determines the only constant log e: a by the method of least squares according to (7) $[\Sigma \log {}'u(T) - n \log \tilde{u}(1)] : [\Sigma \lg \tau(T) - 2.0466\, n] = \log e : a$." (Compiler's translation). The author illustrates his theory by an example using data [see Saville (1936)] for the Tennessee River at Chattanooga during the period 1875-1931. He finds that the agreement between the theory and the observations is satisfactory. He states that there is a probability of three-fourths that the flood discharge for the next 100 years will not exceed the largest exceptional value observed up to the present.

**References:** Fuller (1914), Saville (1936), Gumbel (1938a).

**Citation:** Gumbel (1939f).

* Gumbel, E. J. (1938c). Théorie des plus grandes valeurs. Applications à la radioactivité, à la durée de la vie humaine et aux inondations. *Bulletin de la Société Française de Physique* **9** (417), 60 S.

**Summary:** The author writes: "For an initial distribution w(x), having a [cumulative] probability W(x), there exists a distribution of the $m^{th}$ value among N observations, like those of the $m^{th}$ extreme value and of the largest value. Under certain analytic conditions, verified in general, one obtains the mode of the largest value; its probability of being exceeded is 1/N. Under the same conditions, the $m^{th}$ value tends toward a doubly exponential form. The two constants of this distribution are the mode and the logarithmic derivative of the initial distribution of the mode. For comparison with the observations, control functions have been established which correspond to a domain of probability 2/3. The final distributions of the $m^{th}$ extreme values are of three types. If the initial distribution decreases, for increasing values of the variable: (1) at the same rate as, (2) faster than or (3) more slowly than an

exponential distribution, the mode of the $m^{th}$ value, and in like manner the last value, increases: (1) at the same rate as, (2) more slowly than or (3) faster than the logarithm of the number of observations. At the same time, the standard deviation of these distributions: (1) remains constant, (2) decreases or (3) increases. This signifies that the distribution: (1) does not change form, (2) narrows or (3) spreads, for an increasing number of observations. The first type is realized by radioactive emissions, which one assumes to be independent and which follow an exponential initial distribution. So one can check this theory by observations taken on the extreme interval. The general agreement of the observations with the theory favors the independence of the emissions. The second type is realized by the distribution of ages at the moment of death, around the normal age (75 years). The distribution of greatest ages agrees well with that of the largest values, which allows one to calculate the limiting age and to predict how this age evolves as a function of time. The predictions are verified by observations made in Switzerland and in Sweden. The third type is realized by the distribution of discharges of rivers. For example, the distribution of the flood discharges of the Rhine follows well that of the largest values. For the prediction of floods, one establishes the median of the greatest discharge, as a function of time, assuming a Galton [lognormal] initial distribution. The increase of the standard deviation with the number of observations agrees with current opinions on the question. The determination of the constants, with the aid of least squares, according to the maxima of the floods on the Rhine in the course of the last century, leads to a good agreement between the theory and the observed facts." (Compiler's translation).

**Citation:** Gumbel (1939f).

* Hertzman, Max (1938). Two equations for the study of variability. *American Journal of Psychology* **51**, 571-574. (JFM 65, 1355-1356).

**Summary:** In this note the author discusses the mean difference and the mean squared difference of a set of n scores. The latter is simply twice the standard deviation or $2n/(n-1)$ times the standard deviation, depending on whether the standard deviation is defined with n or $(n-1)$ in the denominator. The author (pp. 571-572) develops a convenient method of computing the mean difference: "The scores in a distribution may be arranged in their order of magnitude from the highest score to the lowest. These may be represented by the following series: $k_n, k_{n-1}, k_{n-2}, \ldots, k_1$ [1] where $k_n$ is the largest score, $k_{n-1}$ is the next largest, and so on. The sum of all the possible differences among these numbers may be represented as $D = (k_n - k_{n-1}) + (k_n - k_{n-2}) + \ldots + (k_{n-1} - k_{n-2}) + \ldots + (k_2 - k_1)$. [2]. It should be noted that $k_n$ will appear $(n-1)$ times and will always be positive; $k_{n-1}$ will have positive values in $(n-2)$ instances and negative values in 1 instance; $k_{n-2}$ will be positive $(n-3)$ times and negative 2 times, etc. Therefore, by rearranging the sums and differences we obtain $D = (n-1)k_n + (n-3)k_{n-1} + (n-5)k_{n-2} + \ldots + (1-n)k_1$. [3]. In order to obtain the average value we need only to divide the equation by $n(n-1)/2$. The latter value will give us the number of differences that exist. It should be noted that equation [3] is symmetrical so that $(n-1) = -(1-n)$; $(n-3) = -(3-n)$; etc. The equation may therefore be simplified to $D=(n-1)(k_n - k_1) + (n-3)(k_{n-1} - k_2) + \ldots$ [4]. These combinations may be made in all cases since in a series with an odd number of terms the middle term will be zero. In equation [4] the last term [after correction of obvious typographical errors] will be $2(k_{(n+3)/2} - k_{(n-1)/2})$ if the series is odd and $(k_{(n+2)/2} - k_{n/2})$ if the series is even." The author gives a numerical example of the use of equation [4].

**Comments:** The mean difference is defined for the unordered observations, but is more conveniently computed when expressed in terms of the order statistics (equation [3]) or the quasi-ranges (equation [4]). The latter equation was developed by von Andrae (1872), of whose work the author was apparently unaware.

* Jeffreys, Harold (1938). The law of error and the combination of observations. *Philosophical Transactions of the Royal Society of London* (A) **237**, 231-271. (JFM 64, 1205).

**Summary:** The author discusses the limitations of the theoretical grounds for accepting the normal law of errors of observation and examines seven series of observations capable of providing tests of its truth. The following statements (pp. 231-232) are of interest: "The normal or Gaussian law of error rests partly on a particular hypothesis about the nature of error, that the error of any individual observation is the resultant of a large number of comparable and independent components; and partly on comparisons with frequencies in actual series of observations. Both arguments are defective. The theory does not prove that the law is true for errors of any magnitude, even if its fundamental hypothesis is accepted. It involves a number of approximations, and when the effects of these are examined it is found that the law should hold only up to a moderate multiple of the standard error. . . . The three hypotheses (1) that the normal law is correct, (2) that the mean is the best value, (3) that the mean square residual provides the best standard of accuracy, are all equivalent; any one of them implies the other two. . . . The rejection of the postulate of the mean is not the same as the rejection of observations." As examples of error distributions for which the arithmetic mean should not be used, the author mentions the Cauchy and the rectangular. He points out that Fisher (1922) has shown (p. 321) that the distribution of the mean of a sample of any size from a Cauchy distribution is the same as that of a single observation and (pp. 348-351) that for a sample from a rectangular distribution the extreme observations contain the whole of the relevant information in the sample; as the sample size n increases, the mean square error of the midrange decreases like $n^{-1}$ and that of the mean only like $n^{-1/2}$.

**References:** Brunt (1917) [1931], Fisher (1922), Whittaker & Robinson (1924), E. S. Pearson (1926), Bond (1935).

**Citations:** Hulme & Symms (1939), Jeffreys (1939), Kendall (1946).

* Kimball, Bradford F. (1938). Probability-distribution-curve for flood-control studies. *Transactions of the American Geophysical Union* **19**, 460-477; discussion, 477-483.

**Summary:** The author reviews the history of attempts since 1896 to apply the statistical method to the occurrence of hydrological phenomena. In that year, at the request of George W. Rafter, Robert E. Horton prepared an analysis, based on the Gaussian curve of error, of the rainfall and runoff of the Upper Hudson Drainage Basin. Between 1900 and 1908, he plotted many flood-records on logarithmic paper. [See Horton's contribution to the discussion of a paper by Fuller (1914)]. Fuller and Hazen also used similar methods and designed logarithmic probability graph paper for this purpose. [See Fuller (1914) and Hazen (1930)]. Another method was developed by Horton (1924). Foster (1924) used Pearson curves (types I and III). Slade (1934, 1936) proposed a transformation $u = c \log [(b+x)/(g-x)]$ such that as x varies from its lower limit $-b$ to its upper limit $+g$, u will vary from $-\infty$ to $+\infty$, and used the mean and standard deviation of x to determine the constants in this equation. All of these developments were reviewed by Saville (1936). The author proposes a modification of the Slade method in which the constants are determined from the mean and standard deviation of the transformed variable u. He applies his method to the 57-year record of the annual flood-peaks of the Tennessee River at Chattanooga, Tennessee (1875-1931). On page 474 the author states: "After the best theoretical probability-curve has been determined, there is a question of the interpretation of this curve which is of some pertinence. Suppose the flood-control policy of the State is to provde against 'all floods that are likely to occur in the next thousand years'. By the same reasoning which we have applied to plotting the points designating the original record of data on the probability-paper [the author uses plotting positions of $1/2N, 3/2N, \ldots, (2N-1)/2N$], we should use the reading of the curve at a percentage-of-time figure halfway between 0.1 per cent and 0.0 per cent, in other words at 0.05 per cent, rather than the reading of the curve at the percentage-of-time figure equal to 0.1 per cent. . . ." The discussion includes comments by Slade and Saville and a rejoinder by Kimball.

**References:** Fuller (1914), Foster (1924), Horton (1924), Hazen (1930), Slade (1934), Horton (1936), Saville (1936), Slade (1936).

* **Citations:** Gumbel (1942c), Kimball (1942).

* Savur, S. R. (1938). The median versus mean or any other statistic in tests of significance (abstract). *Sankhyā* **4**, 92.

**Summary:** "It is shown in this paper that a general test of significance, that is a test of significance which is applicable to all cases irrespective of the frequency distributions, is better than a special significance test, that is a test which is accurate only for a particular frequency distribution or distributions. Of all the general tests that can be visualized at present the 円 test is the simplest. It is also simpler than the existing special tests. Hence the suggestion is made that the 円 test [based on the median; see Savur (1937b)] may be used more and more widely in place of the existing tests [based on the arithmetic mean]." (Author's abstract).

* Thompson, William R. (1938). Biological applications of normal range and associated significance tests in ignorance of original distribution forms. *Annals of Mathematical Statistics* **9**, 281-287. (JFM 64, 1222).

**Summary:** In Section 1, the author develops the theory of range estimation. Let $x_k$ be the $k^{th}$ order statistic of a sample of size n from an unknown infinite universe U. Let $p_k$ be the probability that a random value x from U shall be less than $x_k$. The mean value of $p_k$ in repeated random sampling of n values from U is $\bar{p}_k = k/(n+1)$ and the variance of $p_k$ is $\sigma^2_{p_k} = k(n-k+1)/[(n+1)^2(n+2)]$. In an earlier paper [Thompson (1936)], the author has shown that for a finite population of size N, $\bar{p}_k = [k/(n+1)](1+1/N)-1/N$ and $\sigma^2_{p_k} = \{[k(n-k+1)]/[n+1)^2(n+2)]\}(1+1/N)(1-n/N)$. On page 283 the author writes (for the case of an infinite universe): "Now suppose that we want to find a range $(\alpha,\beta)$ such that, in random drafts from U, the theoretical relative frequency of drawing x less than $\alpha$ is g, and the same as that of drawing x greater than $\beta$. $(\alpha,\beta)$ may be called a *central confidence range* with a *confidence* f = $1-2g$ that x drawn at random will lie within the range. For $g = k/(n+1)$ we may take the range $R_f = (x_k, x_{n-k+1})$; and likewise with g = 5% we may estimate, or approximate by interpolation where $20k > n+1 > 20(k-1)$, a range $R_f$ for normal . . . variation of a specified character, and this may be called briefly the estimated 90% *central normal range*". The remainder of Section 1 is devoted to an example. Section 2 deals with distribution-free significance tests which are based on ranks, *not* on order statistics.

**Reference:** Thompson (1936).

**Citations:** Scheffé (1943), Kendall (1946), Wilks (1948).

* Waschakidse, D. (1938). Über das Maximum der Abweichung des theoretischen Verteilungsgesetzes von der entsprechenden empirischen Kurve. (Russian. German summary). *Akademiia Nauk Gruzinskoi SSR, Matematicheskii Institut, Zbirnik* **4**, 101-122. (JFM 64, 1193). *Moscow Universitet, Uchenye Zapiski* **15**, 183-190. (JFM 65, 560).

**Summary:** "Let there be n independent trials carried out, in which the random variable X follows the same distribution law $F(x) = P\{X \leq x\}$. Let the values of X obtained as a result of the trials be arranged in increasing order: $x_1 \leq x_2 \leq \ldots \leq x_n$. We set $F(x) = 0$ for $x < x_1$, $F(x) = k/n$ for $x_k \leq x < x_{k+1}$, $F(x) = 1$ for $x \geq x_n$. Our problem consists of the evaluation of the upper limit of the difference $D_n = \sup|F_n(x)-F(x)|$. The upper limit will be taken here over all values of x. Let $\Phi_n(\lambda)$ denote the probability of the inequality $D_n \leq \lambda/\sqrt{n}$. As shown by Kolmogoroff [(1933),], the probability $\Phi_n(\lambda)$, in the case of a continuous function F(x), is independent of the form of this function and converges, as $n \to \infty$, uniformly in $\lambda$, to the limiting value $\Phi(\lambda) = \sum_{-\infty}^{\infty} (-1)^k e^{-2\lambda^2 k^2}$. The difficulty in evaluating the remainder term makes the use of this asymptotic expression for the probability $\Phi_n(\lambda)$ burdensome. On the other hand, V. Glivenko [(1933)] has proved the inequality $\Phi_n(\lambda) > 1-1/\lambda^6 n^2$. The inequality of V. Glivenko is sufficient to prove that $D_n$

converges to zero with probability 1; however, this estimate of the probability $\Phi_n(\lambda)$ is very rough. I have succeeded in showing that, for $2 \leqslant \lambda \leqslant \sqrt{n}/10$ (which is possible for $n \geq 400$), $\Phi_n(\lambda) > 1 - 39\sqrt{n}\, e^{-2\lambda^2}$. . . ." (Compiler's translation of author's German summary, pp. 189-190 of the second version, which differs in some details from the first). The author gives an outline of the proof of the inequality $\Phi_n(\lambda) > 1 - 39\sqrt{n}\, e^{-2\lambda^2}$.

**References:** Glivenko (1933), Kolmogoroff (1933).

* Wilks, S. S. (1938). Fiducial distributions in fiducial inference. *Annals of Mathematical Statistics* **9**, 272-280. (JFM 64, 1204).

**Summary:** After an introduction to the fiducial method of argument, the author gives the following example (pp. 273-274): "If a sample of n individuals is drawn from a population whose distribution law is $f(x,\theta) = 1/\theta$, where only values of x between 0 and $\theta$ are considered, (that is, a rectangular distribution having range $\theta$) the probability that the range r of the sample lies between r and r+dr is $\phi(r,\theta)dr$, where (1) $\phi(r,\theta) = n(n-1)(\theta - r)^{n-2}/\theta^n$. Here $\theta$ is the parameter under question, and r is the estimate; r is the difference between the largest and smallest variate in the sample. Thus, for a given value of $\theta$, say $\theta_0$, $\phi(r,\theta_0)$ is a sampling distribution law defined for given values of r on the range $r = 0$, to $r = \theta_0$. If we let $r/\theta = \psi$, then (2) $\phi(r,\theta)dr = n(n-1)(1-\psi)\psi^{n-2}d\psi = G(\psi)d\psi$, which, from a statistical point of view, shows that if we should take an aggregate of randomly drawn samples (of n items each) from rectangular populations and calculate $\psi$ for each *sample-population combination*, then the distribution of $\psi$ will be that given in (2). By a *sample-population combination* in this example we mean any rectangular population that may arise and a 'randomly drawn' sample from it. Thus if $\psi_a$ is such that (3) $n(n-1)\int_0^{\psi_a}(1-\psi)\psi^{n-2}d\psi = \alpha$, i.e. $\psi_\alpha^{n-1}[n-(n-1)\psi_a] = \alpha$, and if we draw a sample of n from a rectangular population, we can claim that the probability is $1-\alpha$ that the $\psi$ produced by this sample-population combination will satisfy the inequality (4) $\psi_\alpha < \psi < 1$. . . . Inequality (4) is equivalent to each of the following inequalities (5) $\psi_\alpha < r/\theta < 1$, $r/\psi_a > \theta > r$. . . . . The interval between r and $r/\psi_a$ is called a *confidence interval or fiducial interval*; $1-\alpha$ is known as the *confidence coefficient* associated with the limits." The author continues with a look at fiducial limits from the geometric viewpoint, a summary of principles, a consideration of a large-sample case, and extensions of the fiducial argument.

**References:** Fisher (1933), Neyman (1937b).

**Citations:** Eisenhart (1939), Kullback & Frankel (1940), Cramér (1946), Kendall (1946).

* Wishart, John (1938). Field experiments of factorial design. *Journal of Agricultural Science* **28**, 299-306.

**Summary:** The author writes (pp. 300-304): "The purpose of this paper is to direct attention to the examination for significance of treatment effects based upon single degrees of freedom. . . . The modern method of calculating treatment effects, and the mean squares (1 D.F.) based on them, has rather directed attention away from the mean square for treatment generally, based upon all degrees of freedom available. With eight treatments, there will be a mean square based on 7 degrees of freedom, available for comparison by a z- or F-test with an error mean square, which, if there are four randomized blocks, will have 21 degrees of freedom. If the comparison shows that the treatment mean square is significantly larger than the error mean square, then the hypothesis that treatments are without effect is disproved, and we are entitled to examine individual effects in relation to their standard error, a process which is exactly equivalent to testing the mean squares for the separate effects (1 D.F. each) against error by the z-test. . . . It has been urged on many occasions that the z-test for treatments as a whole should logically precede the t-comparison of treatment mean differences, for if the latter comes first there is a danger that certain differences may be asserted to be significant when they are in reality

the differences of extreme members of a normally varying sample. Nevertheless, there was a reason for the decomposition of the total treatment mean square into component parts, for each comparison involved all plots, and it was quite likely that, in cases where the total mean square was insignificant, a significant effect might be missed through being watered down by inclusion with a number of insignificant effects. If considerably more than one-seventh of the treatment sum of squares is due to a single degree of freedom, the mean square for this effect may well be significant, even though a higher value of z or F is required, since $n_1$ [the number of D.F. for the numerator] is 1 and not 7. . . . It is evident that some sort of supplementary test should be devised which will test whether the set of treatment mean squares, taken as a whole, are homogeneous or not. . . . The supplementary test that we have indicated above as necessary is evidently supplied by following Bartlett's procedure, and should be applied to the group of treatment mean squares. In fact, the whole group of . . . mean squares [including the one for error] could be examined for homogeneity. . . . It is not claimed that such a test of homogeneity of estimated variances will be appropriate in all cases. It may sometimes be necessary to develop a test for the significance of the highest, say, of a number of mean squares. [See Fisher (1935)]. . . ."

**Reference:** Fisher (1935).

**Citations:** Chandra Sekar & Francis (1941), Kendall (1946), Nair (1948a).

* d'Addario, R. (1939). Un metodo per la rappresentazione analitica delle distribuzioni statistiche. *Annali dell'Istituto di Statistica della Reale Università di Bari* **16**, 3-56.

**Summary:** In Chapter I, the author considers the function $\phi(x)$ (the mean of values not less than x) and its general properties, passage from the function $\phi(x)$ to the corresponding p.d.f. f(x), the method of the mean of exceeding values, the parametric equation of the superior curve of concentration, and the analytic general expression of $\phi(x)$ and its expansion in series. In Chapter II, he deals with the hypothesis that $\phi(x)$ is a linear function $\phi(x) = A_0 + A_1(x - x_0)$, where $x_0 \geq 0$ and $A_0 > 0$. He considers three cases: 1) $x_0 \leq x < \infty$, $A_1 > 1$; 2) $x_0 \leq x < \infty$, $A_1 \approx 1$; 3) $x_0 \leq x \leq k$, $A_1 = (k - A_0)/(k - x_0) < 1$. He relates the first case to the Pareto distribution and the second to the Pearson type III distribution, and shows that the third can take various forms, depending upon the value of $A_1$. In Chapter III, he considers the hypothesis that $\phi(x)$ is a parabolic function $\phi(x) = A_0 + A_1(x - x_0) + A_2(x - x_0)^2$. He relates this case to the Pearson system of frequency curves, the type depending upon the relations among $A_0$, $A_1$ and $A_2$. In Chapter IV, he applies the method of the mean of exceeding values to various sets of income data. He shows that the parameters of Pareto's second approximation $N(x) = N(x_0)[(a + x_0)/(a + x)]^{\alpha}$ can be estimated as follows: $x_0 = \bar{x}_0$, $N(x_0) = \overline{N}(x_0)$, $\alpha = (\text{med } \bar{\delta})/(\text{med } \bar{\delta} - 1)$ and $a = \overline{A}_0/(\text{med } \bar{\delta} - 1) - \bar{x}_0 \text{ med } \bar{\delta}$, where $\bar{x}_0$ is the observed minimum income, $\overline{A}_0$ is the mean of the observed incomes, $\overline{N}(x_0)$ is the observed number of incomes, and med $\bar{\delta}$ is the median of the values of $\delta = \alpha/(\alpha - 1)$ calculated for incomes above $x = x_0, x_1, \ldots, x_s$. He uses these results to estimate the parameters $x_0$, $N(x_0)$, $\alpha$ and a for the data considered. In Chapter V, he considers the problem of the choice of interpolating functions, and notes that by considering values of income (or some other character) not more than x instead of those not less than x, one can transform the method of the mean of exceeding values to the method of the mean of nonexceeding values.

**Citation:** de Vergottini (1940).

* Bresciani-Turroni, C. (1939). Annual survey of statistical data: Pareto's law and the index of inequality of incomes. *Econometrica* **7**, 107-133.

**Summary:** The author summarizes the literature on Pareto's law and his index $\alpha$ of the inequality of incomes. On page 110 he notes that the relative average deviation may be taken as an alternate measure of inequality. In a footnote on the same page, he notes that D, the relative average deviation divided by 2, is given by $D = (\alpha - 1)^{\alpha - 1}/\alpha^{\alpha}$ or by $D = [(\alpha - 1)/\alpha]^{\alpha - 1}[(h + a)/(\alpha h + a)]$, according as the distribution of

income follows Pareto's first or second law, $N_x = A/x^\alpha$ or $N_x = A/(x+a)^\alpha$, where $N_x$ is the number of individuals with income greater than or equal to x, and A, $\alpha$ and a are constants. He states that it can be easily shown that D varies from 0 (perfect equality) to 1 (maximum inequality). In another footnote on the same page, he also mentions the concentration ratio R proposed by Gini [(1914a)], which measures [twice] the area between the Lorenz curve and the line of equidistribution [and also varies from 0 to 1], and states that discussions as to the best single index of inequality have centered in Italy on the respective merits and defects of D and R.

**References:** Pareto (1897), Gini (1910, 1912), Pietra (1915), Ricci (1916), Winkler (1924), Amoroso (1925), d'Addario (1930), von Bortkiewicz (1931), d'Addario (1934a,b), Mortara (1934b), Pietra (1935b), d'Addario (1936), Bresciani-Turroni (1937).

* Coutagne, A. (1939). La "distribution statistique" du maximum annuel des précipitations journalières. *La Météorologie* (3) **1939**, 93-98.

**Summary:** The author begins with the following introduction (page 93): "In a preceding note . . . I have shown that the law of Galton—logarithmic variant of the law of Gauss—leads to a very satisfactory analytic interpretation of the frequency curve of annual, monthly and daily precipitations. The object of the present note is to study—from an analogous point of view—the distribution of maximum annual values of the daily precipitation, to see if the law of Galton faithfully interprets the observed data and moreover to compare it with the theory of the largest value, recently formulated by Gumbel, a theory which, applied upon my instigation to the study of flood flows of streams, leads to interesting results. We shall treat today a single example, taking as statistical data the observations of the observatory of Lyon-Fourvière (49 years—1888-1936). Table I [p. 98] contains, arranged in decreasing order, the annual maxima of daily precipitations for this series of years." (Compiler's translation). He closes as follows (page 97): "In summary—and we have made the same ascertainment in studying the flood flows of several streams— the laws of Galton and of Gumbel seem to lead to identical results agreeing with the observed data furnished by observations of the maxima of precipitations. Both lead—for small probabilities—to the exponential law of Fuller. In the case of streams the asymmetry of the frequency curve may decrease . . . and then the law of Gauss, limiting case of the law of Galton, is practically applicable. Moreover, we propose at a later date to examine thoroughly the question, by studying other statistics, as it concerns flood flows and annual maxima of daily precipitations." (Compiler's translation).

**References:** Gumbel (1934f, 1935a, 1936a,b).

**Citation:** Gumbel (1939e).

* Creager, William P. (1939). Possible and probable future floods. *Civil Engineering* (New York) **9** (11), 668-670.

**Summary:** The author states (pp. 668-670): "About 1914 the theory of probabilities was applied to flood studies; that is, curves were derived indicating by past records on a stream the frequency with which, during a long period, a given flood should be expected. . . . Recently, however, it has been proved by advanced studies and a greater accumulation of data, that the probability method is entirely inadequate. . . . Thus floods have occurred on rivers which, based upon probability studies of prior records of considerable length, would have a frequency, not of the usually adopted 1,000 to 10,000 years, but of once in millions or even billions of years. . . . Much has been done by modern methods to overcome the uncertainties regarding probable floods. However, we have yet to overcome the handicap of lack of long-term records. Modern methods used in flood studies involve as basic data the worst recorded meteorological and hydrological phenomena which have been experienced in the past. In many cases

such phenomena, occurring within a division of the country surrounding the site, and not too large to be reasonably indicative of comparable conditions, have been plotted and enveloping curves drawn. Data from such enveloping curves, adjusted in some cases for special peculiarities of the drainage area in question, form the basic data from which to compute the flood. The flood so computed then corresponds to what would happen at the site under the worst meteorological and hydrological conditions of record. This flood may be termed the 'computed spillway flood.' Since the computed spillway flood corresponds to past experience in that region, and since the worst experience that has occurred within our very limited period of observation surely must be exceeded in the future, it remains to determine what margin of safety should be adopted—that is, how much the computed spillway flood must be increased to obtain what may be termed the 'spillway design flood', or the flood that must be passed safely at the site. . . . Increased data have raised enormously the records of flood flows for all drainage areas. . . . There is no evidence that climatological conditions are changing perceptibly, the upward trend . . . being due solely to an increasing number of gaging stations and an increasing period of record. Therefore, the occurrence of greater floods as time passes must be according to the laws of probability and of chance. The writer suggests for further research the investigation of probabilities in connection with future floods. By this he does not refer to the now obsolete probability curve previously mentioned, but the probability of a storm, of greater magnitude than any that has occurred in the past, centering directly over a given drainage area."

**References:** Fuller (1914), Hazen (1930).

**Citations:** Grant (1940), Lane & Lei (1949).

* Derevitsky, N. (1939). On the rejection of field experiment data and their subsequent analysis. (Russian. English summary). *Acta Universitatis Asiae Mediae* (Tashkent) (V a. Mathematica) Fasciculus 22, 21 pp. (JFM 65, 592).

**Summary:** "The author considers analysis of data given by a field experiment consisting of m treatments and n replications of every treatment by the method of analysis of variance. Some faulty data are supposed and the criterium of rejection given by Chauvenet is applied for their rejection. The resulting incomplete data are treated with the aid of Wishart, Allan and Yates' formulae and the author's own method of successive approximations, a modification of Yates' method. An illustration is given representing analysis of an experiment with cotton made in 1935 in the Azerbaidjan cotton research institute." (Author's English summary, p. 21).

* Eisenhart, Churchill (1939). The interpretation of certain linear regression methods and their use in biological and industrial research. *Annals of Mathematical Statistics* **10**, 162-186. (JFM 65, 600).

**Summary:** This paper deals with the interpretation of relations between two variables or among more than two variables, and the procedure for identifying one of them as the dependent variable, whose values are determined (not exactly, but within limits) from the values that are assumed by the other (independent) variable(s) in the data under consideration. In fitting a straight line (or a curve) to data on two variables, x and y, one may take y (x) as the dependent variable and minimize the sum of squares (or some other appropriate function) of the vertical (horizontal) deviations of the data points from the fitted line or curve. If the values of one variable have been selected by the research worker and the corresponding values of the other variable have been observed (subject to error), then the first should be taken as the independent variable and the second as the dependent variable, but if both variables are subject to error, the choice of dependent and independent variables is not clear. The author discusses the regression question mainly under the assumption that the arithmetic mean is taken as the measure of central tendency and the method of least squares is employed. He points out (p. 170), however, that if the median is used as the measure of central tendency, one should minimize the sum of absolute deviations from the regression line instead of the sum of their squares.

**References:** Shewhart (1931), Fisher (1933), Pearson (1935), Neyman (1937b), Wilks (1938).

**Citations:** Wald (1940), Kendall (1946).

* Fisher, R. A. (1939). The sampling distribution of some statistics obtained from non-linear equations. *Annals of Eugenics* **9**, 238-249. (JFM 65, 572; MR 1, 248).

**Summary:** Section 1, which deals with the sampling theory of Fisher's z-test, is not relevant to our subject. Section 2 deals with Fisher's test of significance of a harmonic component. Let there be s different periods with two components each. The sum of two squares for each period constitutes a certain fraction of the whole. Fisher (1929) has shown that if we choose that period out of s available which makes the largest contribution, the fraction g taken by this period is distributed so that the probability of exceeding any value g is given by $P = s(1-g)^{s-1} - [s(s-1)/2](1-2g)^{s-1} + \ldots + (-1)^{k-1}[s!/(s-k)!k!](1-kg)^{s-1}$, in which k is the greatest integer less than 1/g. The author finds that the mean of the sampling distribution of g is $\bar{g} = (1+1/2+1/3+\ldots+1/s)/s$. Section 3 is devoted to a brief derivation of the simultaneous sampling distribution of the roots of the equation (*) $|a_{ij}-\theta(a_{ij}+a'_{ij})| = 0$, where $\|a_{ij}\|$ and $\|a'_{ij}\|$, (i,j=1,2, . . . ,p), are matrices of second order product sums based on $n_1 \geqslant p$ and $n_2 \geqslant p$ degrees of freedom, respectively, such that the elements of each matrix are distributed according to Wishart's joint distribution law, the two sets of elements being independent in the probability sense. In Section 4, several special cases of the general sampling distribution of the roots of (*) are considered, including a derivation of the distribution of the roots of the equation $|a_{ij}-\lambda\alpha_{ij}| = 0$ for the case in which $\|\alpha_{ij}\|$ is the matrix of variances and covariances in the normal population underlying the Wishart distribution. A more formal demonstration of the new results is given by Hsu (1939).

**References:** Fisher (1929), Hotelling (1933, 1936b), Fisher (1938), Hsu (1939).

**Citations:** Hsu (1939), Cochran (1941), Hsu (1941a), Roy (1942a,b), Wilks (1943), Anderson (1945), Roy (1945), Kendall (1946), Tintner (1946), Bartlett (1947a,b), Anderson (1948), Rao (1948).

* Gini, Corrado (1939). *Memorie di Metodologia Statistica*, Volume I: *Variabilità e Concentrazione*. Editore Dott. A. Giuffrè, Milano. (JFM 65, 1350-1351).

**Summary:** Boldrini (1942) writes (p. 701, compiler's translation): "This volume collects, in revised and augmented form, the numerous memoirs on the subject published by the author between 1910 and 1930, constituting the greatest modern contribution to the empirical measurement of statistical variability."

**Note:** The compiler has not seen this volume, so he is not sure just which papers of Gini are included. Apparently the papers by Gini (1910, 1912, 1914a, 1918, 1930) are included, and perhaps others.

**References:** Laplace (1812) [1818], von Andrae (1869), Jordan (1869), von Andrae (1872), Jordan (1872), Galton (1875), Helmert (1876a), Pareto (1897), Lorenz (1905), Gini (1909), Persons (1909), Gini (1910), Furlan (1911), Mortara (1911), Yule (1911), Dettori (1912), Gini (1912), King (1912), Porru (1912), Czuber (1914), Gini (1914a), Savorgnan (1915), Ricci (1916), Gini (1918), Vinci (1918), Amoroso (1925), Saibante (1926), Gini (1930) [and perhaps others].

**Citations:** Pizzetti (1940), de Vergottini (1940), Pizzetti (1941a,b,c), Boldrini (1942), Livada (1942).

* Girshick, M. A. (1939). On the sampling theory of roots of determinantal equations. *Annals of Mathematical Statistics* **10**, 203-224. (JFM 65, 572; MR 1, 32). Related abstract, *Bulletin of the American Mathematical Society* **43** (1937), 172-173.

**Summary:** The first and fourth parts of the paper deal with the joint moments of $q = r_1^2 r_2^2 \ldots r_s^2$ and $z = (1-r_1^2)(1-r_2^2)\ldots(1-r_s^2)$, where $r_1, r_2, \ldots, r_s$ are sample canonical correlation coefficients, and with the sampling theory of several quantities related to q and z. Since the theory does not depend on the order of the $r_i$ ($=1,2,\ldots,s$), these parts are not directly relevant to our subject. The second part is devoted to a study of the relation between the asymptotic distribution of the canonical correlations and that of the latent roots of the matrix of sample variances and covariances of the s variates. In the third part, the author derives asymptotic standard errors of the ordered latent roots of a matrix of sample variances and covariances and of the coefficients of Hotelling's principal components.

**References:** Hotelling (1933), Girshick (1936), Hotelling (1936b).

**Citations:** Wilks (1943), Kendall (1946), Tintner (1946), Bartlett (1947b).

* Gottschick, J. (1939). Zwillingsbefunde und Reinrassigkeitsgrad. *Archiv für Rassen- und Gesellschafts-Biologie* **33** (2), 102-110.

**Summary:** The author compares the mean difference d and the standard deviation $\sigma$ as measures of dispersion. He calculates the ratio $\sigma/d$ for the binomial distribution [expansion of $(1+1)^n$] for n = 1 to n = 10. Rounded to one decimal place, $\sigma/d = 0.5$ for n = 1, 0.7 for n = 2, 0.8 for n = 3, 4 and 0.9 for $n \geq 5$. For large samples, the relation $\sigma/d = 0.9$ compares with the relation $e/d = 0.7$ found by Lenz & von Verschuer (1928), where e is the mean deviation (from the arithmetic mean). All three measures are valid for any distribution, but the relation between any two of them is different for distributions other than the binomial.

**Reference:** Lenz & von Verschuer (1928).

**Citation:** Pizzetti (1941a).

* Gumbel, Emil J. (1939a). Les valeurs de position d'une variable aléatoire. *Comptes Rendus de l'Académie des Sciences* (Paris) **208**, 149-151.

**Summary:** The author states (page 149): "For a theoretical distribution one defines as median $x_0$ and as quartiles $x_1$, $x_2$ the values having the probabilities $\lambda = 1/2; 1/4; 3/4$. In order to know how one should choose, after ordering the observations, the corresponding median and quartiles of a [sample] distribution based on a small number of observations n, we shall define first the corresponding numbers of position m. One takes as the median the value such that it will have the same number of observations to the left as to the right, and as first (or second [third]) quartile the central observations situated to the left (or to the right) of the median. The three quartiles are thus respectively the $(n+1):4^{th}$, $(n+1):2^{th}$, and $(3n+3):4^{th}$ values from below. In an analogous manner, we assign to the value of position of the variable corresponding to any probability $\lambda$ whatever the observed $m^{th}$ value from below, such that (1) $m-1 = n\lambda-(1-\lambda)$; $n-m = n(1-\lambda)-\lambda$. The $m^{th}$ value is thus the $n\lambda+\lambda^{th}$ value from below. . . ." (Compiler's translation). The author goes on to say that the above attribution does not solve the problem completely, since the $m^{th}$ values are random variables. He proposes corrections to the values of position based on the distribution of the population from which the observations have been chosen.

**References:** Eyraud (1934b), Gumbel (1935a).

**Citations:** Gumbel (1939b,c,e, 1942a, 1943a), Kendall (1946).

* Gumbel, Émile-Jules (1939b). La durée de retour, mesure de l'indépendance. *Comptes Rendus de l'Académie des Sciences* (Paris) **208**, 1471-1473.

**Summary:** "For a variable x with probability W(x) subject to n observations at regular intervals, one defines as the mean return period $\bar{F}(x)$ the mean distance between two successive observations exceeding x, such a value not having been observed between the two observations. If the observations are made on independent events and if one takes the distance between two successive observations as the unit of time, one obtains (1) $\bar{F}(x) = 1:[1 - W(x)]$, a quantity independent of the chronological order. . . . We compare the theoretical value (1) with the corresponding value (2) $'\bar{F}(x) = n:(n - m)$ considered as the mean observed return period, $m = n\ 'W(x)$ being the value of position [Gumbel (1939a)] taken in non-decreasing order [in the observed sample]. One assumes therefore that a unit of time passes between the initial instant and the first observation in chronological order. To the first value in the non-decreasing order, $_1x$, corresponds the mean observed return period $'\bar{F}(_1x) = n:(n-1)$, to the next-to-last, $_{n-1}x$, corresponds $'\bar{F}(_{n-1}x) = n$. One may attribute to every observed value (except the last) its mean return period (2). . . . The expressions (1) and (2) do not depend on the chronological order. Their comparison is in the logical category of criteria common in statistics. If the mean periods calculated from (2) for small values of x are smaller (or larger) than the theoretical numbers, there exists a systematic difference between the probability W(x) and the frequencies 'W(x), namely an attraction (or repulsion) for small values of the variable, and a repulsion (or attraction) for large ones. In order to check the agreement, one could calculate the standard deviation of the mean return period $\sigma(\bar{F})$ or the probable error of the distribution of periods. But the probability that the mean is contained in the interval $\bar{F} \pm \sigma(\bar{F})$ depends on x. On the other hand the mean is not necessarily contained between the two quartiles. It is therefore preferable to attribute to each value of x its standard deviation $s_x$ by considering x as an $m^{th}$ value [Gumbel (1934e)]. For distributions which obey l'Hopital's rule, one obtains asymptotically (3) $\bar{F}(x) = -d \log w(x)/dx\ w(x)$. The Gaussian distribution, for example, leads to (3') $\lim_{x \to \infty} \lg \bar{F}(x) = x^2$. For the exponential distribution which governs the distances between radioactive emissions, one obtains the straight line (4) $x = \bar{x} \lg \bar{F}(x)$. In order to test the length of mean observed distances, one attributes to x the standard deviations of $m^{th}$ values of an exponential distribution [Gumbel (1937b)]. Thus one has an efficient and most simple method of deciding whether radioactive emissions are dependent". (Compiler's translation of pp. 1471-1473).

**References:** Gumbel (1934e, 1937b, 1939a).

**Citations:** Gumbel (1939c, 1940).

* Gumbel, Émile-J. (1939c). La durée de retour des plus grandes valeurs. *Comptes Rendus de l'Académie des Sciences* (Paris) **208**, 2041-2043.

**Summary:** "In what follows, we apply the criterion of the return period [Gumbel (1939b)] to largest values governed by the probability (1) $W(x) = e^{-e^{-y}}$, $y = \alpha(x - u)$, $1/\alpha = \sqrt{6}\, s/\pi$, u being the mode and s the standard deviation. The arithmetic mean [Gumbel (1935a)] is (2) $\bar{u} = u + 0.45005s$. The mean return period of a largest value greater than x (3) $T(x) = 1:[1 - W(x)]$ tends, for large values of x, to the line (3') $x = u + [\lg T(x)]:\alpha$, a formula due to Coutagne [(1937)], valid except for reduced values $y \geqslant 2.5$, that is, for about 8% of the observations. The logarithm of the mean return period of a largest value greater than x tends toward a linear function of x involving the mode and a multiple of the standard deviation. In his fundamental study of flood flows, W. E. Fuller [(1914)] proposed a linear interpolation formula (4) $\overline{U}(x):\bar{u} = 1 + 0.8 \log T(x)$, where $\overline{U}(x)$ denotes the mean of annual floods greater than x. Saville [(1936)] set, in a more general manner, (4') $\overline{U}(x):\bar{u} = a + b \log T(x)$. One easily finds a theoretical basis for these purely empirical formulas, the meaning of the constants and the domain of validity, if one assumes that the floods x are governed by the probability (1). Then one obtains for $\overline{U}(x)$, asymptotically, $-\int_x^x z\,d[1 - W(z)]:[1 - W(x)] = x + 1/a$. By introducing the arithmetic mean $\bar{u}$, the standard deviation s, the coefficient of variation $\nu = s:\bar{u}$ and the asymptotic expression (3') one finds (5) $\overline{U}(x):\bar{u} = 1 + \nu/3 + 1.79\nu \log T(x)$. The

variation of annual floods enters only as a constant. The value of Fuller, supposed to be universal, is $\nu = 0.4456$. But these formulas are valid only asymptotically. Also T(x) does not correspond to $\overline{U}(x)$ but to x. That is why we prefer the calculation of the return period (3) of floods greater than x which does not require the calculation of the means $\overline{U}(x)$. One obtains, for example, for the Merrimack River, Lawrence, Mass. 1846-1933 [Hazen (1930), supplemented by later date], the mean annual flood flow $\bar{u}$ and the two quartiles $u_1$, $u_2$

$$\bar{u} = 45478.7,\ u_1 = 32675.8,\ u_2 = 54380.7$$

cubic feet per second, from which results, according to the method of values of position [Gumbel (1939a)] $1{:}\alpha = 13803$; $u = 37456$, $\nu = 0.38972$, a value near the assumption of Fuller. The figure [p. 2042] shows the observed and theoretical length of the mean return periods of a flood greater than x with control curves which one draws from the theory of $m^{th}$ values [Gumbel (1934e)]. This agreement gives a new justification [Gumbel (1936a)] for the application to floods of the theory of largest values." (Compiler's translation of pp. 2041-2043).

**References:** Fuller (1914), Hazen (1930), Gumbel (1934e, 1935a, 1936a), Saville (1936), Coutagne (1937), Gumbel (1939a,b).

**Citations:** Gumbel (1940, 1943c).

* Gumbel, E. J. (1939d). La probabilité des hypothèses. *Comptes Rendus de l'Académie des Sciences* (Paris) **209**, 645-647. (JFM 65, 597; MR 1, 152).

**Summary:** Let x be a random variable with pdf f(x) and cdf $y = F(x) = \int_{-\infty}^{x} f(x)dx$. Let $x_1, x_2, \ldots, x_n$ be an ordered sample of n independent observations on x. Suppose we wish to test the hypothesis h that x follows a rectangular distribution between 0 and 1, i.e. $f(x) = 1$, $0 \leq x \leq 1$. Let $y_m$ be the value of y corresponding to $x_m$, the $m^{th}$ order statistic of the sample. Then the expected value of $y_m$ is $\bar{y}_m = m/(n+1)$. The author proposes a test of the hypothesis h based on a comparison of the observed values $y_m$ and their expectations $\bar{y}_m$.

**References:** Gumbel (1936b), Neyman (1937a).

* Gumbel, E. J. (1939e). Les plus grandes précipitations. *La Météorologie* (3) **1939**, 17-28.

**Summary:** The author begins with the following introduction (page 17): "In what follows, we shall treat precipitations as random variables obeying a distribution law. We introduce the abstraction of secular trends. A very usual assumption concerning the convergence of this distribution toward large values of the variable permits the application of the theory of the largest value. The purpose of the following lines is to show schematically that the distribution of the largest precipitations can be explained by this theory. Since we have expounded the method in several previous works, we shall content ourselves here with recalling those of the results of which we make use." (Compiler's translation). On page 28 the author states his conclusions: "In applying the theory of the largest value to monthly precipitations, one assumes that the number 12 is already very large, which is, evidently, a slight exaggeration. But the purpose of this article being only to show it in principle, we shall content ourselves with monthly observations. Coutagne [(1939)] has followed a better route by studying maxima of daily precipitations which assumes that $N = 365$ is very large. It follows from four criteria that observations bearing upon maximum monthly precipitations can be interpreted in a quite satisfactory manner by the theory of the largest value." (Compiler's translation).

**References:** Gumbel (1934e, 1935a, 1936a), von Mises (1936), Gumbel (1937-38), Coutagne (1939), Gumbel (1939a).

**Citation:** Gumbel (1943c).

* Gumbel, E. J. (1939f). Statistische Theorie der grössten Werte. *Zeitschrift für Schweizerische Statistik und Volkswirtschaft* **75** (2), 250-271. (JFM 65, 1353).

**Summary:** This expository paper discusses, from a popular viewpoint, statistical distributions and the theory of the largest value, with applications to radioactivity (longest intervals between emissions), age limits (longest human lives), and floods (largest discharges of rivers). Numerous references to earlier work by the author and others are given.

**References:** Fuller (1914), von Bortkiewicz (1922a), von Mises (1923), Fréchet (1927), Fisher & Tippett (1928), Gumbel (1932, 1933a,b,c,d,e,f,g, 1934a,b,c,d,f,g, 1935a,b,c,d, 1935-36, 1936a,b,c, 1936-37), von Mises (1936), Gumbel (1937a,b,d,e, 1937-38), Brenet & Armand (1938), Gumbel (1938a,b,c).

**Citation:** Mosteller (1946).

* l'Hermite, R. (1939). Étude statistique des bétons de chantier. *Annales de l'Institut Technique du Bâtiment et des Travaux Publics* (Paris) **4**, 66-75.

**Summary:** The author compares the strength under compression of concrete cubes, prisms, and cylinders of various dimensions. He finds (p. 67) that the arithmetic mean of the results of trials on cubes decreases when the edge of the cube increases. In a number of cases he determines the median as well as the arithmetic mean of the results of a number of trials on specimens of the same size, shape, composition, and age. On p. 70 he states: "The agreement of the arithmetic mean with the median shows the symmetry of the distribution of the population and confirms the application of the law of dispersion of Gauss." (Compiler's translation).

**Comments:** The author's results concerning the size effect are in agreement with what one would predict from the "weakest link" theory. With regard to the comparison between the median and the arithmetic mean, the former is actually slightly less than the latter in almost every case, which indicates that the distribution is not normal (Gaussian) or even symmetric, but slightly skewed to the right. However, the skewness is so small that use of the normal distribution probably does not lead to any serious errors, except perhaps in estimating extreme fractiles of the distribution, which are quite sensitive even to mild skewness.

* Hsu, P. L. (1939). On the distribution of roots of certain determinantal equations. *Annals of Eugenics* **9**, 250-258. (JFM 65, 572-573; MR 1, 248).

**Summary:** The author begins (page 250): "The extension (Fisher, 1938) of Fisher's discriminant analysis to more than two multivariate samples has directed attention to the problem of the exact distribution of the roots of a certain type of determinantal equation, which are required for various significance tests. . . . The purpose of the present paper is to give a complete demonstration of the analytic solution, including the case in which the number of variates, p, exceeds one of the sample numbers $n_1$." He proceeds to state and prove three theorems. Theorem 1 gives the simultaneous distribution of the roots $\theta_1, \theta_2, \ldots$, in order of descending magnitude, of the determinantal equation $|a_{ij}-\theta(a_{ij}+b_{ij})| = 0$, where $a_{ij} = \Sigma_{r=1}^{n_1} y_{ir}y_{jr}$, $b_{ij} = \Sigma_{t=1}^{n_2} z_{it}z_{jt}$, and $y_{ir}$, $z_{it}$ $(i=1,2,\ldots,p;\ r=1,2,\ldots,n_1;\ t=1,2,\ldots,n_2)$ are a set of $p(n_1+n_2)$ random variables following the distribution law const. $\exp[-\Sigma_{i,j=1}^{p} \alpha_{ij}(a_{ij}+b_{ij})/2]|H|dydz$. If the $p(p+1)/2$ covariances $s_{ij}$ $(i \leq j = 1,2,\ldots,p)$ are such that $||s_{ij}||$ is always non-singular and if their distribution depends only on the latent roots $\lambda_i$ of $||s_{ij}||$, then Theorem 2 gives the joint distribution of the latent roots $\lambda_1, \lambda_2, \ldots, \lambda_p$ arranged in order of descending magnitude. Theorem 3 states, in effect, that Hotelling's canonical correlations between two sets of random variables are distributed as the square roots of the $\theta$'s, and gives the form of the distribution function of the $\theta$'s, arranged in order of descending magnitude, under the condition that the first set of variables be normally distributed.

**References:** Hotelling (1936b), Fisher (1938, 1939).

**Citations:** Fisher (1939), Hsu (1941a), Roy (1942a,b), Wilks (1943), Anderson (1945), Roy (1945), Kendall (1946), Tintner (1946), Bartlett (1947a,b), Anderson (1948), Nanda (1948a,b), Rao (1948).

* Hulme, H. R.; Symms, L. S. T. (1939). The law of error and the combination of observations. *Monthly Notices of the Royal Astronomical Society* **99**, 642-649. (JFM 65, 606).

**Summary:** The authors summarize three objections that have been raised against the normal or Gaussian law of error; (1) Experimental results for the distribution of errors often show that large errors are far more numerous than the normal law indicates, a point which has been treated at length by Newcomb (1886); (2) the normal law is also defective from the theoretical point of view, as pointed out by Jeffreys (1938), Newcomb (1886), and Eddington (1933); (3) the normal law does not offer any really satisfactory criterion for the rejection of large observations. The authors attack the problem of finding the actual error law which best fits the residuals and then obtaining the system of weighting corresponding to this law, since the arithmetic mean is the best value only if the errors follow the normal law. They develop a method, similar to but simpler than those of Newcomb (1886) and Jeffreys (1932, 1939), which allows a system of continuous weighting to be found for a series of observations. They apply this method to an extensive set of latitude observations at Greenwich, and conclude that it leads to a small but definite improvement of the results.

**References:** Newcomb (1886), Whittaker & Robinson (1924), Eddington (1933), Jeffreys (1938).

* Jeffreys, Harold (1939). *Theory of Probability*. Oxford University Press, London. (JFM 65, 546). Second edition, 1948; third edition, 1961.

**Summary:** The author gives the following discussion of the use of the median (pp. 211-213): "If the law of error is unknown and the observations are too few to determine it, we can use the median observation as a statistic for the median of the law. We can then proceed as follows. Let $\alpha$ be the median of the law; we want to find a range such that the probability, given the observations, that $\alpha$ lies within it has a definite value. Let a be a possible value of $\alpha$ such that $\ell$ observations exceed a and $n-\ell$ fall short of it. Then $P(\ell|\alpha,n,H) = {}^nc_\ell(1/2)^n = (2/\pi n)^{1/2}\cdot \exp\{-2(\ell-n/2)^2/n\}$ (1) nearly; and if the a priori probability of $\alpha$ is taken uniform, $P(d\alpha|\ell,n,H)\propto(2/\pi n)^{1/2}\exp\{-2(\ell-n/2)^2/n\}$. (2) Thus the posterior probability density of $\alpha$ is a maximum at the median, and if we take $\ell = n/2\pm\sqrt{n}/2$ as limits corresponding to the standard error, the corresponding values of $\alpha$ will give a valid uncertainty, whatever the law and the scale parameter. . . . It is often advocated that the uncertainty of the true value, as estimated from the mean, should be got from the average residual without regard to sign instead of the mean square residual, on the ground that the former is less affected by a few abnormally large residuals than the latter is. But if the mean of the observations is taken as the estimate of the mean of the law, its uncertainty is correctly estimated from $\mu_2$ if the latter exists, and if it does not exist the uncertainty will not be proportional to $n^{-1/2}$. For all laws such that $\mu_2$ exists the mean square residual gives an unbiased estimate of $\mu_2$. . . . If the mean is taken as the estimate, there is no alternative to the mean square residual to provide an estimate of uncertainty when the law is in doubt. On the other hand, it is only for the normal law that the mean is actually the best estimate, and for other laws we are entitled to consider other estimates that may be more efficient. One interesting case is the law $P(dx|m,a,H) = (1/2)\exp(-|x-m|/a)dx/a$. (3) Here we find easily that the likelihood is a maximum if m is taken equal to the median observation, and if a is the average residual without regard to sign. This law is therefore known as the median law. Given any of the three properties the other two can be deduced provided the chance is a function $x-m$ only. It is only subject to this law that the average residual leads to the best estimate of uncertainty, and then the best estimate of the location parameter is provided by the median observation and not by the mean. The interest of the law is reduced somewhat by the fact that there do not appear to be any cases where it is true. . . . There is . . . much to be said for the use of the median as an estimate when the form of the law is unknown; it loses some accuracy

in comparison with the best methods, but increase of the uncertainty is often unimportant, and varies little with the form of the law, and the uncertainty actually obtained is found easily by the rule (2)." The author (pp. 212, 318-319, 386) criticizes Peirce's and Chauvenet's criteria for rejection of observations, and offers (pp. 316-317) a modified form of the alternative which he proposed earlier [Jeffreys (1932)], with a table of weights for its implementation.

**Note:** References to specific page numbers are to those in the third edition.

**References:** Chauvenet (1863), Newcomb (1886), Keynes (1921), Jeffreys (1938).

**Citations:** Kendall (1943), Cramér (1946), Kendall (1946).

* Lévy, Paul (1939). Sur la division d'un segment par des points choisis au hasard. *Comptes Rendus de l'Académie des Sciences* (Paris) **208**, 147-149. (JFM 65, 553).

**Summary:** Let $X_0, X_1, \ldots, X_n$ denote the lengths of $n+1$ intervals which are separated by $n$ points $x_1, x_2, \ldots, x_n$ chosen at random on the segment (0,1), the probability distribution being uniform. The author proves certain theorems concerning $Y_n$ and $Z_n$, where $Y_n$ is the smallest and $Z_n$ is the largest of the $X_\nu$ ($\nu = 0,1, \ldots, n$).

**Reference:** Gumbel (1937b).

**Citations:** Garwood (1940), Kendall (1946).

* Newman, D. (1939). The distribution of range in samples from a normal population, expressed in terms of an independent estimate of standard deviation. *Biometrika* **31**, 20-30. (JFM 65, 573; MR 1, 153).

**Summary:** The author begins with a review of the work of Tippett (1925) on the mean range of normal samples, of E. S. Pearson (1932) on the standard deviation and percentage points of the range, and of Pearson & Haines (1935) on the use of the mean range of a number of small samples to estimate the population standard deviation $\sigma$. He recalls the suggestion of "Student" (W. S. Gosset) [see Pearson (1939)] concerning the use of the ratio $q = w/s$ (the studentized range), where $w$ is the range of a sample of $n$ observations from a normal population with standard deviation $\sigma$ and $s^2$ is an *independent* and unbiased estimate of $\sigma^2$ based on $f$ degrees of freedom, obtained from a sum of squares in the usual manner. Suppose that, as a result of an experiment, $n$ treatment means $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n$ are available along with an independent estimate, $s^2$, of their sampling variance. Then Student's idea was that one could test whether any treatment differences exist by comparing the difference between the highest and lowest treatment means, say $w = \bar{x}_n - \bar{x}_1$, with $s$. If this difference, with due regard to the values of $n$ and $f$, is found to be clearly significant, then one can set aside the more divergent of the extremes, say $\bar{x}_1$, and compare $\bar{x}_n - \bar{x}_2$ with $s$, using $n-1$ and $f$, and so on. The procedure suggested by Student (1927), based on $w/\sigma$ instead of $w/s$, is a limiting case ($f \to \infty$) of this procedure. In order to facilitate the use of tests based on the studentized range, the author tabulates one-decimal-place (2DP for $n = 2$ and/or $f = \infty$) 5% and 1% points of $q = w/s$ for $n = 2(1)12, 20$ and $f = 5(1)20,24,30,40,60,\infty$. Exact results are available for $n = 2$ and $n = 3$, but those for $n > 3$ are based on numerical integration. He works out three numerical examples to illustrate use of the tables.

**References:** Tippett (1925), E. S. Pearson (1926), Student (1927), K. Pearson (1931), E. S. Pearson (1932), McKay and Pearson (1933), Fisher (1935) [1937], Pearson & Haines (1935), Snedecor (1937).

**Citations:** Snedecor (1937) [1946], Pearson & Hartley (1943), Hartley (1944), Baker (1946), Kendall (1946), Egudin (1947), Tukey (1948c, 1949a).

* Pearson, E. S. (1939). "Student" as statistician. *Biometrika* **30**, 210-250. (JFM 65, 18—listing only).

**Summary:** The author gives a summary and evaluation of the statistical work of "Student" (the late W. S. Gosset). Included are extracts from several previously unpublished letters of Student. Of particular interest is Letter III, concerning which the author writes (page 245): "The suggestion in Letter III of 1932 ultimately led to the production of tables of percentage limits of the ratio of (a) range in a sample of n observations to (b) an independent estimate of standard deviation, which are to be published shortly in *Biometrika* [Newman (1939)]. From the beginning of his analysis of the results of chess-board experiments, Gosset had wondered how best to judge what differences among variety means were significant. While the ratio of (a) the difference between any two means selected at random to (b) the estimate of the standard error could be referred to 'Student's' distribution or, if desired, the significance of the set as a whole could be judged by Fisher's z-test, it was not possible to treat selected differences in either of these ways. In the article in *Baillière's Encyclopedia* . . . [Student (1931)] he refers to a method suggested by Fisher of taking the differences between individual variety yields and the mean yield. He felt however that a knowledge of probability levels of 'studentized' range would in addition be very useful; on this could be based a rough test of the kind he had suggested in his paper on 'Errors of routine analysis' . . . [Student (1927)]." The letter itself, dated January 29, 1932, is quoted on pp. 245-246. The relevant paragraph reads: "I have been meaning to write to you for some time re the proposals for the use of range and sub-range which I made in my last letter to you. Of course there is a serious crab which I had at one time recognised and then forgotten in that the thing would have to be 'Studentised': the only measure of the S.D. is provided by a limited number of degrees of freedom. Whether one could get an approximate correction for this with moderately small numbers by reducing still further the degrees of freedom or whether it would be necessary as Fisher suggested when I mentioned the matter to him (he was here [Dublin] lecturing) to dive into the depths of hyperspace to produce the jewel I am not clear, but obviously something would have to be done about it."

**References:** Helmert (1876a), Student (1927), Sophister (1928), Student (1931).

**Citation:** Pearson & Hartley (1943).

* Pitman, E. J. G. (1939). The estimation of the location and scale parameters of a continuous population of any given form. *Biometrika* **30**, 391-421; note by E. S. Pearson, 471-474. (JFM 65, 582-583).

**Summary:** "The main problem considered is the location and scaling of the distribution of a continuous chance variable X. We suppose that the probability function of x is $(1/c)f\{(x-a)/c\}$, $c > 0$, where the function $f(x)$ is known but one or both of the parameters a, c, which determine respectively the location and scale of the distribution, is unknown. We have a sample of n independently observed values of X, and from these we have to estimate the unknown parameter or parameters. Any function of the sample values whose value may be used as an estimate of the unknown parameter is called an estimator of that parameter. The paper shows how to determine an estimator with any required property, such as minimum mean absolute error, or minimum mean square error. In particular, the closest estimator is determined; this is an estimator whose median value is the true value of the parameter and which is likely to be closer to the true value than any other estimator. It is shown that in certain cases a best estimator exists. Fiducial limits for the unknown parameter are determined, and what is called the fiducial distribution of the parameter is defined. It is shown that problems of estimation can be dealt with very simply, and completely, by means of fiducial distributions. For a population of any given form, the fiducial distribution of a, when both a and c are unknown, provides us with a test which corresponds to 'Student's' test for significance of the mean of a sample from a normal population. The estimation of the difference between the location parameters of two populations of similar forms is discussed." (Author's summary, pp. 420-421). The largest and smallest sample values figure prominently in the theory. The note by Pearson is devoted mainly to a refutation of the author's statement (p. 392, footnote) that there is no difference between the theory of confidence intervals and the theory of fiducial probability, and is not directly relevant to our subject.

**References:** Neyman & Pearson (1936), Pitman (1936), Neyman (1937b), Pitman (1937).

**Citations:** Cramér (1946), Kendall (1946).

* Roy, S. N. (1939). P-statistics or some generalizations in analysis of variance appropriate to multivariate problems. *Sankhyā* **4**, 381-396.

**Summary:** In the introduction (pp. 381-382), the author writes: "In an earlier paper worked out by the author jointly with Mr. R. C. Bose the problem of discrimination between and classification of different multivariate populations (with the same sets of variances and covariances but with different sets of means) was . . . tackled by defining the Studentized $D^2$-statistic and obtaining its sampling distribution. . . . It was noted that if following Fisher one started from a linear compound, with arbitrary coefficients, of the variates and so chose the compounding coefficients as to make a maximum the ratio between the square of the difference of the sample means (for the compound character) and the within variance of the samples for the same character, one could easily see after maximisation that this ratio is proportional to the studentised $D^2$-statistic referred to in that paper. In the present paper starting from a similar linear compound of the variates and so choosing the compounding coefficients as to make a maximum the ratio of the variance of the first sample (for the compound character) and the variance of the second sample for the same character one finds that the quest for the maximised value of the ratio spoken of above leads one in the case of p-variate normal populations to as many as p studentised statistics [roots of a certain determinantal equation] instead of to merely one studentised statistic as in the previous case. Of these statistics it is found that one is a true maximum, another is a true minimum and the rest are what are known in analysis as stationary values. . . . The joint sampling distribution of the statistics is worked out for the case where the two populations in question are identical in the sets of variances and covariances. The statistic corresponding to the maximum value of the ratio is here called the maximum statistic and that for the minimum value the minimum statistic. . . . The sampling distribution of either of these can be obtained by considering the joint distribution of the p-statistics and integrating out for all the others between suitable limits. Work has been carried up to this point in the present paper." Let the p-statistics be $K_1 \geq K_2 \geq \ldots \geq K_p$ and let $K_r$ be the statistic which (or whose reciprocal) is closest to unity. The author proposes tests based on $K_r$ and max $(K_1, 1/K_p)$.

**Citations:** Roy (1940a), Hsu (1941a), Roy (1942a,b, 1945), Kendall (1946), Roy (1946a,b), Bartlett (1947a,b), Rao (1948).

* Ruark, Arthur (1939). The time distribution of so-called random events. *Physical Review* (2) **56**, 1165-1167. (JFM 65, 1344).

**Summary:** The author writes (pp. 1165-1167): "It is shown that in a sequence of random events, the expected distribution of interval-sizes in a run of given duration differs from the Bateman [Poisson] distribution when the counting rate is different from the average one. The formula for 'local' size-distribution to be expected for such a run is derived by the use of Bayes' theorem. It is pointed out that this theorem is useful in discussing the properties of any statistical system which is known to be in a condition differing from the idealized equilibrium state. . . . A distribution of events is said to be random in time if the chance that one of them occurs in time dt is a constant, f dt. Bateman's formula gives the chance $W_n$ that n events will occur in time t, namely $W_n = (ft)^n e^{-ft}/n!$ (1). Then the chance that an interval between two events will exceed t is $W_0 = e^{-ft}$ (2). . . . Now the problem we want to solve is this: The average counting rate is f and a certain run has yielded n counts in time D. n is not equal to the expected value fD. What is the distribution function G(t) replacing (2) during the interval D? . . . To summarize: if we deal with a limited portion of a random distribution and find that the local counting rate is higher or lower than the average, the length-distribution law of the intervals differs from Eq. (2), and the expected local form of this distribution is given by Eq. (7) $[G \cong \{1/(1-e^{-fD})\}(n/fD)(1-T/D)^{n-1}; (n=2,3,\ldots)]$, which is fundamental in the sense that it determines the expected local values of all properties in the sequence of events."

**Reference:** Ruark & Devol (1936).

**Citation:** Feller (1940).

* Smirnov, N. [Smirnoff, N.] (1939a). Sur les écarts de la courbe de distribution empirique. (Russian. French summary). *Matematicheskii Sbornik* **48** (N. S. **6**), 3-26. (JFM 65, 561-562; MR 1, 246).

**Summary:** "Let $F(x)$ be the cumulative distribution function of a random variable x. We denote by $S_n(x)$ the curve formed from n steps, which represent the results of n independent trials. We assume that the function $F(x)$ is continuous and increasing for all values of x. One knows [Glivenko (1933)] that the probability of uniform convergence of the series $S_1(x), S_2(x), \ldots, S_n(x), \ldots$ to $F(x)$ is equal to one. In this work we shall study in a more detailed manner the character of the approximation which the functions $S_n(x)$ give for large values of n. Let $v_n$ be the number of cases where the deviation $S_n(x)-F(x)$ changes sign by passing from negative to positive values and let $\Phi_n(t) = P(v_n \leq t\sqrt{n})$ $(t \geq 0)$. One can prove the following theorem: Theorem I. Whatever be the function $F(x)$ the series of functions $\Phi_n(t)$ converges for $n \to \infty$ to the limiting distribution $\Phi(t) = 1-e^{-t^2/2}$ $(t>0)$ and $\Phi(t) = 0$ $(t \leq 0)$. Let $\lambda > 0$ and $F(x,\lambda) = F(x) + \lambda/\sqrt{n}$; denote by $v_n(x)$ the number of points of intersection of $S_n(x)$ and $F(x,\lambda)$. $v_n(x)$ represents the number of cases where $S_n(x)-F(x)$ changes sign by passing from $-$ to $+$. By setting $\Phi_n(t,\lambda) = P\{v_n(\lambda) \leq t\sqrt{n}\}$ $(t>0)$, we shall have Theorem II. Whatever be the function $F(x)$, the series of functions $\Phi_n(t,\lambda)$ converges for $n \to \infty$ to the limiting law $\Phi(t,\lambda)$ defined by the equations $\Phi(t,\lambda) = 0$ for $t<0$, $\Phi(0,\lambda) = 1-e^{-2\lambda^2}$, $\Phi(t,\lambda) = 1-e^{-(t+2\lambda)^2/2}$ for $t>0$. In consequence of Theorem II we shall have $P\{\sup_{-\infty<x<\infty} S_n(x)-F(x) < \lambda/\sqrt{n}\} \underset{n}{\rightrightarrows} 1-e^{-2\lambda^2}$. Let $V_n(\lambda)$ be the number of points of intersection of the curves $S_n(x)$ and $y = F(x) \pm \lambda/\sqrt{n}$, and $\tilde{\Phi}(t,\lambda) = P\{V_n(\lambda) \leq t\sqrt{n}\}$. We shall establish the following proposition: Theorem IIIa. For $n \to \infty$ $\tilde{\Phi}(t,\lambda)$ converges to the limit $\tilde{\Phi}(t,\lambda) = 1-2\sum_{m=0}^{\infty}[(-1)^m/m!](d^m/dt^m)e^{-[t+2\lambda(m+1)]^2/2}t^m$. Theorem IIIa includes as a special case Theorem III. For $n \to \infty$ one has $P\{\sup|S_n(x) - F(x)| \leq \lambda/\sqrt{n}\} \to \tilde{\Phi}(0,\lambda) = \Phi(\lambda)$ where $\Phi(\lambda) = 1-2\sum_{m=1}^{\infty}(-1)^m e^{-2m^2\lambda^2}$. This theorem was first established by A. Kolmogoroff [(1933)]." (Compiler's translation of author's French summary, pp. 25-26).

**Note:** Wald & Wolfowitz (1941) have pointed out that the author's formula for $\Phi(\lambda)$ contains a typographical error; the exponent of $(-1)$ should be $m-1$ instead of m, making the formula read $\Phi(\lambda) = 1-2\sum_{m=1}^{\infty}(-1)^{m-1}e^{-2m^2\lambda^2}$.

**References:** Glivenko (1933), Kolmogoroff (1933).

**Citations:** Smirnov (1939b), Kolmogoroff (1941), Wald & Wolfowitz (1941), Wilks (1943), Smirnov (1944), Feller (1948), Wilks (1948), Chung (1949), Doob (1949), Maniya (1949), Noether (1949).

* Smirnov, N. (1939b). On the estimation of the discrepancy between empirical curves of distribution for two independent samples. *Bulletin Mathématique de l'Université de Moscou* (Série internationale) **2** (Fascicule 2), 16 pp. (JFM 65, 1356-1357; MR 1, 345).

**Summary:** This paper deals mainly with the two-sample Kolmogorov-Smirnov statistic $D(n_1,n_2) = \sup_{-\infty<x<\infty}|S_{n_1}(x)-S_{n_2}(x)|$, where $S_{n_1}(x)$ and $S_{n_2}(x)$ are the cumulative distribution functions of two independent samples, of sizes $n_1$ and $n_2$, from the same continuous population. Among other results, he proves that if the ratio $n_2/n_1 = \tau$ remains constant as $n_1 \to \infty$ and $n_2 \to \infty$ and if $n = n_2/(1+\tau)$, then the probability that $D(n_1,n_2) \leq \lambda/\sqrt{n}$ $(\lambda>0)$ tends to $\Phi(\lambda) = \sum_{-\infty}^{\infty}(-1)^k e^{-2k^2\lambda^2}$, whatever be the continuous distribution law of the underlying population. Thus the limiting distribution of the two-sample Kolmogorov-Smirnov statistic is of the same form as that of the one-sample statistic $\sqrt{n}\sup|S_n(x) - F(x)|$, where $S_n(x)$ and $F(x)$ are the cumulative distribution functions of a random sample of size n and of the continuous population from which the sample was drawn, respectively, which was found by Kolmogoroff (1933). The author tabulates $\Phi(\lambda)$ to six decimal places for $\lambda = 0.28(0.01)2.49$, to 7DP for $\lambda = 2.50(0.05)2.80$ and to 8DP for $\lambda = 2.85(0.05)3.00$.

**Comments:** The two-sample Kolmogorov-Smirnov statistic, unlike its one-sample counterpart, is not directly relevant to our subject, since it depends only on the ranks of the observations from the two samples in the combined sample, and not on the values of any of the order statistics of either sample or of the combined sample. However, because both statistics have the same limiting distribution, this paper and others dealing primarily with the two-sample statistic are included in this bibliography.

**References:** Kolmogoroff (1933), Smirnov (1939a,c).

**Citations:** Kolmogoroff (1941), Scheffé (1943), Smirnov (1944), Feller (1948), Smirnov (1948), Wilks (1948), Doob (1949), Wolfowitz (1949).

* Smirnov, N. V. (1939c). Sur un théorème limite dans un schème d'épreuves indépendantes. (Russian. French summary). *Izvestiia Akademiia Nauk SSSR* (Seriia Matematicheskaiia) **3** (3), 319-328. (JFM 65, 561).

**Summary:** "For each value of x ($0<x\leqslant n$) let $T_n(x)$ be the number of times that an event A occurs in a system of independent trials of which the indices do not exceed x, $P(A) = pn$, $\overline{T}_n(x) = E\,T_n(x)$ so that $\overline{T}_n(x) = kp$ ($k\leqslant x<k+1$). Let us denote by $V_n(x)$ the number of times that the curve $T_n(x)$ lies outside the band $U_\lambda$ bounded by the curves $y_\lambda = \overline{T}_n(x) \pm\lambda\sqrt{np(1-p)}$ in the interval (0,n). The following limit theorem holds: Theorem. For $n\to\infty$ one has $P\{V_n(\lambda) \leqslant t\sqrt{np(1-p)}\} \to F(t,\lambda)$, where $F(t,\lambda) = 1-2\sqrt{2/\pi}\ \Sigma_{m=0}^{\infty}$ $[(-1)^m/m!](d^m/dt^m)t^m\int_t^\infty e^{-[u+(2m+1)\lambda]^2/2}du (t\geqslant 0)$. As a consequence of this theorem one obtains $P[\sup_{0<x<\infty}$ $|T_n(x)-\overline{T}_n(x)| \leqslant \lambda\sqrt{np(1-p)}\} = P\{V_n(\lambda) = 0\} \underset{n\to\infty}{\to} F(0,\lambda)$." (Compiler's translation of author's French summary, p. 328).

**Citation:** Smirnov (1939b).

* Stevens, W. L. (1939). Solution to a geometrical problem in probability. *Annals of Eugenics* **9**, 315-320. (JFM 65, 554; MR 1, 245).

**Summary:** The author solves the following problem, as stated on page 315: "On the circumference of a circle of unit length, n arcs each of length x are marked off at random. What is the probability that every point of the circle is included in at least one of the arcs?" Let k be the greatest integer less than 1/x. Then the frequency distribution of the number of gaps is given by $f(0) = 1-n(1-x)^{n-1} + [n(n-1)/2]$ $(1-2x)^{n-1}-\ldots \pm [n(n-1)\ldots(n-k+1)/k!](1-kx)^{n-1}$, $f(1) = n\{(1-x)^{n-1} -(n-1)(1-2x)^{n-1} + \ldots \pm [(n-1)$ $(n-2)\ldots(n-k+1)/(k-1)!](1-kx)^{n-1}\}$, $f(2) = [n(n-1)/2]\{(1-2x)^{n-1} - (n-2)(1-3x)^{n-1} + \ldots \pm[(n-2)$ $(n-3)\ldots(n-k+1)/(k-2)!](1-kx)^{n-1}\}$, $\ldots$ $f(i) =[n!/i!(n-i)!]\{(1-ix)^{n-1}-(n-i)[1-(i+1)x]^{n-1} + \ldots \pm$ $[(n-i)\ldots(n-k+1)/(k-i)!](1-kx)^{n-1}\}$, $\ldots$ $f(k) = [n(n-1)\ldots(n-k+1)/k!](1-kx)^{n-1}$.

**Citations:** Fisher (1940), Garwood (1940), Finney (1941), Greenwood (1946), Kendall (1946), Domb (1947).

* de Vergottini, Mario (1939). Über einige Konzentrationsindizes. *Deutsches Statistisches Zentralblatt* **31** (2), 47-52. (JFM 65, 1356).

**Summary:** "The concentration indices B and R are means of the relative deviations of special arithmetic means from the general arithmetic mean; between them there exists a very simple relation, and indeed that between two arithmetic means (the unweighted and the weighted). [$B = \Sigma_1^{n-1}(M-M_s)/(n-1)M$, where M represents the general mean and $M_s$ the special arithmetic mean formed from the first s members, when these members are ordered according to increasing magnitude; $R = \Sigma_1^{n-1}(p_s-q_s)/\Sigma_1^{n-1}p_s$, where $p_s = N_s/N$, i.e. the ratio of the number of members with smaller intensity than s ($N_s$) to the total

number of members (N), and $q_s = A_s/A$, i.e. the corresponding ratio of the intensity ($A_s$) to the general intensity of the phenomenon (A).] The two above-named concentration indices can thus be characterized as indices of relative variability of the special arithmetic mean with respect to the general arithmetic mean, and as such they show a great resemblance to indices of relative variability or dissimilarity (mean deviation from the arithmetic mean, etc.). Nevertheless, their areas of application are different. Recently it was remarked [by Mortara (1934a)]: 'Dissimilarity is a property of the observed distribution of successive values, concentration a property of the integral curve itself.' " (Compiler's translation of author's summary, p. 52).

**References:** Gini (1914a), Mortara (1934a).

**Citation:** de Vergottini (1940).

* Wald, Abraham (1939). Contributions to the theory of statistical estimation and testing hypotheses. *Annals of Mathematical Statistics* **10**, 299-326. (JFM 65, 585; MR 1, 152).

**Summary:** Many aspects of the theory of the closely related problems of statistical estimation and testing hypotheses are discussed. The best estimate of a parameter is defined as the one which minimizes the risk (expected loss), and depends on the weight assigned to the various errors. Two examples are given on pages 322-323. The second of these involves the use of order statistics. Consider the family of variates $X(\theta)$ with probability density $f(x,\theta) = 1$ if $\theta - 1/2 \leq x \leq \theta + 1/2$, $f(x,\theta) = 0$ otherwise. Let $x_1$ denote the smallest and $x_n$ the greatest value in a sample of size n. The best estimate $\bar{\theta}$ of the parameter $\theta$ depends on the weight function. The author shows that if the weight function is $W(\theta,\bar{\theta}) = \theta - \bar{\theta}$ if $\bar{\theta} \leq \theta$ and $W(\theta,\bar{\theta}) = 2(\bar{\theta} - \theta)$ if $\bar{\theta} > \theta$, the best estimate of $\theta$ is given by the expression $\bar{\theta} = (x_1 + 2x_n - 1/2)/3$.

**References:** Neyman & Pearson (1936), Neyman (1937b).

**Citations:** Wilks (1943), Kendall (1946), Duncan (1947).

* Wald, A.; Wolfowitz, J. (1939). Confidence limits for continuous distribution functions. *Annals of Mathematical Statistics* **10**, 105-118. (JFM 65, 585). Abstract, *Bulletin of the American Mathematical Society* **45**, 232-233.

**Summary:** In their introduction (page 105) the authors write: "Let E be the point in an n-dimensional Euclidean space which corresponds to the observed values $x_1, \ldots, x_n$ of the n stochastic variables $X_1, \ldots, X_n$ of which we know only that they constitute an element of the family G, i.e., that they are independently distributed with the same continuous distribution function. Let us denote their distribution function by f(x); the probability that $X_i < x$ is f(x), $i = 1, \ldots, n$. Let $\alpha$ be a number such that $0 < \alpha < 1$. To each pair $E,\alpha$ we shall construct two functions $\bar{\ell}_{E,\alpha}(x)$ and $\underline{\ell}_{E,\alpha}(x)$, with the following property: The probability is $\alpha$ that, if we were to draw a sample from the system $X_1, \ldots, X_n$, we would get a system of observations $E = (x_1, \ldots, x_n)$ such that f(x) lies entirely between $\bar{\ell}_{E,\alpha}(x)$ and $\underline{\ell}_{E,\alpha}(x)$ (i.e., that $\underline{\ell}_{E,\alpha}(x) \leq f(x) \leq \bar{\ell}_{E,\alpha}(x)$ for all x). We shall call $\bar{\ell}_{E,\alpha}(x)$ and $\underline{\ell}_{E,\alpha}(x)$ the upper and lower confidence limits, respectively, corresponding to the confidence coefficient $\alpha$." Given a sample of size $n = 6$, the authors construct the confidence limits, which are, like the sample cumulative distribution function, step functions with the steps occurring at values of the abscissa equal to the order statistics of the sample.

**Citations:** Kolmogoroff (1941), Wald & Wolfowitz (1941), Scheffé (1943), Kendall (1946), Wilks (1948), Noether (1949), Wolfowitz (1949).

* Weibull, W. (1939a). A statistical theory of the strength of materials. *Ingeniörs Vetenskaps Akademien Handlingar*, Number 151. Generalstabens Litografiska Anstalts Förlag, Stockholm.

**Summary:** "Part I. The classical theory of strength is obviously incompatible with numerous results of experimental research. This discrepancy may be bridged over by considering as an essential element of the problem the dispersion obtained in experimental measuring of the ultimate strength. Viewed from this standpoint, the ultimate strength of a material can not be expressed by a single numerical value, as has been done heretofore, and a statistical distribution function will be required for this purpose. The application of the calculus of probability leads to the fundamental law of the theory, viz., that the probability of rupture (S) at any given distribution of stress ($\sigma$) over a volume (V) is determined by the equation $\log(1-S) = -\int_V n(\sigma)\,dv$ where $n(\sigma)$ is a function characteristic of each particular material. This fundamental formula allows us to compute the influence of the volume on the ultimate strength, the relation between tensile, bending, and torsional strength, etc. An experimental substantiation of the theory is provided by observations obtained from tensile, bending, and torsional tests on rods made of stearic acid and plaster-of-Paris. Part II. In this part of the paper a description is given of the graphic method used for the statistical treatment of the observations and of some measuring series relating to strength of materials under the action of mechanical and electrical forces. It is shown that the material function may be expressed by the formula $n(\sigma) = [(\sigma-\sigma_u)/\sigma_0]^m$ where $\sigma_u$, $\sigma_0$ and m are constants [location, scale and shape parameters, respectively] characteristic of the material. This formula applies to statistically homogeneous materials." (Author's summary, p. 45).

**Note:** See comments on next paper (Weibull (1939b)].

**Reference:** Griffith (1920).

**Citations:** Weibull (1939b), Afanassiev (1940), Kontorova & Frenkel (1941), Tucker (1941, 1945a, 1945b), Oding (1946), Davidenkov, Shevandin & Wittmann (1947), Gurney (1947), Epstein (1948a,b), Frankel (1948), McAdam, Geil, Woodard & Jenkins (1948), van Meer & Plantema (1949), Weibull (1949).

* Weibull, W. (1939b). The phenomenon of rupture in solids. *Ingeniörs Vetenskaps Akademien Handlingar*, Number 153. Generalstabens Litografiska Anstalts Förlag, Stockholm.

**Summary:** "Experimental investigations intended to serve as verifications of a statistical theory of strength of materials have shown that the rupture in solids may follow two fundamentally divergent courses resulting in different mathematical expressions for the probability of rupture. Formulae for these two possibilities are deduced for isotropic and anisotropic materials. With a view to facilitating the numerical computation of the distribution constants by arithmetical methods, formulae are deduced and values tabulated. The theoretical investigations deal also with series comprising two or more components. As experimental evidence for the theory, two test series are shown, which illustrate the two above-mentioned possibilities." (Author's summary, p. 55).

**Comments:** Epstein (1948a) writes (p. 141): "The application of essentially the same ideas [the 'weakest link' theory of Peirce (1926)] to the study of the strength of materials is found in two papers by the Swedish engineer, Weibull [(1939a,b)]. Unlike Peirce who assumed a Gaussian distribution of strengths, he makes the assumption that $F_0(\sigma)$, the probability of breakage of a unit volume as a function of the stress $\sigma$ is given by $F_0(\sigma) = 1-\exp[-(\sigma/\sigma_0)^m]$ where $\sigma_0$ and m are unknown parameters which may depend on the characteristics of the material under test. . . . The term probability of breakage of a unit volume is purely conceptual. . . . He is interested in finding out how the probability of rupturing a specimen of volume V depends on V. Weibull shows that the strength of specimens of volume V is proportional to $\sigma_0 V^{-1/m}$. It is apparent from his calculations that one of the primary motivations for his particular choice of $F_0(\sigma)$ is to make it easier to obtain results in closed form." The statistical distribution which was introduced in these two papers by Weibull, and which has come to be called by his name, is also known as the third asymptotic distribution of smallest values, and presumably would be known exclusively by that name except for his influence. Nevertheless, engineering statistics owes him a great debt for pointing out its broad applicability.

**References:** Yule (1911) [(Yule & Kendall (1937)], Weibull (1939a).

**Citations:** Daniels (1945), Fowler (1945), Gurney (1947), Epstein (1948a,b), Frankel (1948), van Meer & Plantema (1949), Peterson (1949a,b), Weibull (1949).

* Zappa, Guido (1939). Sull'estensione del concetto di mediana alle serie sconnesse. *Società Italiana per il Progresso delle Science, Atti del Riunione* **27** (5), 50-53. (JFM 65, 1355).

**Summary:** The author extends the concept of the median to two-dimensional series of observations by conserving the property, characteristic of the one-dimensional median, that the sum of the distances from all points in the series to their median is a minimum. Except for the case of a very small number of points (3 or 4) an attempt to use this property to determine the two-dimensional median leads in practice to a set of equations of high degree which cannot be solved. The author obtains a solution in the case of three points, not only for the case of a single observation at each point, but also for the case of frequencies $f_1$, $f_2$ and $f_3$ at points $P_1$, $P_2$ and $P_3$ respectively.

**Comments:** Neither the definition of the two-dimensional median nor the solution for the weighted median of three distinct two-dimensional observations is new, though the latter may not have been published before. The definition is that of Gini & Galvani (1929). Ross (1930) notes that the problem of finding the point the sum of whose distances from three non-collinear points is a minimum was proposed by Fermat and solved by Torricelli during the first half of the seventeenth century and that a solution was published by Goursat in 1902. Ross gives a solution submitted by E. B. Wilson for the case of single observations at each of three points, and mentions that Wilson has pointed out that for any non-reëntrant quadrilateral the point is the intersection of the diagonals and has given a simple procedure (not published by Ross) for the location of the point in the triangular case where populations of different sizes, instead of individuals, are located at the vertices (the case considered by Zappa).

**Reference:** Cisbani (1938).

* Afanassiev, N. N. (1940). Statistical theory of the fatigue strength of materials. (Russian). *Zhurnal Tekhnicheskoĭ Fiziki* [*Journal of Technical Physics* (USSR)] **10**, 1553-1568.

**Summary:** The author reviews previous results on the statistical theory of fatigue strength, including those of Weibull (1939a), none of which he finds satisfactory. He proposes a new theory which is not based on order statistics.

**Reference:** Weibull (1939a).

**Citation:** Oding (1946).

* Arley, Niels (1940). On the distribution of relative errors from a normal population of errors. *Mathematisk-Fysiske Meddelelser udgivet af det Kgl. Danske Videnskabernes Selskab* **18** (3), 62 pp. (JFM 66, 656-657; MR 2, 232).

**Summary:** The author generalizes the results of W. R. Thompson (1935). Let $x_1, x_2, \ldots, x_n$ be independent normal variates with mean $\varepsilon(x_i) = \sum_{j=1}^{m} a_{ij}p_j$, where the a's are known coefficients and the p's unknown parameters. Let the variance of $x_i$ be $\sigma_i^2 = \sigma^2/P_i$, where $\sigma^2$ is unknown but the weights $P_i$ are known. Finally let $\xi_i$ denote the estimate of $\varepsilon(x_i)$ and $S_i^2$ the estimate of the variance of $x_i - \xi_i$, both obtained by the method of least squares. The author shows that the probability density function of $r_i = (x_i - \xi_i)/S_i$ is given by $p(r) = (\text{const.})(n-m-r^2)^{(n-m-3)/2}$ for $|r| < \sqrt{n-m}$. He applies this result to obtain a criterion for the rejection of observations, which he compares with criteria proposed by various other authors, including Chauvenet, Vallier, Heydenreich, Mazzuoli, Rohne, and Topsøe-Jensen. He cites Czuber (1899) as the source of much of his information about these criteria.

**References:** Cranz (1896), Czuber (1899), Thompson (1935), Pearson & Chandra Sekar (1936), Fisher & Yates (1938), Arley & Buch (1940).

**Citations:** Arley & Buch (1940), Cramér (1946).

* Arley, Niels; Buch, K. Rander (1940). *Calculus of Probability with Applications on Statistics, Theory of Errors and Theory of Adjustment.* (Danish). Copenhagen. Second edition, 194_; third edition, 194_. English translation (1950), *Introduction to the Theory of Probability and Statistics*, John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London. (MR 11, 187).

**Summary:** Chapter 5 (pp. 55-71) deals with various measures of central tendency (mean, median, and mode) and of dispersion (standard deviation, mean deviation, and range) as well as quartiles, deciles and percentiles. Chapters 11 (pp. 152-180) and 12 (pp. 181-209) deal with application of the theory of probability to the theory of errors and to the theory of adjustment, respectively. On p. 152, the authors state: "By the theory of errors—a somewhat unfortunate but now generally adopted name—is understood that special branch of statistics which deals with the numerical determination of physical quantities." On p. 153, they divide errors into three groups: (I) Coarse or gross errors. (II) Systematic errors. (III) Statistical or random errors. In what follows [except in Sec. 17 (pp. 172-173), which deals with rejection of stragglers], they assume that observations encumbered with coarse errors have already been discarded and that the measurements have been corrected for known systematic errors, and concentrate their attention on random errors. They base their theory of errors entirely on the normal distribution, on the ground that "experience has shown that most physical measurements—corrected for coarse and systematic errors—are approximately normally distributed, a fact which may be explained theoretically" (pp. 154-155). Hence they present only the method of least squares and omit its alternatives based on order statistics. They do, however, consider (Example 3, p. 160) use of the sample midrange and range to estimate the mean and the standard deviation. Most of Chapter 11 deals with direct measurements, but indirect measurements are discussed in Sec. 22 (pp. 178-180). On p. 181, the authors state: "In the theory of errors, discussed in Chapter 11, we have considered the simple case of n independent measurements of one and the same physical quantity. In the **theory of adjustment** we consider the more general problem of n independent normally distributed measurements of *different* quantities [e.g. the three angles of a triangle] which are not free but which, as we know beforehand, are subjected to certain relations. . . . ." No use is made in Chapter 12 of methods based on order statistics.

**Note:** All references to specific chapter, section, and page numbers are to those in the English translation.

**References:** Helmert (1872) [1907], Czuber (1891a), Cranz (1896) [1925], Czuber (1899, 1903 [1908-10]), Kapteyn (1903) [Kapteyn & van Uven (1916)], Yule (1911) [Yule & Kendall (1940)], Fisher (1922), Whittaker & Robinson (1924), von Mises (1931), Fisher & Yates (1938), Arley (1940), {Kendall (1943), Cramér (1946)}.

**Citation:** Arley (1940).

* Dodd, Edward L. (1940). The substitutive mean and certain subclasses of this general mean. *Annals of Mathematical Statistics* **11**, 163-176. (JFM 66, 631). Abstract, *Bulletin of the American Mathematical Society* **46**, 414.

**Summary:** The author discusses various types of means which belong to the class of substitutive means, which are means such that if one substitutes a constant c for each of a set of numbers, their mean also takes the value c. Most of the means discussed are not based on order statistics, exceptions being the median and its generalization due to Jackson (1921). The author points out that the median of an even number of values is indeterminate over an interval, and that Jackson generalized the definition in order to indicate one method of selecting a single value from this interval of indeterminateness. Consider the Jackson median, M, of four numbers $x_1 \leq x_2 \leq x_3 \leq x_4$, viz. $M = (x_4x_3 - x_2x_1) / (x_4 + x_3 - x_2 - x_1)$. The author

points out that a direct substitution of $x_i = c$ (i=1,2,3,4) renders M indeterminate. But if $x_i \to c$ (i=1,2,3,4), indeed, if merely $x_2 \to c$ and $x_3 \to c$, so also does M.

**References:** Dunkel (1909), Jackson (1921), Dodd (1922), Pietra (1925), Galvani (1927), Gini & Galvani (1929), de Finetti (1931b), Cisbani (1938), Gini (1938).

* Feller, Willy (1940). On the time distribution of so-called random events. *Physical Review* (2) **57**, 1165-1167. (JFM 65, 1344).

**Summary:** The author gives a discussion and a complete solution of a problem treated by Ruark (1939). The author (p. 906) states the problem as follows: "Consider a sequence of 'random events', that is to say, suppose that the probability $W_n(t)$ of the occurrence, in a run of duration t, of exactly n events is given by the Poisson formula: $W_n(t) = (ft)^n e^{-ft}/n!$, $W_0(t) = e^{-ft}$, where $f > 0$ is a constant characteristic of the process. Now, out of a series of observations, pick out at random some time-interval of length D; let n be the number of actually observed events during that time. Pick out, again at random, two consecutive events among these. What is the probability that the time elapsed between their occurrence will exceed a given number $T < D$? In other words: we wish to calculate the conditional probability P(T;n,D) that the interval between two consecutive events will exceed T, under the hypothesis that it is known that these events occurred in a run of duration D during which there occurred exactly $n \geq 2$ events." Using Bayes' theorem, Ruark found that $P(T;n,D) = [1/(1-e^{-fD})]\,(n/fD)\,(1-T/D)^{n-1}$, but the author shows that the correct solution is $P(T;n,D) = (1-T/D)^n$.

**Reference:** Ruark (1939).

* Fisher, R. A. (1940). On the similarity of the distributions found for the test of significance in harmonic analysis, and in Stevens's problem in geometrical probability. *Annals of Eugenics* **10**, 14-17. (MR 1, 347).

**Summary:** "The identity of the solutions obtained for two problems apparently quite unconnected is shown to be no coincidence, but due to their intrinsic equivalence. The further formulae developed by Stevens are also relevant to one of the cases which may arise when harmonic components other than the greatest suggest the reality of periodic disturbances." [Author's summary, p. 17]. The two problems referred to are those of testing the significance of the largest of the harmonic components into which a series of observations may be analyzed [Fisher (1929)] and the probability that every point of a circle of unit length is included in at least one of n arcs, each of length x, marked off at random on the circumference [Stevens (1939)]. A table for testing the significance of the largest and second largest harmonic components is given. The tabular values for second largest components were computed by W. L. Stevens, who also extended the author's table in his earlier paper [Fisher (1929)] for the largest component.

**References:** Fisher (1929), Stevens (1939).

**Citations:** Garwood (1940), Finney (1941), Kendall (1946), Moran (1947).

* Fréchet, Maurice (1940a). Sur une limitation très générale de la dispersion de la médiane. *Journal de la Société Statistique de Paris* **81**, 67-76; discussion, 76-78. (JFM 66, 1304—listing only; MR 10, 722).

**Summary:** The author compares certain measures of dispersion of the sample median and the sample mean for large samples from unimodal distributions with finite variance. Let X be a random variable with unimodal distribution, $\sigma_X$ its standard error, $\theta_X$ its mean absolute error and $E_X$ its quartile deviation, and let M be the median and V the arithmetic mean of a sample of size n from this distribution. The author shows that for sufficiently large n, (i) $\sigma_M \leq 1.74\sigma_X/\sqrt{n}$, (ii) $\theta_M \leq 1.60\,\theta_X/\sqrt{n}$, (iii) $E_M \leq 1.35$

$E_X/\sqrt{n}$. For V, there is the equality $\sigma_V = \sigma_X/\sqrt{n}$ corresponding to (i), but there are no inequalities for V corresponding to (ii) and (iii). The author concludes that the sample median should be more widely used except when the distribution is known to be such that the sample mean is better. He points out that, in choosing between the median and the mean, one should take account not only of the distribution law but also of the use that is to be made of the result.

**References:** Fréchet (1935), Gumbel (1937c).

**Citations:** Fréchet (1940b).

* Fréchet, Maurice (1940b). Comparaison des diverses mesures de la dispersion. (English summary). *Revue de l'Institut International de Statistique* **8**, 1-12. (JFM 66, 631).

**Summary:** The author obtains the following inequalities for the measures of dispersion of the random variable X itself, the notation being the same as in the preceding paper: $0 \leq \theta_X/\sigma_X \leq 1$; $0 \leq E_X/\theta_X \leq 2$; $0 \leq E_X/\sigma_X \leq \sqrt{2}$. If $\Delta_X$ is the upper bound of the mean probability density of X between $x_1$ and $x_2$: $\text{Prob}\{x_1 < X < x_2\}/(x_2 - x_1)$ when $x_1$, $x_2$ vary, then he shows that the following inequalities hold: $0 \leq 1/\Delta\theta_X \leq 4$; $0 \leq 1/\Delta\sigma_X \leq 2\sqrt{3}$; $0 \leq 1/\Delta E_X \leq 4$. Moreover, he shows that none of the above twelve inequalities can be replaced by a sharper one.

**Reference:** Fréchet (1940a).

* Garwood, F. (1940). An application of the theory of probability to the operation of vehicular-controlled traffic signals. *Journal of the Royal Statistical Society, Supplement* **7**, 65-77.

**Summary:** "The traffic at a simple cross-roads, consisting of a main stream and a subsidiary stream crossing it, is supposed to be controlled by vehicle-actuated signals operating on the following principle. A vehicle in the subsidiary stream will be kept waiting until a time interval I, known as the vehicle interval, occurs between successive main-road vehicles, except that in any case it will not be kept waiting longer than M, the maximum period. Tables and curves give, for various values of I, M and the intensity of the main-road traffic, the probability that the waiting period runs to various times up to and including the maximum and also the average value of the waiting period. The main-road traffic is assumed to follow the Poisson distribution, and it is found that observational values agree with the theory. The calculations involve the following problem of the random division of an interval. A number of points are placed at random on a line: what is the probability that none of the parts into which the line is divided is more than a given fraction of the whole? . . ." [Author's summary, pp. 72-73]. Appendix (i) deals with the problem of the random division of an interval and the contributions of various authors to its solution. Appendices (ii) and (iii) deal with the probability function P(T) of the waiting period and the mean waiting period $\overline{T}$ respectively.

**References:** Whitworth (1897), Fisher (1929), Baticle (1933a,b, 1935), Lévy (1939), Stevens (1939), Fisher (1940).

**Citations:** Finney (1941), Greenwood (1946), Kendall (1946), Moran (1947).

* Gillett, H. W. (1940). Size effect in fatigue. *Metals and Alloys* **11**, A19, 94.

**Summary:** The author discusses the size effect in fatigue from the point of view of the "weakest-link" theory (the theory of extreme values). He writes (p. A19, 94): "It seems to be fundamental in fatigue that the nucleus for failure starts at the sorest spot . . . ., i.e., wherever the relatively highest stress gets to work on the relatively weakest spot. . . . Suppose we have one big balloon of 1000 sq. ft. surface and 1000 toy balloons, each of 1 sq. ft. surface. Make a single pin-prick in each case. The big balloon has failed, one of the toy balloons has failed, but the other 999 are all right. Thus, if we have sparsely distributed weak

spots in a material, we may take fatigue specimens No. 1-999 from it, all of which behave well, while No. 1000 will fall below the S-N curve. With weak spots occurring more commonly, a sample of a dozen fatigue specimens may run into enough of them at the highly stressed volumes in the test pieces to show a wide S-N 'scatter band' instead of an S-N line, or it may not, according to chance. . . . In the testing of a large specimen, with a relatively great volume stressed to the maximum, the weak spots get in their work and make it impossible to cover up. Thus, tests on large specimens tend to give the lower limit of the scatter band that small-specimen tests would show if thousands and thousands were tested."

* Grant, Eugene L. (1940). The probability viewpoint in hydrology. *Transactions of the American Geophysical Union* **21**, 7-12; discussion, 12-13.

**Summary:** The author defends the probability viewpoint in hydrology against its critics, but at the same time warns that extrapolation can be dangerous, as in trying to estimate a 1000-year flood from 20 to 40 years of record. He writes (page 12): "In general, the conclusion of this paper is that estimates of the expected frequencies of various extreme conditions should be recognized as a desired objective in most hydrologic studies, even though it is not always practicable to realize this objective. Whenever possible, investigators should endeavor to develop methods of analysis which permit estimating these frequencies by interpolation rather than by extrapolation." The discussion consists of an exposition by L. S. Hall of the use of the probability viewpoint in water-studies of large storage reservoirs.

**References:** Hazen (1930), Shewhart (1931), Creager (1939).

**Citation:** Gumbel (1941a).

* Gumbel, E. J. (1940). La durée de retour des températures annuelles extrêmes. *Comptes Rendus de l'Académie des Sciences de Paris* **210**, 468-470. (JFM 66, 603).

**Summary:** The author writes (pp. 468-470): "Let $W(x)$ be the probability of a value less than x, x being a stationary random variable of which one has observations made at constant time intervals, taken as unity. One defines the return periods $\overline{T}(x)$ [or $\underline{T}(x)$] of a value greater (or less) than x, by (1) $\overline{T}(x) = 1:[1-W(x)]$; $\underline{T}(x) = 1:W(x)$. For a variable unlimited in both directions, the first function increases, for increasing values of x, from the initial value 1 to infinity, the second decreases from $\infty$ to 1. . . . Observed return periods are obtained by the aid of the distribution curve. If the variable is symmetric with respect to zero, which is the mathematical expectation, one has certainly, according to (1), (3) $\underline{T}(-x) = \overline{T}(x)$. . . . For two random variables ${}_1y$ and $y_1$ with probabilities ${}_1W({}_1y)$ and $W_1(y_1)$ such that (4) ${}_1W(-{}_1y) = 1-W_1(y_1)$, one obtains the relation, analogous to (3), (5) ${}_1\underline{T}(-{}_1y) = \overline{T}_1(y_1)$. Equations (4) and (5) are satisfied by $m^{th}$ extreme values from above and from below of an initial distribution of the exponential type, if the number of observations N is very large. In particular, for $m = 1$, the probabilities ${}_1W(y)$ and $W_1(y)$ of the smallest and largest values are (6) ${}_1W(-y) = 1-W_1(y)$; $W_1(y) = e^{-e^{-y}}$, where the reduced variable y is a linear function of the variable x. . . . Let x be the daily temperature ($N=365$); then it follows from (6) that the number of years showing a maximum annual temperature less than $-y$ should be equal to the number of years showing a maximum annual temperature greater than y. The return period of a minimum annual temperature less than $-y$ should be equal to the return period of a maximum annual temperature greater than y. A first control of this theory is furnished by the postulate that the standard deviations s of the two observed distributions of extreme annual temperatures should be equal. Let ${}_1\bar{u}$ and $\bar{u}_1$ be the observed means of extreme temperatures; then one obtains the theoretical minimum and maximum temperatures ${}_1x$ and $x_1$ corresponding to the reduced temperatures $-y$ and y according to (7) ${}_1x = {}_1\bar{u}+C(\sqrt{6}/\pi)s-y(\sqrt{6}/\pi)s$; $x_1 = \bar{u}_1-C(\sqrt{6}/\pi)s+y(\sqrt{6}/\pi)s$, where C denotes Euler's constant. One may trace on the same scale the minimum and maximum annual temperatures by taking as a common point of departure the theoretical most probable values of minimum temperature ${}_1u$ and maximum temperature $u_1$, which are (8) ${}_1u = {}_1\bar{u}+C(\sqrt{6}/\pi)s$; $u_1 = \bar{u}_1-C(\sqrt{6}/\pi)s$. The comparison of the observed numbers of years showing minimum temperature less than ${}_1x$ and maximum greater than $x_1$

with each other and with the common theoretical frequency, also the observed return periods of minimum temperatures less than $_1x$ and maxima greater than $x_1$ with each other and with the common theoretical return period, furnish criteria for the deviations of observations from theory." (Compiler's translation).

**References:** Gumbel (1939b,c).

**Citation:** Gumbel (1942b).

* Kendall, M. G. (1940). Note on the distribution of quantiles for large samples. *Journal of the Royal Statistical Society, Supplement* **7**, 83-85. (JFM 66, 638; MR 2, 231).

**Summary:** The author gives the joint distribution function of $x_\ell$ and $x_m$ $(1 \leq \ell < m \leq n)$, where $x_1, x_2, \ldots, x_n$ are the values of x (arranged in order of increasing magnitude) in a sample from a continuous distribution. He shows that if $\ell$ and m are taken as fixed proportions of n, then $x_\ell$ and $x_m$ are asymptotically (for large n) distributed according to a bivariate normal distribution whose variances and covariances he gives.

**Reference:** Gumbel (1935a).

* Kontorova, T. A. (1940). Statistical theory of strength. (Russian). *Zhurnal Tekhnicheskoĭ Fiziki* [*Journal of Technical Physics* (USSR)] **10**, 886-890. (JFM 66, 1177—listing only).

**Summary:** The author reviews the work of Reinkober (1931) and of Alexandrov & Žurkov (1933) on the effect of size on strength. The latter writers developed the empirical formula $F = a + b/r$ (A), where a and b are constants, F is the strength, and r is the radius of the cross-section of the item. They attributed this effect to the occurrence of surface cracks—the greater the cross-section the greater the probability of a dangerous surface crack. In the literature, there are also indications that the strength depends on length, but experimental data show that the strength decreases much more slowly with increasing length than with increasing radius. The author develops a statistical theory based on the length of the most dangerous crack which yields a result different from formula (A). The mean of the strengths of a group of objects of size n V should be the same as that of the minimal strength of n groups of objects of size V. The author's theory is based on the fact that the larger the specimen, the greater the probability that it contains a dangerous inhomogeneity: "The probability $P_{\ell_{max}}$ of the presence in the crystal of an inhomogeneity whose size lies in the interval between $\ell_{max}$ and $\ell_{max} + \delta\ell$ may be determined by a function of the form (1) $P_{\ell_{max}} = C \exp[-(\ell_{max} - \bar{\ell})^2/(\overline{\Delta\ell})^2]\,\delta\ell$, where $C = 1/\sqrt{2\pi(\overline{\Delta\ell})^2}$, $\bar{\ell}$ being the mean size of the inhomogeneity, $\Delta\ell$ the effective width of the region which includes possible values of $\ell$. From this the number of inhomogeneities of size $\ell_{max}$ contained in the model is $n_{\ell_{max}} = n_0 P_{\ell_{max}}$ (2), where $n_0$ is the total number of inhomogeneities. A necessary and sufficient condition for the rupture of the object is the presence of one or more such inhomogeneities. Using this condition and replacing $n_0$ by $\bar{n}$ V, where $\bar{n}$ is the mean number of inhomogeneities occurring in unit volume of the material and V is the size of the specimen, we obtain $\bar{n}\,V\,C \exp[-(\ell_{max} - \bar{\ell})^2/(\overline{\Delta\ell})^2]\,\delta\ell = 1$. (4). From this $(\ell_{max} - \bar{\ell})^2 = (\overline{\Delta\ell})^2 [\lg V + \lg \bar{n} + \lg \delta\ell + \lg C]$, (5), or approximately, neglecting $\bar{\ell}$ in comparison with $\ell_{max}$ and omitting $\lg \delta\ell$: $\ell^2_{max} \cong (\overline{\Delta\ell})^2 [\lg V + \lg \bar{n} + \lg C]$. (5a). To determine the quantitative relation between the dimension of the inhomogeneity $\ell_{max}$ and the corresponding strength of the specimen $F_{min}$, we use considerations developed in the time of Griffith, which yield $W_1 = F^2\ell^3/2E$ (F = tension, E = modulus of elasticity, $W_1$ = elastic energy) and $W_2 = \sigma\ell^2$ ($\sigma$ = surface tension). This gives us $\ell = 2E\sigma/F^2$. (8). Substituting in Eq. (5a) $\ell^2_{max} = 4E^2\sigma^2/F^2_{min}$ and $\Delta\ell = -2E\sigma\Delta F/F^3$, (8a), we find finally $F_{min} = 1/\sqrt[4]{\alpha \lg \beta V}$, (9), where $\alpha = 4\,(\overline{\Delta F/F^6})$, $\beta = \bar{n}\,C$. The strength must therefore be inversely proportional to the logarithm of the fourth root of the size V. This relation (9) has, however, no relation to the empirical formula (A)." (Translation by Mark Breiter of pp. 889-890).

**References:** Reinkober (1931), Alexandrov & Žurkov (1933).

**Citations:** Kontorova & Frenkel (1941), Kontorova (1943, 1946), Oding (1946), Gensamer, Saibel & Lowrie (1947), Epstein (1948a,b), Kontorova & Timoshenko (1949).

* Kullback, S.; Frankel, A. (1940). A simple sampling experiment on confidence intervals. *Annals of Mathematical Statistics* **11**, 209-213. (JFM 66, 635; MR 1, 347).

**Summary:** Consider a population which is uniformly distributed on the interval $(0,\theta)$, the range $\theta$ being unknown. Interval estimation of $\theta$ may be based on the range of a sample of size n [see Wilks (1938)] or on the sample sum [or mean] or the larger (largest) sample value [see Neyman (1937b)]. The authors set up a simple sampling experiment, using Tippett's tables of random numbers, for estimating $\theta$ from samples of sizes 2 and 4. They drew 3900 samples of size 2 and computed 81% confidence intervals based on sample mean, range, and largest variate, and found the correspondence between the theoretical and observed results to be very close. Similar results were obtained for 82.08% confidence intervals from samples of size 4. The larger (largest) value is much the best (in the sense that the average length of the confidence interval based on it is by far the shortest) for both $n = 2$ and $n = 4$; the sum is much better than the range for $n = 2$, but not quite as good as the range for $n = 4$.

**References:** Neyman (1937b), Wilks (1938).

* Lawley, D. N. (1940). The estimation of factor loadings by the method of maximum likelihood. *Proceedings of the Royal Society of Edinburgh* **60,** 64-82. (JFM 66, 653, 1304; MR 2, 110).

**Summary:** "The method of maximum likelihood is used to estimate the loadings of a battery of tests in the factors which are assumed to be present. In a simple numerical example the results obtained are compared with those found by using various other methods. A method, suitable for large samples, is then given for testing the truth of any hypothesis made concerning the number of factors required." (Author's summary, p. 82). The matrix equations obtained by the method of maximum likelihood for the estimation of the factor loadings from observed test scores do not have a unique set of solutions, but the author picks out a particular set by the use of the largest latent roots of a pair of matrices associated with the equations of condition. The resulting method resembles Hotelling's method of principal components.

**References:** Fisher (1922), Thomson (1934), Aitken (1937).

* Nair, K. R. (1940a). The median in tests by randomization. *Sankhyā* **4,** 543-550. (JFM 66, 1305—listing only; MR 4, 108).

**Summary:** The author proposes the use of the median instead of the mean in randomization tests, since the distribution of the median of pair differences for paired samples or of the difference of medians for independent samples (for odd sample sizes) comes out simply, as does the distribution of range or midrange, whereas the corresponding distributions for means are difficult to find. A numerical example of the use of the median in a randomization test is given.

**References:** Fisher (1935), Thompson (1936), Savur (1937b).

**Citations:** Scheffé (1943), Wilks (1948).

* Nair, K. R. (1940b). Table of confidence intervals for the median in samples from any continuous population. *Sankhyā* **4,** 551-558. (JFM 66, 1304—listing only; MR 4, 165).

**Summary:** The author discusses the results of Thompson (1936) and Savur (1937b) on confidence intervals for the median when nothing is known about the population distribution except that it is continuous. He gives a table of confidence coefficients for intervals between pairs of symmetrically placed order statistics, based on Thompson's results, for samples of various sizes. He observes that when the population form is known it may be possible to work out the sampling distribution of the median in random samples, but points out that this has been done only for rectangular and exponential populations [see Pearson and Pearson (1931, 1932)]. For populations for which the median is an efficient statistic for estimating the center of location, a confidence interval based on the sampling distribution of the median for the population under consideration can be expected to be shorter than the distribution-free confidence interval obtained by Thompson and Savur. However, the greatest merit of their results is precisely that they do not depend on the form of the population.

**References:** Pearson & Pearson (1931, 1932), Anderson (1935), Thompson (1936), Savur (1937b).

**Citations:** Scheffé (1943), Kendall (1946), Bickerstaff (1947), Wilks (1948), Noether (1949).

* Paulson, Edward (1940). A note on the distribution of the median. *National Mathematics Magazine* **14,** 379-382. (JFM 66, 636).

**Summary:** The author finds the distribution of the median of a random sample of size $(2n+1)$, where n is an integer, from a symmetric population (whose mean and median are both zero) in terms of the incomplete Beta function, which has been tabulated by Karl Pearson. He discusses the use of the median in testing hypotheses and in interval estimation. He suggests that his results are especially useful in sampling from populations such as the Cauchy distribution, for which the mean is not a consistent statistic.

**References:** Hojo (1931), Craig (1932a), Thompson (1936).

* Pizzetti, Ernesto (1940). Medie ascendenti e medie discendenti. *Metron* **14,** 55-66. (JFM 66, 1303).

**Summary:** Let $a_0, a_1, \ldots, a_n$ be a sequence of positive numbers in nondecreasing order. The author gives the following definitions of their ascending and descending arithmetic means of order k, $C_n^{(k)}$ and ${}^{(k)}C_n$; ascending and descending geometric means of order k, $G_n^{(k)}$ and ${}^{(k)}G_n$; and ascending and descending harmonic means of order k, $H_n^{(k)}$ and ${}^{(k)}H_n$:

$$C_n^{(k)} = \sum_{m=0}^{n} \binom{n-m+k-1}{k-1} a_m \Big/ \binom{n+k}{k}, \quad {}^{(k)}C_n = \sum_{m=0}^{n} \binom{n-m+k-1}{k-1} a_{n-m} \Big/ \binom{n+k}{k} = \sum_{m=0}^{n} \binom{m+k-1}{k-1} a_m \Big/ \binom{n+k}{k};$$

$$G_n^{(k)} = \left[\prod_{m=0}^{n} a_m^{\binom{n-m+k-1}{k-1}}\right]^{1/\binom{n+k}{k}}, \quad {}^{(k)}G_n = \left[\prod_{m=0}^{n} a_{n-m}^{\binom{n-m+k-1}{k-1}}\right]^{1/\binom{n+k}{k}} = \left[\prod_{m=0}^{n} a_m^{\binom{m+k-1}{k-1}}\right]^{1/\binom{n+k}{k}};$$

$$H_n^{(k)} = \binom{n+k}{k} \Big/ \sum_{m=0}^{n} \left[\binom{n-m+k-1}{k-1} / a_m\right], \quad {}^{(k)}H_n = \binom{n+k}{k} \Big/ \sum_{m=0}^{n} \left[\binom{n-m+k-1}{k-1} / a_{n-m}\right] = \binom{n+k}{k} \Big/ \sum_{m=0}^{n} \left[\binom{m+k-1}{k-1} / a_m\right].$$

The author states and proves various properties of ascending and descending means. As n increases from 0 to $\infty$, for fixed k, $C_n^{(k)}$ is monotone increasing but ${}^{(k)}C_n$ is monotone decreasing. As k increases from 1 to $\infty$, for fixed n, $C_n^{(k)}$ is monotone decreasing, with $\lim_{k\to\infty} C_n^{(k)} = a_0$, but ${}^{(k)}C_n$ is monotone increasing, with $\lim_{k\to\infty} {}^{(k)}C_n = a_n$. Similar relations hold for ascending and descending geometric and harmonic means. Other properties include: $C_n^{(2)} + {}^{(2)}C_n = 2C_n^{(1)}$; ${}^{(2)}C_n - C_n^{(2)} = [n/(n+2)]\triangle$, where $\triangle = \sum_{m=0}^{[n/2]} (n-2m)(a_{n-m} - a_m)/[n(n+1)/2]$ is Gini's mean difference; ${}^{(3)}C_n - C_n^{(3)} = (3/2)[n/(n+2)]\triangle$; $G_n^{(2)} \cdot {}^{(2)}G_n = [G_n^{(1)}]^2$; $\log[{}^{(2)}G_n / G_n^{(2)}] = [n/(n+2)]\triangle$; also special properties when the a's are in arithmetic progression or in geometric progression. In the former case $C_n^{(1)}$ and ${}^{(1)}C_n$ coincide with the median (as well as with the arithmetic mean), $C_n^{(3)}$ [the author erroneously writes $C_n^{(4)}$] and ${}^{(3)}C_n$ with the first and third quartiles, $C_n^{(9)}$ and ${}^{(9)}C_n$ with the first and ninth deciles, etc. The author works out a numerical example in which he computes ascending and descending arithmetic, geometric, and harmonic means of orders $k = 1(1)4,10$ for $n = 6$ numbers and verifies several of the above properties.

**References:** Czuber (1914), Pietra (1925), Gini (1939).

**Citation:** Boldrini (1942).

* Roy, S. N. (1940a). Distribution of p-statistics on the non-null hypothesis. *Science and Culture* **5,** 562-563.

**Summary:** The author writes (p. 562): "If $k_i^2$ (i = 1,2, . . . ,p) be the p roots of the determinantal equation $|a_{ij} - k^2 a'_{ij}| = 0$, (1), where $||a_{ij}||$ and $||a'_{ij}||$ are the dispersion matrices of two samples $\Sigma$ and $\Sigma'$ of sizes n and n′ supposed to have been drawn from the same p-variate normal population with dispersion matrix $||\alpha_{ij}||$, then the distribution I had obtained [see Roy (1939)] was of the form Const. $(k_1^2 - k_2^2) \ldots (k_1^2 - k_p^2)(k_2^2 - k_3^2) \ldots (k_2^2 - k_p^2) \ldots (k_{p-1}^2 - k_p^2) \Pi_{i=1}^{p} [k_i^{n-p-1}/(1 + \lambda^2 k_i^2)^{(n+n'-2)/2}] dk_i$, (2), $\lambda^2$ being equal to n/n′. . . . If $\xi_i$ (i = 1,2, . . . ,p) denote the p roots of the determinantal equation $|\alpha_{ij}\xi^2 \alpha'_{ij}| = 0$, (3), and if $\lambda^2 = n/n'$ and if we define quantities $\zeta_{ij}$ such that $\zeta_{ij}^m = \Gamma(m/2)$, $\zeta_{ij}^m \zeta_{i'j'}^n = \Gamma(m/2)\Gamma(n/2)$ where $i \neq i'$ and $j \neq j'$ but $\zeta_{ij}^m \zeta_{i'j'}^n = \Gamma[(m+n)/2]$ when $i = i'$ and $j = j'$, (4), then the joint distribution of p-statistics $k_i$ (i = 1,2, . . . ,p) comes out in the form Const. $(k_1^2 - k_2^2) \ldots (k_1^2 - k_p^2)(k_2^2 - k_3^2) \ldots (k_2^2 - k_p^2) \ldots (k_{p-1}^2 - k_p^2) E^{n+n'-p-2} \{\Pi_{i,j=1}^{p} \zeta_{ij}/(\xi_i^2 + \lambda^2 k_j^2)^{1/2}\} \{\Pi_{i=1}^{p} k_i^{n-p-1} dk_i\}$, (5), where E is the determinant $|\zeta_{ij}/(\xi_i^2 + \lambda^2 k_j^2)^{1/2}|$. It is easily seen that if the two populations have the same dispersion matrix, i.e. if $||\alpha_{ij}|| = ||\alpha'_{ij}||$, then $\xi_i$ (i = 1,2, . . . ,p) = 1 and the joint distribution (5) is easily seen to reduce to (2) as it should. The $k_i$'s are supposed to be arranged in descending order of magnitude starting with $k_1$. . . ."

**Note:** Certain obvious typographical errors have been corrected in the above quotation.

**Reference:** Roy (1939).

**Citations:** Roy (1940b, 1942a,b).

* Roy, S. N. (1940b). Distribution of certain symmetric functions of p-statistics on the null hypothesis. *Science and Culture* **5,** 563.

**Summary:** The author writes (p. 563): "If we consider the distribution (2) of [Roy (1940a)] and define a new set of quantities $u_1, u_2, \ldots, u_p$ which are the symmetric functions of the roots $k_i^2$ of the determinatal equation (1) of the foregoing note, and which are given by $u_1 = (1/p) \Sigma_{i=1}^{p} k_i^2$, $(u_2)^2 = (1/{}^pC_2) \Sigma_{i,j=1}^{p} k_i^2 k_j^2$, . . . , $(u_p)^p = (1/{}^pC_p) k_1^2 k_2^2 \ldots k_p^2$, (1), then the joint distribution of $u_i$'s (i = 1,2, . . . ,p) has been found out in the form Const. $u_2 (u_3)^2 (u_4)^3 \ldots (u_{p-1})^{p-2} u_p^{[p(n-p)-2]/2} \Pi_{i=1}^{p} du_i / [1 + \lambda^2 {}^nC_1 u_1 + \lambda^4 {}^nC_2 (u_2)^2 + \ldots + \lambda^{2p} {}^nC_p (u_p)^p]^{(n+n'-2)/2}$, (2), where $\lambda^2 = n/n'$. Just as in (2) of the foregoing paper $k_p$ varies from 0 to $k_{p-1}$, $k_{p-1}$ varies from 0 to $k_{p-2}$ and so on, and finally $k_2$ varies from 0 to $k_1$ and $k_1$ from 0 to $\infty$, so here it has been found that $u_p$ varies from 0 to $u_{p-1}$, $u_{p-1}$ from 0 to $u_{p-2}$ and so on, and finally $u_2$ from 0 to $u_1$ and $u_1$ from 0 to $\infty$. The distribution of the $u_i$'s on the non-null hypothesis is being worked out and will be published in due course. The set of $k_i$'s defined in (1) of the foregoing note may be useful for purposes of testing one category of hypotheses and the set of $u_i$'s defined in (1) of this note may be useful for testing another category of hypotheses and other functions of the $k_i$'s may be useful for other purposes."

**Reference:** Roy (1940a).

**Citation:** Roy (1942a).

* Singleton, Robert R. (1940). A method for minimizing the sum of absolute values of deviations. *Annals of Mathematical Statistics* **11,** 301-310. (JFM 66, 634).

**Summary:** The author points out that the method given by Rhodes (1930) for estimation of the parameters in a regression equation by minimizing the sum of absolute values of deviations is iterative and recursive, and is presented without proof. The author uses geometric methods and terminology to

develop proofs for various methods and to obtain a new method which reduces the labor by eliminating the recursive feature, while retaining the iterative approach. For purposes of comparison of his new method (steepest descent) and Rhodes' method, he solves Rhodes' illustrative problem by both methods.

**Reference:** Rhodes (1930).

* de Vergottini, Mario (1940). Sul significato di alcuni indici di concentrazione. *Giornale degli Economisti e Annali di Economia* (5) **2,** 317-347. (JFM 66, 1303-1304).

**Summary:** Two indices of concentration, introduced by Bonferroni and Gini, respectively, are $B = \Sigma_{s=1}^{n-1}(M-M_s)/n-1)M$ and $R = \Sigma_{s=1}^{n-1}(p_s-q_s)/\Sigma_{s=1}^{n-1}p_s$, where M is the general arithmetic mean and $M_s$ is the arithmetic mean of the s smallest of n observations, $p_s = s/n$ and $q_s = sM_s/nM$. The author has discussed the interpretation of B and R in an earlier paper [de Vergottini (1939)]. Here he shows $B > R$ except in the extreme cases (no concentration and maximum concentration), in which both take the values 0 and 1, respectively. The author also discusses two other indices of concentration, $B' = \Sigma_{s=1}^{n-1}(M'_s-M)/M[1+n\Sigma_{s=2}^{n-1}(1/s)]$ and $\beta = \Sigma_{s=1}^{n-1}(M'_s-M_s)/nM\Sigma_{s=1}^{n-1}(1/s)$, where $M'_s$ is the arithmetic mean of the s largest of n observations, and discusses the relations between B and B', B' and R, and $\beta$ and R.

**References:** Gini (1914a), Dalton (1920), Gini (1930), Castellano (1933b), Mortara (1934a), Castellano (1935), d'Addario (1939), Gini (1939), de Vergottini (1939).

**Citation:** Boldrini (1942).

* Wald, Abraham (1940). The fitting of straight lines if both variables are subject to error. *Annals of Mathematical Statistics* **11,** 284-300. (JFM 66, 638; MR 2, 108-109).

**Summary:** The author considers two sets of random variables $x_i$, $y_i$ ($i=1,2,\ldots,N$; N even). Neither the true values $X_i$, $Y_i$ nor the coefficients $\alpha$ and $\beta$ of the linear relation [$Y_i = \alpha X_i + \beta$] between them are known. As an estimate of $\alpha$ [cf. Lambert (1765b) and van Beeck Calkoen (1816)] he uses $a = a_2/a_1$, where $a_1 = [(x_1 + \ldots + x_m) - (x_{m+1} + \ldots + x_N)]/N$, $a_2 = [(y_1 + \ldots + y_m) - (y_{m+1} + \ldots + y_N)]/N$ and $m = N/2$. He points out that the greater $|a_1|$ the more efficient is the estimate a of $\alpha$, and that $|a_1|$ is a maximum when one orders the observations so that $x_1 \leq x_2 \leq \ldots \leq x_N$.

**Reference:** Eisenhart (1939).

**Citations:** Deming (1943), Nair & Banerjee (1943), Tintner (1945), Kendall (1946), Kimball (1946a), Tintner (1946), Housner & Brennan (1948), Bartlett (1949).

* Wilson, Edwin B. (1940). The sampling error of the median. *Science* **92,** 58-59. (JFM 66, 637).

**Summary:** The author points out that the usual formula for the standard error of the median M of random samples of size n, $\sigma_M = 1/2\phi_M\sqrt{n}$, where $\phi_M$ is the value of the probability density function at the median, unlike the corresponding formula $\sigma/\sqrt{n}$ for the mean, is not universally valid. The author explores various pathological cases for which it gives incorrect results. He then sets out to find a true expression for $\sigma_M$, restricting himself to samples of odd size $n = 2k+1$ (k an integer) from a population, symmetric about the origin, with probability density function $\phi(x)$ and cumulative distribution function $\Phi(x)$ [not the author's notation]. The probability density function of the median is then $\psi(x) = [(2k+1)!/(k!)^2][\Phi(x)]^k[1-\Phi(x)]^k\phi(x)$, its mean is zero, and its variance is given by $\sigma_M^2 = \int x^2\psi(x)dx$, where the integration extends over the whole range of the function $\phi(x)$. The author applies this result to show that the standard error of the median of samples of size n from the Cauchy distribution $\phi(x) = 1/\pi(1+x^2)$ is infinite for $n = 3$, but finite for $n = 5, 7, \ldots$.

* Zappa, Guido (1940). Osservazioni sopra le medie combinatorie. *Metron* **14,** 31-53. (JFM 66, 1303).

**Summary:** The author discusses in some detail combinatorial means and combinatorial power means as defined by Gini (1938). He gives various limiting forms, which are functions of the largest and smallest observations, to which power means and combinatorial means tend. Several of these limits are due to Galvani (1927) and Cisbani (1938). The author gives examples of the application of combinatorial power means.

**References:** Galvani (1927), Cisbani (1938), Gini (1938).

**Citation:** Boldrini (1942).

Brookner, R. J. (1941). A note on the mean as a poor estimate of central tendency. *Journal of the American Statistical Association* **36,** 410-412.

**Summary:** The author shows that, for a random sample of size N from a rectangular population of length 1 around an unknown value $\theta$ [that is, $f(x) = 1$ if $\theta - 1/2 \leq x \leq \theta + 1/2$, $f(x) = 0$ otherwise], the variance of the sample midrange t is $1/[2(N+1)(N+2)]$. This compares with a variance of $1/12N$ for the sample mean $\bar{x}$. Both the mean and the midrange are unbiased estimators of $\theta$. The ratio of the variance of t to that of $\bar{x}$ is $6N/[(N+1)(N+2)]$, which is less than one for $N > 2$ and approaches zero as $N \to \infty$, so the mean is a poor estimator of central tendency for a rectangular population.

* Chandra Sekar, C.; Francis, Mary G. (1941). A method to get the significance limit of a type of test criteria. *Sankhyā* **5,** 165-168. (MR 4, 165).

**Summary:** Let $m_1, m_2, \ldots, m_k$ be k mean squares, each with f degrees of freedom. Under the null hypothesis of no treatment effect, each $m_i$ will be an unbiased estimator of the same variance $\sigma^2$ and will be distributed as $\chi^2\sigma^2/f$. The authors give a method of finding critical values, at specified levels of significance (for example, 5% and 1%) for the test criterion $M = m_{max}/\Sigma_{i=1}^{k} m_i$, where $m_{max}$ is the largest of the mean squares. They show that, in the interval [1/2,1] the probability density function of M is equal to k times the probability density function of $M_i = m_i/m_{max} (i=1,2,\ldots,k)$. Since the distribution of $M_i$ is relatively simple, one can easily find the critical value of M corresponding to any specified level of significance, provided it lies in the interval [1/2,1]. The authors tabulate, to four decimal places, the upper 5% points for f = 1, k = 2(1)13 and the upper 1% points for f = 1, k = 2(1)18. Results for larger values of k when f = 1 cannot be found by the authors' method, since they do not lie in the interval [1/2,1].

**References:** Fisher (1929, 1935 [1937]), Thompson (1935), Pearson & Chandra Sekar (1936), Wishart (1938).

**Citation:** Kendall (1946).

* Cochran, W. G. (1941). The distribution of the largest of a set of estimated variances as a fraction of their total. *Annals of Eugenics* **11,** 47-52. (MR 3, 171).

**Summary:** "For a set of n independent estimates of the same variance, each based on two degrees of freedom, Fisher (1929) obtained the distribution of the largest estimate as a fraction of the total. By means of a theorem in probability which may be frequently useful in problems of this type, this result is extended to a set of estimates each based on r degrees of freedom. The distribution has the same general form as Fisher's result, but is simplest in the case that he considered. A table is given of the 5% significance levels of the largest ratio, for small values of r [1(1)8,10] and n [3(1)10]. The limiting distribution when n is large is briefly discussed." (Author's summary, p. 52).

**Comments:** In his review (MR 3, 171), W. G. Madow calls attention to the relation of the author's result to certain theoretical results in a paper by the reviewer [Madow (1937]. The author's results are a generalization of those of Fisher (1929) for the special case r=2, as the author himself points out, and also of those of Chandra Sekar and Francis (1941) for r=1. In all cases of overlap, the tabular values of the author agree with those of Fisher and of Chandra Sekar & Francis to within a unit in the fourth decimal place.

**References:** Fisher & Tippett (1928), Fisher (1929, 1939).

**Citations:** Finney (1941), Kendall (1946), Eisenhart & Solomon (1947), Nair (1948a).

* Daniels, H. E. (1941). A property of the distribution of extremes. *Biometrika* **32,** 194-195. (MR 3, 171).

**Summary:** The author gives an empirical approximate relation $M = 2\cot(\pi\sigma/2)$ between the mean M and the standard deviation $\sigma$ of the largest value in a random sample of size n from a standard normal distribution [N(0,1)]. He admits that his formula has no obvious mathematical derivation, but shows, by comparison with exact values of M given by Tippett (1925), that for $n \leq 1000$ the greatest discrepancy is only about 1.5%. For $n > 1000$, he compares his approximation with penultimate values given by Fisher & Tippett (1928) and finds that the maximum discrepancy occurs for n = 7228, for which his approximation exceeds the penultimate value by about 3%. He speculates that the real discrepancy may be somewhat less, since a table given by Fisher & Tippett suggests that the penultimate values may be underestimated. The error at $n = \infty$ is less than 1% since, as $n \to \infty$, the true value of M tends to $\pi/\sigma\sqrt{6} \doteq 1.282/\sigma$ and the approximate value to $4/\pi\sigma \doteq 1.274/\sigma$.

**References:** Tippett (1925), Fisher & Tippett (1928).

**Citation:** Kendall (1946).

* Davis, Harold T. (1941). *The Analysis of Economic Time Series.* Principia Press, Inc., Bloomington, Indiana. (JFM 67, 530).

**Summary:** Concerning tests of significance in harmonic analysis, the author writes (pp. 186-189): "In order to understand the problem of determining significance in harmonic analysis, let us consider the time series, y(t). Let the values $A_n$ and $B_n$ [Fourier coefficients] be computed by the formulas $A_n=(2/N)\Sigma_{t=0}^{N-1}y(t)\cos n\pi t/N$, $B_n=(2/N)\Sigma_{t=0}^{N-1}y(t)\sin n\pi t/N$, and let the squares of the amplitudes of the Fourier sequence be defined . . . by $R^2=A_n^2+B_n^2$. Schuster's test of significance may then be formulated as follows: Let $R_M^2=4\sigma^2/N$ be the mean value of the squares of the amplitudes of the periodogram sequence $R_n^2$. Then the Schuster probability, $P_S$, that any squared amplitude, $R^2$, chosen at random, will exceed $\kappa R_M^2$ is given by $P_S=e^{-\kappa}$. . . . Sir Gilbert Walker [(1914)] was the first to call attention to the inadequacy of Schuster's test of significance. His argument ran as follows: Suppose that a large value of $R^2$ has been found in the total Fourier sequence of the N/2 independent terms necessary to represent N observations. Schuster's test merely gives the probability that an $R^2$ chosen at random shall exceed $\kappa R_M^2$. But what is really required is the probability that some $R^2$ among the total number in the Fourier sequence shall exceed $\kappa R_M^2$. We may state Walker's test as follows: The Walker probability, $P_W(\kappa)$, that at least one $R^2$ among the total Fourier sequence of N/2 independent values representing N observations will exceed $\kappa R_M^2$ is given by $P_W(\kappa)=1-(1-e^{-\kappa})^{N/2}$. . . . The tests of Schuster and Walker were derived on the assumption that the $R^2$ to be tested is derived from a series whose observations are random selections from a normal universe with known variance equal to $\sigma^2$. But when the unknown variance must itself be estimated from samples then these tests must be modified to take account of this fact. The analysis necessary to establish the criterion in this case was carried out by R. A. Fisher [(1929)]. The test may be formulated as follows: Let g' be defined by the ratio $g'=R^2/2\sigma^2$, where $R^2$ is the largest among the squares of the amplitudes of the Fourier sequence. Then, if $n=(N-1)/2$, where N is the number of observations, the Fisher probability, $P_F$, that g' will exceed some critical value g is given by the formula $P_F = n(1-g)^{n-1} - [n(n-1)/2!](1-2g)^{n-1} + \ldots + (-1)^{m-1}[n!/m!(n-m)!](1-mg)^{n-1}$, where m is the

greatest integer less than 1/g." The author tabulates the Walker probability function, $P_W$, on pp. 583-599 and the Fisher probability function, $P_F$, on pp. 601-605. In Chapter 9 (pp. 386-443), the author discusses the Pareto, lognormal and other distributions, and also the Lorenz curve and Gini's concentration ratio, as applied to distributions of wealth and income.

**References:** Galton (1869), Pareto (1897), Lorenz (1905), Czuber (1914), Walker (1914), Pietra (1915), Ricci (1916), Gini (1918), Vinci (1918), Dalton (1920), Amoroso (1925), Gini (1926), Saibante (1926, 1928), Fisher (1929), Gini (1930), von Bortkiewicz (1931), Gini (1932), Castellano (1933a), Yntema (1933), Wold (1935), Gini (1936), Bresciani-Turroni (1937).

**Citations:** Boldrini (1942), Kendall (1946), Tintner (1946).

* Finney, D. J. (1941). The joint distribution of variance ratios based on a common error mean square. *Annals of Eugenics* **11,** 136-140. (MR 3, 172).

**Summary:** In the introduction the author states (p. 136): "Fisher (1929) derived the exact distribution of the ratio of the largest of a set of sums of squares, each based on two degrees of freedom, to the total of the set. The solution to a geometrical problem, given by Stevens (1939), was later (1940) shown by Fisher to be an equivalent result, and also to give the distribution of the second, third, etc., largest of the set. Garwood (1940) has shown that the same mathematical solution applies to yet a third problem. Cochran (1941) has discussed the generalization of the problem to sums of squares based on any number of degrees of freedom. It is the purpose of the present note to show the solution to the analogous problem of the distribution of the ratio of the largest of a number of mean squares, all based on the same number of degrees of freedom, to an independent 'error' mean square. . . ." The author summarizes his results as follows (p. 140): "The distribution has been obtained of the largest of a number of variance ratios, such as occur in an analysis of variance when a number of independent mean squares with the same number of degrees of freedom are all compared with the same error mean square. It is found that the computation of exact significance levels is a lengthy process on account of slowness of convergence, but the approximation resulting from an assumption of complete independence of the ratios is satisfactory providing that the error degrees of freedom are sufficiently numerous. The solution of the less interesting problem of obtaining significance levels for the least of the variance ratios is arithmetically much simpler, and the similar approximation very satisfactory."

**References:** Fisher (1929), Fisher & Yates (1938), Stevens (1939), Fisher (1940), Garwood (1940), Cochran (1941).

**Citations:** Kendall (1946), Nair (1948a), Hartley (1949).

* Fréchet, Maurice (1941). Sur la correspondance entre certaines lois d'erreurs et certaines définitions de la distance. *Revue Scientifique* (Paris) **79,** 3-14. (JFM 67, 1059-1060).

**Summary:** The author examines two methods of demonstrating the validity of the normal law of error—the method of Gauss (1809) based on the postulate that the arithmetic mean is the best average of a set of equally reliable observations and the method [due to Hagen (1837), whom the author does not mention] based on the composition of a large number of small elementary errors. He does not find either method convincing, and concludes that verification is possible only experimentally, by comparison with actual data. This can be accomplished in various ways—by comparing theoretical and observed frequencies in the various classes of a grouped frequency distribution; by computing the Pearsonian measure of kurtosis $\beta_2=\alpha_4=\mu_4/\mu_2^2$, where $\mu_r$ is the $r^{th}$ moment about the mean, for the data and comparing it with the theoretical value [3 for the normal (second Laplacean) law, 6 for the first Laplacean law]; or by computing the ratios $D=(q_2-q_1)/(d_2-d_1)$ and $C=(q_2-q_1)/(c_2-c_1)$, where $q_1$ and $q_2$ are the first and last

quartiles, $d_1$ and $d_2$ are the first and last deciles, and $c_1$ and $c_2$ are the first and last centiles, and comparing them with the theoretical values [D=0.5263 and C=0.2899 for Laplace's second law, D=0.4307 and C=0.1772 for Laplace's first law]. The author applies the last method to data on artillery fire. Returning to theory, he shows the correspondence between laws of error and measures of central tendency and dispersion, the arithmetic mean and the standard deviation corresponding to Laplace's second law and the median and the mean absolute deviation (from the median) to Laplace's first law, etc. He extends this correspondence to the multivariate case (two or more dimensions).

**References:** Bertrand (1889), Czuber (1891a), Poincaré (1896), Wilson & Hilferty (1929).

* Gnedenko, B. V. (1941). Limit theorems for the maximal term of a variational series. *Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS* (N. S.) **32,** 7-9. (MR 3, 169).

**Summary:** The author writes (p. 7): "Consider the sequence $\xi_1,\xi_2,\ldots,\xi_n,\ldots$ of mutually independent random variables, subjected to one and the same distribution law $F(x)$. Form the random quantities $\eta_n=\max(\xi_1,\xi_2,\ldots,\xi_n)$. (1) It is easily seen that the distribution law of the $\eta_n$ is $F^n(x)$. M. Fréchet [(1927)] and R. A. Fischer [sic] and L.H.C. Tippett [(1928)] have established that the distribution function of the quantities $(\eta_n-b_n)/a_n$(2) can converge, for an appropriate choice of the constants $a_n>0$ and $b_n$, only to distribution functions of the following types $\Phi_\alpha(x)=0$ for $x<0$, $\Phi_\alpha(x)=e^{-x^{-\alpha}}$ for $x>0$; $\Psi_\alpha(x)=e^{-(-x)^{-\alpha}}$ for $x<0$, $\psi_\alpha(x)=1$ for $x\geq 0$ where the constant $\alpha>0$; and $\lambda(x)=e^{-e^{-x}}$. R. [von] Mises [(1936)] has given certain sufficient conditions for the convergence of distribution functions of the quantities (2) to each of the indicated limit laws for a definite choice of the constants $a_n>0$ and $b_n$." After some intervening discussion of the law of large numbers, he continues (p. 8): "We shall say that the distribution function $F(x)$ belongs to the domain of attraction of the limit law $\Omega(x)$, if for an appropriate choice of the constants $a_n>0$ and $b_n$ the distribution functions of the random variables (2) converge to $\Omega(x)$ with $n\to\infty$. As regards the domains of attraction, the following theorems hold. Theorem 3. In order that the distribution function $F(x)$ should belong to the domain of attraction of law $\Phi_\alpha(x)$, it is necessary and sufficient that: a) $F(x)<1$ for every x; b) for every constant $k>0$ $\lim_{x\to\infty}\{[1-F(x)]/[1-F(kx)]\}=k^\alpha$. Theorem 4. In order that the distribution $F(x)$ should belong to the domain of attraction of the law $\Psi_\alpha(x)$, it is necessary and sufficient that: a) there should exist an $x_0$ such that $F(x_0)=1$ and $F(x)<1$ for every $x<x_0$; b) for every constant $k>0$ $\lim_{x\to-0}\{[1-F(kx+x_0)]/[1-F(x+x_0)]\}=k^\alpha$. Theorem 5. In order that the distribution function $F(x)$ [$F(x_0)=1$, $F(x)<1$ for $x<x_0$, $x_0\leq\infty$] should belong to the domain of attraction of the law $\lambda(x)$, it is necessary and sufficient that there should exist such a positive continuous function $A(z)$, for which $\lim_{z\to x_0-0}A(z)=0$, that for all real x $\lim_{z\to x_0-0}\{[1-F(z(1+A(z)x))]/[1-F(z)]\}=e^{-x}$. From Theorem 5 the following result may be deduced. Corollary. If $F(x)<1$ for all x and $F(x)$ belongs to the domain of attraction of the law $\lambda(x)$, then the sequence of the maxima (1) is relatively stable". [i.e., for an appropriate choice of positive constants $B_n$ the relation $\lim_{n\to\infty}P\{|\eta_n/B_n-1|<\epsilon\}=1$ holds for every $\epsilon>0$].

**References:** Fréchet (1927), Fisher & Tippett (1928), von Mises (1936).

**Citation:** Smirnov (1949b).

* Gumbel, E. J. (1941a). The return period of flood flows. *Annals of Mathematical Statistics* **12,** 163-190. (JFM 67, 481; MR 3, 10—listing only).

**Summary:** The author summarizes his results as follows (pp. 187-189): "The theoretical return period $T(x)$, the mean number of years between two annual flood discharges greater than or equal to x, is a statistical function. . . . The exceedance interval $'T(x_m)$[$=n/(n-m)$], and the recurrence interval $''T(x_m)$ [$=n/(n-m+1)$]; $x_m$ being the $m^{th}$ flood discharge, where m is counted from below. . . . The return period $T(x)$ of a flood discharge x is found by [the] formula [$T(x)=1/(1-e^{-e^{-y}})$, where $y=\alpha(x-u)$]. For large values of x the flood discharge converges toward a linear function [$\log_{10}T(x)=0.434294\,\alpha(x-u)$] of the logarithm of the return period. . . . The two constants of our formula u are $1/\alpha$ are, respectively, the most probable annual flood discharge and a multiple of the standard deviation [$s=\pi/\alpha\sqrt{6}$]. . . . Our method can be

summarized by the following rules: (1) For each year find the maximum daily discharge $x_m$ (do not use momentary peaks) and arrange these n data in increasing magnitudes. (2) Calculate for each discharge $x_m$ ($m=1,2,\ldots,n-1$) the values $\log T(x_m) = \log n - \log(n-m)$ and plot the curves $x_m$, $\log[n/(n-m)]$ and $x_{m+1}$, $\log[n/(n-m)]$. These are the observed exceedance and recurrence intervals. (3) Calculate the annual mean flood $\bar{u}$ and the annual mean squared flood $\overline{u^2}$; determine ... the standard deviation $s=\sqrt{[1+1/(n-1)]}\,[\overline{u^2}-\bar{u}^2]$ and the two constants $1/\alpha=0.77970s$, $u=\bar{u}-0.57722/\alpha$. (4) The theoretical flood discharges x corresponding to the logarithm of the return period T(x) ... are obtained by the linear transformation $x=u+y/\alpha$ .... Plot x as a function of log T(x). For large values of x and for extrapolation it is sufficient to use the linear asymptote obtained graphically. The linear part of the theoretical curve (x,log T) permits of two interpretations: First, T is the theoretical return period of a flood greater than or equal to x; second, x is the most probable flood to be reached within T years. The second interpretation holds for the straight line through the point (u,0).... It is our claim that the calculus of probabilities and especially the theory of largest values, is an efficient tool for the solution of certain hydrological problems."

**Comments:** For correctness [and consistency with the author's formula (9'), "$T(x_m)=n/(n-m+1)$, and Table I], in rule (2) m should range from 0 to $n-1$ for the recurrence interval curve $x_{m+1}$, $\log[n/(n-m)]$, or better still, the curve should be given as $x_m$, $\log[n/(n-m+1)]$, $m=1,2,\ldots,n$.

**References:** Fuller (1914), Jarvis (1926) [discussion], Fréchet (1927), Fisher & Tippett (1928), Hazen (1930), Gibrat (1932b), Gumbel (1935a, 1935-36), Horton (1936), von Mises (1936), Saville (1936), Slade (1936), Coutagne (1937), Gumbel (1937a,b), Grant (1940).

**Citations:** Gumbel (1941b), Beard (1942), Gumbel (1942b,c), Kimball (1942), Gumbel (1943a,b), Powell (1943), Gumbel (1945a,b), Kendall (1946), Kimball (1946a), Epstein (1948a,b), Epstein & Brooks (1948), Wilks (1948), Langbein (1949), Press (1949).

* Gumbel, E. J. (1941b). Probability-interpretation of the observed return-period of floods. *Transactions of the American Geophysical Union* **22,** 836-849; discussion, 849-850. (MR 4, 29—listing only).

**Summary:** The author states (pp. 836-837): "The purpose of this article is to show by numerical examples how the problem of flood flows can be treated from the viewpoint of probability without making any artificial or arbitrary assumptions whatever. *Methods*—It is customary in hydrology to calculate either the observed recurrence or the observed exceedance-intervals of floods. In a recent article [Gumbel (1941a)] we have shown that there is only one theoretical value which corresponds to both. We call it the return-period. It is a statistical function as is the distribution or the cumulative frequency. Therefore these calculations are related to the distribution of flows. We have reproduced this distribution by the theory of largest values.... The interpretation of the exceedance- and recurrence-intervals of floods as being the observed return-periods of largest values leads to a formula for the theoretical return period T(x) in years, as a function of a reduced flood $y=\alpha(x-u)$ where x is the flood-discharge, u is the most probable annual flood and $(1/\alpha)$ equal to 98 per cent of the mean deviation [78% of the standard deviation]. ..." The method of analysis is summarized by four rules given on pp. 839-840. These results are essentially the same as those given in the author's earlier paper [Gumbel (1941a)]. However, the author makes the following addition to Rule 2: "m is called the serial number. The exceedance intervals start with $x_1$, $\log[n/(n-1)]$ and end with $x_{n-1}$, $\log n$. There is no exceedance-interval corresponding to the largest value $x_n$. The recurrence-intervals start with $x_1$, 0 [=log 1] and end with $x_n$, $\log n$. For given $x_m$, of course, the exceedance-interval surpasses the recurrence-interval. But this difference is small, unless m is large."

**References:** Hazen (1930), Saville (1936), Gumbel (1941a).

**Citations:** Beard (1942), Gumbel (1942c), Kimball (1942), Gumbel (1943b, 1945b).

* Hsu, P. L. (1941a). On the limiting distribution of roots of a determinantal equation. *Journal of the London Mathematical Society* **16,** 183-194.

**Summary:** The author writes (pp. 183-184): "Consider k p-variate populations and the matrix

$$\begin{Vmatrix} \xi_{11} & \xi_{21} & \dots & \xi_{p1} & 1 \\ \xi_{12} & \xi_{22} & \dots & \xi_{p2} & 1 \\ \dots & \dots & \dots & \dots & \dots \\ \xi_{1k} & \xi_{2k} & \dots & \xi_{pk} & 1 \end{Vmatrix}$$

where $\xi_{it}$ is the mean of the i-th variate of the t-th population. . . . We shall assume that the populations are normal ones with the same set of second moments about the means which we shall denote by $\sigma_{ij}=\sigma_{ji}(i,j=1, \dots ,p)$. Suppose that k samples, of sizes $m_1, \dots , m_k$, are drawn respectively from the k populations. . . . Denoting by $x_{1t}, \dots , x_{pt}$ the means of the t-th sample, by $x_1, \dots , x_p$ the grand sample means, by $s_{ijt}(i,j=1, \dots ,p)$ the second moments about the means of the t-th sample, and writing $\bar{a}_{ij}=\sum_{t=1}^{k} m_t (x_{it} - x_i)(x_{jt} - x_j)$, $\bar{b}_{ij} = \sum_{t=1}^{k} m_t s_{ijt}$, $\ell_1 = \min(p,k-1)$, $\ell_2 = \max(p,k-1)$, we may readily see that the matrix $\|\bar{a}_{ij}\|$ is positive and of rank $\ell_1$, and that provided $N-k \geq p$ [where $N = \sum_{t=1}^{k} m_t$], the matrix $\|b_{ij}\|$ is positive definite. Hence the determinantal equation in $\phi$ $|\bar{a}_{ij} - \phi \bar{b}_{ij}| = 0$ has a root zero of multiplicity $p-\ell_1$ and $\ell_1$ real positive roots." The author proves a theorem which gives the limiting distribution of these $\ell_1$ (ordered) roots as $m \to \infty$.

**References:** Fisher (1938, 1939), Hsu (1939), Roy (1939).

**Citations:** Hsu (1941b), Anderson (1945), Kendall (1946), Anderson (1948).

* Hsu, P. L. (1941b). On the problem of rank and the limiting distribution of Fisher's test function. *Annals of Eugenics* **11,** 39-41. (JFM 67, 1058; MR 3, 8).

**Summary:** The author assumes that $d_i = (x_i - \sum_{u=1}^{q} \beta_{iu} w_u)$ $(i = 1,2, \dots ,p)$ are distributed according to a multivariate normal law. He considers the problem of testing the hypothesis H that the rank of the matrix $\|\beta_{ii'}\|$ $(i' = 1,2, \dots ,n_1<q)$ is $\ell$ on the basis of a sample $(x_{ir}, w_{ur})$, $(r = 1,2, \dots ,N \geq p+q; i = 1,2 \dots , p; u = 1,2, \dots ,q)$, where the rank of $\|w_{ur}\|$ is q. Let $\|a_{ij}\| = XMW'_1(W_1MW'_1)^{-1}W_1MX'$ and $\|a_{ij} + b_{ij}\| = XMX'$, i,j = 1,2, . . . , p, where $X = \|x_{ir}\|$, $W_1 = \|w_{i'r}\|$ $(i'=1,2, \dots ,n_1; r=1,2, \dots ,N)$ and $M = I - W'_2(W_2W'_2)^{-1}W_2$, $W_2 = \|w_{i''r}\|$ $(i'' = n_1+1, \dots ,q; r=1,2, \dots ,N)$. The criterion $\Phi$ which the author proposes for testing H is the sum of the $\ell - \min(p,n_1)$ smallest non-vanishing roots of the equation $|a_{ij} - \phi b_{ij}| = 0$. The author shows that, when the hypothesis H is true, the limiting distribution of $N\Phi$ as $N \to \infty$ is the $\chi^2$-distribution with $(p-\ell)(n_1-\ell)$ degrees of freedom.

**References:** Fisher (1938), Hsu (1941a).

**Citations:** Anderson (1945), Tintner (1945), Kendall (1946), Tintner (1946), Bartlett (1947b), Anderson (1948), Geary (1948).

* Kolmogoroff, A. (1941). Confidence limits for an unknown distribution function. *Annals of Mathematical Statistics* **12,** 461-463. (MR 4, 25).

**Summary:** The author writes (pp. 461-463): "Let $x_1, x_2, \dots , x_n$ be mutually independent random variables following the same distribution law (1) $P\{x_i \leq \xi\} = F(\xi)$. A recent paper by A. Wald and J. Wolfowitz [(1939)] deals with the problem of using the observable values of the x's to estimate the function $F(\xi)$. In this connection it may be useful to recall the following results published by me in 1933. Put (2) $F_n(\xi) = N(\xi)/n$ where $N(\xi)$ denotes the number of those x's whose observed values do not exceed $\xi$. Theorem 1: If the function $F(\xi)$ is continuous then the distribution law of the quantities (3) $D_n = \sup |F(\xi) - F_n(\xi)| \sqrt{n}$ does not depend on $F(\xi)$. Denote by $\Phi_n(\lambda)$ the value of the probability $P\{D_n \leq \lambda\}$ which is common to all continuous distribution functions $F(\xi)$. Theorem 2: For n tending to infinity, the

distribution function $\Phi_n(\lambda)$ tends to (4) $\Phi_n(\lambda) = \sum_{k=-\infty}^{+\infty} (-1)^k e^{-2k^2\lambda^2}$ uniformly with respect to $\lambda$. A more elementary proof of Theorem 2 was given by N. Smirnoff [(1939a)]. Another paper by the same author [Smirnoff (1939b)] gives a table of the function $\Phi(\lambda)$. Without the assumption that $F(\xi)$ is continuous, we easily obtain Theorem 3: Whatever be the distribution functio[illegible]$(\xi)$, (5) $P\{D_n \leq \lambda\} \geq \Phi_n(\lambda)$. Theorems 1 and 3 giving the exact lower bound of the probability that $F_n(\xi)$ w[illegible] satisfy the inequality (6) $|F(\xi) - F_n(\xi)| \leq \lambda/\sqrt{n}$ for all values of $\xi$, can be used to establish confidence limits for $F(\xi)$ corresponding to the confidence coefficient (7) $\alpha = \Phi_n(\lambda)$. These confidence limits will be free from any restriction concerning the nature of the function $F(\xi)$. For sufficiently large values of n we can use the limiting distribution (4) and write (8) $\alpha = \Phi(\lambda)$." The author gives a short table, based on that of Smirnoff, of values of $\lambda$ corresponding to $\alpha = .95, .98, .99, .995, .998, .999$. He also gives an extension, due to Smirnoff, of the confidence procedure and the corresponding test (the Kolmogorov-Smirnov test) to the two-sample case.

**Note:** The Kolmogorov-Smirnov test and the corresponding confidence procedure, in the two-sample case, depend only on the *ranks* of the observations in the combined sample, and hence are not really relevant to our subject. Some papers dealing primarily with them are included in this volume because of their relation to the corresponding one-sample test and confidence procedure, which do depend on order statistics.

**References:** Kolmogoroff (1933), Smirnoff [Smirnov] (1939a,b), Wald & Wolfowitz (1939).

**Citations:** Scheffé (1943), Kendall (1946), Feller (1948), Wilks (1948), Noether (1949).

* Kontorova, T. A.; Frenkel, J. I. (1941). Statistical theory of brittle strength of real crystals. (Russian). *Zhurnal Tekhnicheskoĭ Fiziki* [*Journal of Technical Physics (USSR)*] **11** (3), 173-183; English translation, *Journal of Physics* (USSR) **7** (3), 108-114.

**Summary:** The authors review earlier efforts of Reinkober (1931), Alexandrov and Žurkov (1933), Weibull (1939a) and Kontorova (1940) to explain the phenomenon of dependence of unit strength of material on size, and propose to present more rigorous and general results. An analogy is drawn between the most dangerous inhomogeneity in a crystal and the weakest link of a chain. Consider a chain made up of n links, each of which has been chosen at random from a finite population of s links (s>>n) whose strengths are $\phi_1 \leq \phi_2 \leq \ldots \leq \phi_k < \ldots \leq \phi_s$. The probability that a particular link has strength not less than $\phi_k$ is given by $\sum_{i=k}^{s} p_{\phi_i}$, where $p_{\phi_i}$ is the a priori probability of encountering a link having strength $\phi_i$. The probability that the weakest link has strength $\phi_k$ is given by $W_{\phi_k} = np_{\phi_k} (\sum_{i=k}^{s} p_{\phi_i})^{n-1}$. If the number of chains and the number of links in each chain are both sufficiently large, we can replace this discrete distribution by a continuous one. The probability of finding a chain whose strength lies in the interval from $\phi_0$ to $\phi_0 + d\phi$ is given by $W(\phi_0)\,d\phi = np(\phi_0)[I(\phi_0)]^{n-1}\,d\phi$, where $I(\phi_0) = \int_{\phi_0}^{\infty} p(\phi)\,d\phi$ and $p(\phi)\,d\phi$ is the a priori probability that the strength of a single link of the chain lies between $\phi$ and $\phi + d\phi$. Similarly, the probability of finding a crystal whose strength lies in the interval (F', F' + dF) is given by the same formula with $\phi, \phi_0$, n and $d\phi$ replaced by F, F', N and dF, respectively, where N is the number of inhomogeneities present, which is given by $N = \overline{N}V$, where $\overline{N}$ is the mean number of inhomogeneities/cm$^3$ and V is the volume in cm$^3$. Using this and neglecting the one in the exponent, we obtain $W(F')dF = \overline{N}V\,p(F')\,[I(F')]^{\overline{N}V}dF$. We assume that F has a Gaussian distribution with $p(F) = Ce^{-\alpha(F-F_0)^2}$, where $F_0$ is a priori the most probable value of the strength and the coefficients C and $\alpha$ are given by $C = 1/\sqrt{2\pi\overline{(F-F_0)^2}}$ and $\alpha = 1/2\ \overline{(F-F_0)^2}$. If we set $F_0 - F' = \triangle F$, $F - F_0 = y$, then $W(F')\,dF = C\overline{N}V\,e^{-\alpha(\triangle F)^2} \cdot [I(\triangle F)]^{\overline{N}V}dF$, where $I(\triangle F) = C\int_{-\triangle F}^{\infty} e^{-\alpha y^2}\,dy$. In the majority of cases $F' < F_0$ or $\triangle F > 0$. We rewrite the integral for $I(\triangle F)$ in the form $I(\triangle F) = 1 - C\int_{\triangle F}^{\infty} e^{-\alpha y^2}\,dy = 1 - (C/\sqrt{\alpha})\int_{\sqrt{\alpha}\triangle F}^{\infty} e^{-z^2}\,dz \cong 1 - e^{-\alpha(\triangle F)^2}/2 \cdot \sqrt{\pi\alpha}\triangle F$. The probability of finding a specimen of volume V having the brittle strength F' can then be written in the final form $W(F')\,dF \cong C\overline{N}V\,e^{-\alpha(\triangle F)^2}[1 - e^{-\alpha(\triangle F)^2}/2\sqrt{\pi\alpha}\triangle F]^{\overline{N}V}dF$, where $\triangle F$ represents the absolute value of the difference $(F' - F_0)$. We now find the most probable value F* of the strength of a specimen of volume V by differentiating $W(F')dF$ with respect to F', setting the derivative equal to zero, and solving for F',

calling the result F*. (The value $F_0$ of the a priori strength may be considered a constant of the material, while F*, that of the a posteriori strength, is a decreasing function of the volume of the specimen.) The result is $F^* \cong F_0 - \sqrt{A \lg V + B}$, where $A = 1/\alpha$, $B = (1/\alpha) \lg \overline{N}/2\sqrt{\pi}$. The authors compare their results with those of Weibull. The dependence of the probability of brittle rupture of specimens upon their size for a given stress is essentially the same for both theories, but the dependence of the probability of brittle rupture of the specimens upon the stress is completely different for the two theories. Weibull treats the defects as indistinguishable from one another, while the present authors consider their seriousness.

**References:** Reinkober (1931), Alexandrov & Žurkov (1933), Weibull (1939a), Kontorova (1940).

**Citations:** Kontorova (1943, 1946), Oding (1946), Davidenkov, Shevandin & Wittmann (1947), Fisher & Hollomon (1947), Gensamer, Saibel & Lowrie (1947), Epstein (1948a,b), McAdam, Geil, Woodard & Jenkins (1948), Kontorova & Timoshenko (1949), Weibull (1949).

* Mood, A. M. (1941). On the joint distribution of the medians in samples from a multivariate population. *Annals of Mathematical Statistics* **12,** 268-278; related abstract, *Bulletin of the American Mathematical Society* **47,** 410. (JFM 67, 481; MR 3, 172).

**Summary:** The author gives an exact expression for the joint probability density function of the medians $\tilde{x}_1$ and $\tilde{x}_2$ of a sample of size $2n+1$ from a bivariate population with probability density function $f(x_1,x_2)$. He shows that the limiting joint probability density function (as $n \to \infty$) of $y_1 = (2n)^{1/2} a_1 \tilde{x}_1$ and $y_2 = (2n)^{1/2} a_2 \tilde{x}_2$, where $a_1 = \int_0^\infty f(0,x_2)\,dx_2 + \int_{-\infty}^0 f(x_1,0)\,dx_1$ and $a_2 = \int_0^\infty f(x_1,0)\,dx_1 + \int_{-\infty}^0 f(0,x_2)\,dx_2$, is a bivariate normal density function under weak conditions, including the condition that the population medians are zero, with a rather simple covariance matrix. If $f(x_1,x_2)$ is the bivariate normal distribution with correlation coefficient $\rho$, the author shows that the limiting value of the correlation between $y_1$ and $y_2$ (and hence between $\tilde{x}_1$ and $\tilde{x}_2$) is $(2/\pi) \sin^{-1}\rho$. He extends his analysis to the case of k-dimensional distributions, showing that the limiting joint probability density function of k quantities analogous to $y_1$ and $y_2$ is a k-variate normal density function. For $k=3$, he gives the elements of the covariance matrix explicitly.

**Reference:** Craig (1932a).

**Citation:** Wilks (1948).

* Paulson, Edward (1941). On certain likelihood ratio tests associated with the exponential distribution. *Annals of Mathematical Statistics* **12,** 301-306. (JFM 67, 480; MR 3, 174-175).

**Summary:** The author considers the elementary probability law $\sigma^{-1} e^{-(x-B)/\sigma}$, $B \leq x < \infty$, and calculates the power functions of the likelihood-ratio tests for the following three hypothesis: (1) the hypothesis that the location parameter B is equal to some specified value [0, without loss of generality (w.l.o.g.)], it being assumed that the scale parameter $\sigma$ is known; (2) the hypothesis that B is equal to some specified value [0, w.l.o.g.] when nothing is known about $\sigma$; (3) the hypothesis that two samples have been drawn from exponential populations with the same location parameter, assuming it is known that the samples must have come from two exponential distributions with the same scale parameter. He also shows that all three tests are unbiased. In each case the likelihood ratio is a function of the sample order statistics. In the first two cases let $x_1 \leq x_2 \leq \ldots \leq x_n$ be a sample of size n. Then the likelihood ratios are $\lambda_1 = e^{-nx_1}$ and $\lambda_2 = \{1/[1 + nx_1/\sum_{i=1}^n (x_i - x_1)]\}^n$, respectively. In the third case, let $x_1 \leq x_2 \leq \ldots \leq x_{n_1}$ and $y_1 \leq y_2 \leq \ldots \leq y_{n_2}$ be two samples of sizes $n_1$ and $n_2$, $N = n_1 + n_2$. Then the likelihood ratio is $\lambda_3 = [1/(1 + z/u)]^N$, where $z = n_2 (y_1 - x_1)$ if $y_1 > x_1$, $z = n_1 (x_1 - y_1)$ if $x_1 > y_1$, and $u = \sum_{j=1}^{n_1} (x_i - x_1) + \sum_{j=1}^{n_2} (y_j - y_1)$.

**References:** Neyman & Pearson (1928), Sukhatme (1936, 1937).

**Citations:** Kendall (1946), Lehmann (1947).

* Pizzetti, Ernesto (1941a). Betrachtungen über die Messung der Variabilität vermittels der mittleren Differenz nach Gini. *Archiv für Rassen- und Gesellschafts-Biologie* **34**, 321-328. (JFM 67, 475).

**Summary:** The author notes that Gottschick (1939) cites the formula $d = [2/(n^2-n)][p_1(p_2 + 2p_3 + \ldots + (s-1)p_s) + p_2(p_3 + 2p_4 + \ldots + (s-2)p_s) + \ldots + p_{s-1}p_s]$, given by Lenz & von Verschuer (1928), as a measure of variability of a series of values grouped into classes of unit width, where $p_i$ is the frequency of the $i^{th}$ class. He points out that d is nothing but the mean difference defined by Gini (1912) as $\triangle = [2/n(n-1)] \sum_{i=1}^{[n+1)/2} (n+1-2i)(x_{n-i+1}-x_i)$, where $x_j (j = 1, \ldots, n)$ is the $n^{th}$ observation in ascending order. Another formula given by Gini (1912), for grouped data, is $\triangle = [4/n(n-1)] \sum_{i=1}^{s} d_{k,M}|x_k - M|$, where M is the median of the given series and $d_{k,M}$ is the mean graduated distance from M of the $p_k$ quantities in the class with class mark $x_k$, where by the graduated distance of two quantities $x_1$ and $x_2$ we understand the number (increased by 1) of terms of the series lying between them. The author also gives alternate formulas for $\triangle$ due to other writers. Czuber (1914) has shown that $\triangle = 2(S' - S)/n(n-1)$ where $S' = x_n + (x_n + x_{n-1}) + (x_n + x_{n-1} + x_{n-2}) + \ldots + (x_n + x_{n-1} + \ldots + x_1)$ and $S = x_1 + (x_1 + x_2) + (x_1 + x_2 + x_3) + \ldots + (x_1 + x_2 + \ldots + x_n)$. A. de Gleria (1929) has shown that $S + S' = (n-1)\sum_{h=1}^{n} x_h$, so that Czuber's formula may be rewritten in the form $\triangle = 2\{(n+1)\sum_{h=1}^{n} x_h - 2S\}/n(n-1)$. Pietra (1925) and de Gleria (1930) have modified Czuber's formula to accommodate grouped data, obtaining $\triangle = [2/n(n-1)]\{(\sum_{k=1}^{s} p_k + 1)\sum_{k=1}^{s} p_k x_k - \sum_{k=1}^{s} p_k(p_k + 1 - 2\sum_{i=k+1}^{s+1} p_i)\}$, while de Finetti and Paciello (1930) have shown that $\triangle = [2/n(n-1)]\sum_{k=1}^{s+1} q_k \cdot (q - q_k)(x_{k+1} - x_k)$, where $q_k = p_1 + p_2 + \ldots + p_k$ and $q = p_1 + p_2 + \ldots + p_s$. The author applies the various formulas for grouped data to data on the cranial indices of 6209 soldiers grouped into 31 classes, and finds that these formulas require the following numbers of arithmetical operations (additions, subtractions, multiplications, and divisions): von Verschuer & Lenz, 499; Gini, 190; Pietra-de Gleria, 130; de Finetti & Paciello, 97.

**References:** Bowley (1901) [1920], Gini (1912), Czuber (1914), Gini (1914a), Weinberg (1916), Gini (1918), Dalton (1920), Julin (1921), Pietra (1925), Gumbel (1928), Lenz & von Verschuer (1928), de Gleria (1929), de Finetti & Paciello (1930), de Gleria (1930), Gini (1939), Gottschick (1939).

**Citation:** Pizzetti (1941c).

* Pizzetti, Ernesto (1941b). Relazione fra il rapporto de concentrazione e l'indice di concentrazione del Gini. *Atti della II Riunione Scientifica della Società Italiana di Statistica*, pp. 11-24. (JFM 67, 476).

**Summary:** Let $0 \leq a_1 \leq a_2 \leq \ldots \leq a_n$ and $A_i = \sum_{k=1}^{i} a_k$. Gini (1910, 1914a) [see also Gini (1939)] defines the concentration index $\delta$ of the quantities $a_1, a_2, \ldots, a_n$ by $(n-i)/n = [(A_n - A_i)/A_n]^{\delta}$ (so far as $\delta$ remains approximately constant as i varies) and the concentration ratio by $R = \sum_{i=1}^{n-1}(i/n - A_i/A_n)\sum_{i=1}^{n-1}(i/n)$. By binomial expansion of $[(n-i)/n]^{1/\delta}$, the author proves that the relation $R = 1 - 3/2\delta + 7/12\delta^2 - 1/12\delta^3$ is a good approximation for $\delta > 1$ and reduces for sufficiently large n to the less exact relation $R = (\delta-1)/(\delta+1)$ given by Gini. He compares both $\delta$ and R with a third index $\alpha$, where $n - i = H/x^{\alpha}$ and H is a constant, proposed by Pareto (1897). He illustrates his results by a numerical example on distributions of income.

**References:** Pareto (1897), Gini (1910, 1914a, 1939).

* Pizzetti, Ernesto (1941c). Un nuovo aspetto nella misura della variabilità. *Statistica* (Ferrara) **1**, 315-325. (JFM 67, 1058—listing only).

**Summary:** Given a sequence of n positive numbers $x_1 \leq x_2 \leq \ldots \leq x_n$, the author defines the following measures of variability: Mean oscillation, $\Omega = [2/n(n-1)]\sum_{k=2}^{n}(x_k/x_1 + x_k/x_2 + \ldots + x_k/x_{k-1})$; Mean oscillation with repetition, $\Omega_r = [2/n(n+1)]\sum_{k=1}^{n}(x_k/x_1 + x_k/x_2 + \ldots + x_k/x_k)$; Mean quotient, $Q = [(x_2/x_1)(x_3/x_1)(x_3/x_2)\ldots(x_n/x_1)(x_n/x_2)\ldots(x_n/x_{n-1})]^{2/n(n-1)}$; Mean quotient with repetitition, $Q_r = [(x_1/x_1)(x_2/x_1)(x_2/x_2)\ldots(x_n/x_1)(x_n/x_2)\ldots(x_n/x_n)]^{2/n(n-1)}$. He shows that $\Omega_r = [(n-1)/(n+1)]\Omega + 2/(n+1)$ and $\log Q_r = [(n-1)/(n+1)]\log Q$; also that the logarithm of the mean quotient is equal to the mean

difference of the logarithms of the numbers in the sequence. He gives examples of the computation of the above measures for ungrouped and for grouped data, and compares their values in the latter case with those of the mean difference, the mean difference with repetition, the mean deviation from the arithmetic mean, and the root-mean-square deviation from the arithmetic mean. He shows that if the sequence of numbers has a lower limit I and an upper limit L, then $\Omega$, $\Omega_r$, Q and $Q_r$ all attain their maximum values (which he gives) for n/2 numbers equal to I and n/2 numbers equal to L when n is even, and for (n+1)/2 [or (n−1)/2] numbers equal to I and (n−1)/2 [or (n+1)/2] numbers equal to L when n is odd. For calculating Q, he recommends formulas based on those for the mean difference given by Gini (1939), de Gleria (1929), and de Finetti & Paciello (1930), for which the author has discussed the rapidity of calculation in an earlier paper [Pizzetti (1941a)]. For calculation of $\Omega$ for grouped data, he recommends the formula $\Omega = [2/n(n-1)][(f_1/x_1)(f_2x_2 + \ldots + f_sx_s) + (f_2/x_2)(f_3x_3 + \ldots + f_sx_s) + \ldots + f_{s-1}(f_s/x_s) + \binom{f_1}{2} + \ldots + \binom{f_s}{2}]$, where s is the number of distinct values of x and $f_i$ is the number of times $x_i$ occurs in the sequence.

**Note:** The compiler has corrected an obvious typographical error in the author's definition of $Q_r$.

**References:** de Gleria (1929), de Finetti & Paciello (1930), de Gleria (1930), Gini (1939), Pizzetti (1941a).

**Citation:** Boldrini (1942).

* Rajalakshman, D. V. (1941). On the extreme values of samples taken from a rectangular population. *The Mathematics Student* **9,** 103-111. (MR 4, 21).

**Summary:** Let $\mu_m$ and $\nu_m$ be the $m^{th}$ moments of the maximum and minimum values in samples of size n taken from an infinite rectangular population of range $\omega$, about their respective arithmetic means. The author derives the following formulas: $\mu_m = [\omega/(n+1)]^m F(-m,1;n+1;n+1)$; $\nu_m = (-1)^m\mu_m$, where $F(a,b;c;z) = 1 + (a\cdot b/1\cdot c)z + [a(a+1)\cdot b(b+1)/1\cdot 2\cdot c\cdot(c+1)]z^2 + \ldots$ is the hypergeometric function. The author gives two derivations, the first using a result of Pearson & Pearson (1931) and the second using a method similar to that of Sastry (1935). He verifies the results experimentally, and finds that the empirical distributions of extreme values are of Pearson type IX.

**References:** Pearson & Pearson (1931), Sastry (1935).

**Citation:** Rajalakshman (1943).

* Roller, Paul S. (1941). Statistical analysis of size distribution of particulate materials, with special reference to bimodal and frequency distributions. Correlation of quartile with statistical values. *Journal of Physical Chemistry* **45,** 241-281. (JFM 67, 486).

**Summary:** The author has found empirically that the size distribution of particulate materials may be either a univariate distribution with c.d.f. of the form $F(x) = ax^{1/2}e^{-b/x}$, the corresponding p.d.f. being of Pearson type III and a and b being parameters of the distribution, or a bivarate distribution consisting of a mixture of two such distributions, with parameters a′, b′ and a″, b″ respectively. The author states that four statistical constants characterize a material as to size: surface area per gram S, coefficient of uniformity U, coefficient of regression R, and number of particles per gram N. He defines regression as a deficiency of the very coarsest sizes relative to the others, which is unrelated to the usual statistical definition of the term. Given observations which represent a random sample from the size distribution, one can estimate the parameters a, b (or a′, b′, a″, b″ in the case of a bimodal distribution), or he can estimate the median and the quartiles. The statistical constants S, U, R, N can be estimated from the parameters, or S and N can be estimated from the median and U and R from the median and the quartiles. The author defines the quartile coefficient of uniformity by $U_q = 1.2x_{50}/(x_{75}-x_{25})$, where $x_{50}$ is the median and $x_{25}$ and $x_{75}$ are the first and third quartiles, respectively. He states that results based on the quartiles are good when $U_q$ is high. He also considers distributions other than the

Pearson type III for the size of the particles, including the lognormal and the Weibull (third asymptotic distribution of smallest values).

**References:** Pearson (1895), Rosin & Rammler (1934), Bond (1935).

* von Schelling, Hermann (1941). Die Bedeutung der statistischen Methodik für die Biologie. *Ergebnisse des Hygiene, Bakteriologie, Immunitätsforschung und Experimentellen Therapie* **24,** 87-149. (JFM 67, 500).

**Summary:** In this largely expository article on statistical methodology in biology, the author includes a section (Section 12, pp. 141-143) on confidence limits for the true median of a population. He points out that the median, not being influenced by the extreme observations, is often a better average to use in biometry than the arithmetic mean. Given n measurements $\ell_1 \leq \ell_2 \leq \ldots \leq \ell_N$, he notes that i points lie to the left of the division point $\lambda_i = (\ell_i + \ell_{i+1})/2$, $i = 1, 2, \ldots, N-1$. He gives a new method, different from that of Thompson (1936), of finding lower and upper confidence limits, $\lambda'$ and $\lambda''$, for a specified confidence level, by use of the values of $\lambda$ defined above. He applies his results to data on the survival time of 99 rabbits inoculated with tubercle bacilli. He also notes that the middle observation (the 50th of 99) is the observed (sample) median.

**References:** von Mises (1931), Thompson (1936).

* Simon, Leslie E. (1941). *An Engineer's Manual of Statistical Methods.* John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London.

**Summary:** In Chapter VII (pp. 63-70) the author gives a method of process inspection (sampling by variables) based on the use of control charts for average ($\overline{X}$) and standard deviation ($\sigma$). In Appendix C (pp. 188-207) he gives an alternate system based on the range of subgroups. The control limits for averages are given by $\overline{\overline{X}} \pm A\overline{R}$, where $\overline{\overline{X}}$ is the average of the subgroup averages, $\overline{R}$ the average range, and A is given in Table C1 (p. 205). The control limits for ranges are given by $D\overline{R}$, where D is selected in a symmetric manner with respect to probability (e.g. $D_{0.005}$ and $D_{0.995}$) or with respect to multiples of the standard deviation ($D_{-3\sigma}$ and $D_{+3\sigma}$), and D is given in Table C2 (p. 206). Three-decimal-place values are given in Table C1 for $A_{3\sigma}$, $A_{0.001}$, $A_{0.005}$, $A_{0.01}$, $A_{0.05}$ and $A_{0.10}$ and in Table C2 for $D_{-3\sigma}$, $D_{0.005}$, $D_{0.01}$, $D_{0.05}$, $D_{0.10}$, $D_{+3\sigma}$, $D_{0.995}$, $D_{0.99}$, $D_{0.95}$ and $D_{0.90}$ for sample size n=2(1)15. In comparing the two systems, the author writes (pp. 63-64): "The method of process inspection . . . based on standard deviation . . . requires the squaring of numbers and the taking of square roots. . . . The method of standard deviation is offered because of its broad applicability and high efficiency, i.e. greater information from a given number of samples. . . . However, conditions are frequently met in practice where it is more economical to take a few more samples and avoid computational labor. . . . In that event the simple method offered in Appendix C may be used to advantage; this method is based on the range, i.e. the difference between the greatest value and least value in a sample. . . . Range on a sample of 10 may be said to convey approximately the same information as standard deviation on a sample of 9. . . . "In Chapter XII (pp. 133-143) the author discusses various measures of dispersion, including the standard deviation, the range, the mean deviation, the mean successive difference, and the root-mean-square successive difference, together with the relative efficiency (assuming normality) of the various measures.

**References:** Laplace (1812) [1886], Helmert (1876a), Cranz (1896) [Cranz & Becker (1921)], Poincaré (1896) [1912], Yule (1911) [Yule & Kendall (1937)], Keynes (1921), Whittaker & Robinson (1924), Coolidge (1925), Tippett (1925), E. S. Pearson (1926), Student (1927), K. Pearson (1931), Shewhart (1931), Dodge et al. (1933), Davies & Pearson (1934), E. S. Pearson (1935).

**Citations:** Dodge (1933), Knudsen (1943), Pearson & Hartley (1943), Hartley (1944), Grant (1946), Kendall (1946).

* Smirnoff, N. [Smirnov, N. V.] (1941). On the estimation of the maximum term in a series of observations. *Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS* (N. S.) **33** (5), 346-350. (MR 5, 127).

**Summary:** The author writes (p. 346): "Let the sequence $x_1 < x_2 < \ldots < x_n$ (1) represent the results of n independent observations of a random variable x, enumerated in ascending order of magnitude. We suppose that x follows the normal law of distribution with the centre $a = Ex$ and the variance $\sigma^2 = E\{x-a\}^2$. Assume, however, that for some individual observation there may be realized with a small probability another law of distribution, different from the normal one with a larger variance. This fact usually accounts for the appearance at the ends of the series (1) of 'abnormal' results strongly deviating from the mean. Thus, the question arises as to a statistical rule to be followed for the elimination of the extreme terms of the variational series (1) in the analysis of observational data or, in other words, as to the limit of deviations from the mean to be considered practically impossible to pass, provided there exists a normal law of distribution for a given number of observations. A variety of such rules is known in literature [Cranz (1896) {Cranz, Eberhard & Becker (1925)}]. That most frequently used in practice is Chauvenet's [(1863)] rule, the theoretical foundation of which is not satisfactory. A more correct, though incomplete solution of the problem has been obtained by W. R. Thompson [(1935)]. In the present note we obtain the law of distribution of the maximum deviation from the mean upon which the evaluation of the extreme terms is to be based." The author proceeds to find the cumulative distribution function $\tau_n(\lambda)$ of $\xi_n = (x_n - \bar{x})/s$, where $\bar{x}$ is the sample mean and s is the square root of the sample variance. For any fiducial probability $\alpha$, the limits $\lambda_n(\alpha)$ of significance for $\xi_n$ are determined by the equation $\tau_n(\lambda) = \alpha$. The author tabulates these limits (2DP) for n = 3(1)20 and $\alpha$ = 0.9, 0.925, 0.95, 0.975. He states that, for small n, his limits $\lambda_n(\alpha)$ do not differ much from those of W. R. Thompson.

**References:** Chauvenet (1863), Cranz (1896) [Cranz, Eberhard & Becker (1925)], Thompson (1935).

* Thompson, Catherine M. (1941). Tables of percentage points of the incomplete Beta-function. Prefatory note by E. S. Pearson; description of the calculation by L. J. Comrie and H. O. Hartley; methods of interpolation by H. O. Hartley. *Biometrika* **32,** 151-181. (MR 3, 153).

**Summary:** The percentage points of the incomplete Beta-function (i.e., the values of x satisfying the equation $[\Gamma(p+q)/\Gamma(p)\Gamma(q)]\int_0^x x^{p-1}(1-x)^{q-1}dx = P$) are tabulated (to five significant figures) for P = 0.005, 0.01, 0.025, 0.05, 0.10, 0.25 and 0.50; 2q = 1(1)10, 12, 15, 20, 24, 30, 40, 60, 120, $\infty$, and 2p = 1(1)30, 40, 60, 120, $\infty$. The purpose of the table, the methods of calculation, and appropriate methods of interpolation are explained in the introductory material.

**Note:** These tables are relevant to our study because the incomplete Beta-function ratio $I_x(p,q)$ tabulated is the $\alpha$ probability point of the $r^{th}$ order statistic of a sample of size n for the uniform distribution with p.d.f. $f(x) = 1$, $0 \leq x \leq 1$ for $x = \alpha$, $p = r$, $q = n-r+1$.

**References:** Yule (1911) [Yule & Kendall (1937)], K. Pearson (1931), Fisher & Yates (1938).

**Citations:** Fisher & Yates (1938)[1943], Mathisen (1943), Hartley (1944), Kendall (1946), Eisenhart & Solomon (1947), Murphy (1948), Nair (1948a), Noether (1948, 1949).

* Tucker, John, Jr. (1941). Statistical theory of the effect of dimensions and of method of loading upon the modulus of rupture of beams. *Proceedings of the American Society for Testing Materials* **41,** 1072-1088; discussion, 1089-1094.

**Summary:** "There is an inherent difference in the strength of duplicate test specimens no matter how carefully these specimens are made or tested. Such differences are a natural characteristic of the materials and are more pronounced in some than in others. The paper shows how the variations in the strength of small elements of volume within a specimen will affect the modulus of rupture of beams of

different dimensions and beams subjected to different loading. For example, the modulus of rupture of a beam will be decreased with beam length and with beam depth and will be greater in centrally loaded beams than in similar beams loaded at third points." (Author's synopsis, p. 1072). The discussion includes comments by Gerald Pickett and G. R. Gause and a closure by the author. Pickett writes (p. 1089): "Mr. Tucker . . . has questioned the application of the weakest-link theory to elements in parallel and has proposed instead the strength-summation theory. . . . His analysis of beams differing in width on the basis of this summation theory is excellent. However, the writer sees no reason why the theory cannot also be used for beams of varying depth . . . ." Gause writes (p. 1091): "In comparing Mr. Tucker's treatment of the statistical viewpoint with the treatment used by Weibull the outstanding difference to be noted is the introduction by Mr. Tucker of the so-called 'strength-summation theory'. As stated, this theory appears to apply to certain problems and to give results which agree better with test data. However, certain applications of this theory will lead to inconsistencies. . . ." In his closure, the author comments at some length on these and other points raised by the discussants.

**References:** Griffith (1920), Tippett (1925), Tucker (1927), Weibull (1939a).

**Citations:** Tucker (1945a,b), Epstein (1948b), Frankel (1948), van Meer & Plantema (1949).

* Wald, A.; Wolfowitz, J. (1941). Note on confidence limits for continuous distribution functions. *Annals of Mathematical Statistics* **12,** 118-119. (MR 3, 9).

**Summary:** The authors call attention to papers by Kolmogoroff (1933) and Smirnoff (1939a) containing results which anticipate some of those in an earlier paper by the authors [Wald & Wolfowitz (1939)]. The results of Kolmogoroff and Smirnoff provide an easy method of constructing tables for computing confidence limits for a cumulative distribution known to be continuous. The authors correct a typographical error in an equation given by Smirnoff (1939a).

**References:** Kolmogoroff (1933), Smirnoff [Smirnov] (1939a), Wald & Wolfowitz (1939).

**Citations:** Scheffé (1943), Kendall (1946), Wilks (1948).

* Wilks, S. S. (1941). Determination of sample sizes for setting tolerance limits. *Annals of Mathematical Statistics* **12,** 91-96. (JFM 67, 481-482; MR 3, 9).

**Summary:** "A method based on truncated sample ranges [quasi-ranges] for determining size of sample required for setting tolerance limits [$L_1 = x_r$, $L_2 = x_{n-r+1}$, where $x_i (= 1,2,\ldots,n)$ is the $i^{th}$ order statistic of a sample of size n] on a random variable x having any unknown continuous distribution f(x) and having a given degree of stability is given. A method for setting tolerance limits [$L'_1 = \bar{x} - ks$, $L'_2 = \bar{x} + ks$, where $\bar{x}$ and s are the sample mean and standard deviation] corresponding to a given degree of stability in case f(x) is normal is discussed and a comparison of the stabilities of the tolerance limits set by the two methods in the normal case is made. Illustrative examples of the methods are given." (Author's summary, p. 96). If P is the proportion of the normal universe in $(x_r, x_{n-r+1})$ and P' that in $(\bar{x} - ks, \bar{x} + ks)$ when $E(P) = E(P') = a$, then, to within terms of order $1/n$, $\sigma_P^2 = a(1-a)/n$ and $\sigma_{P'}^2 = t_a^2 e^{-t_a^2}/\pi n$, where $t_a = k\sqrt{n/(n+1)}$. For a large sample of a given size, say $n \geq 100$, and $a = 0.99$, the efficiency ratio $\sigma_{P'}^2 / \sigma_P^2$ is 0.28.

**References:** Shewhart (1931), Gumbel (1935a).

**Citations:** Wald (1942), Wilks (1942), Scheffé (1943), Wald (1943), Wilks (1943), Robbins (1944a), Grant (1946), Kendall (1946), Bickerstaff (1947), Tukey (1947), Wilks (1948), Wolfowitz (1949).

* Aldanondo, I. (1942). Cuestiones resueltas (número 18). *Revista Matematica Hispano-Americana* (4) **2,** 232-241. (JFM 68, 282).

**Summary:** Let the random variable x have a distribution whose probability density function (p.d.f.) is $C\,x\,e^{-x^2/2}$, $0 \leq x \leq \infty$. The author finds that the normalizing constant C is 1, the median value of x is $\sqrt{\ln 4}$, its mean value is $\sqrt{\pi/2}$, its standard deviation is $\sqrt{(4-\pi)/2}$, and its mode is 1. For a random sample of size n from this distribution, let u be the smallest value (first order statistic) and v the largest value ($n^{th}$ order statistic). The author shows that the p.d.f. of u is $nue^{-nu^2/2}$ and its cumulative distribution function (c.d.f.) is $1-e^{-nu^2/2}$, while the p.d.f. of v is $nv[1-e^{-v^2/2}]^{n-1}e^{-v^2/2}$ and its c.d.f. is $[1-e^{-v^2/2}]^n$. The author shows that the median value of v is $v_n=\{2\ln[2^{1/n}/(2^{1/n}-1)]\}^{1/2}$, and its limiting value as $n\to\infty$ is $\lim_{n\to\infty} v_n = \sqrt{2\ln(n/\ln 2)}$; moreover, for any $\varepsilon > 0$, $\lim_{n\to\infty} P(v_n-\varepsilon \leq v \leq v_n+\varepsilon) = 1$. Now let $\varepsilon$ be a number with unspecified sign and let $v = v_n+\varepsilon$, with $\varepsilon = w/v_n$; the author shows that the c.d.f. of w is $(1/2)e^{w}$.

* Beard, L. R. (1942). Statistical analysis in hydrology. *Proceedings of the American Society of Civil Engineers* **68,** 1077-1088; discussion, **68,** 1458-1464, 1661-1669 and 1847-1849; **69** (1943), 299-305 and 995-1005. *Transactions of the American Society of Civil Engineers* **108** (1943), 1110-1121; discussion, 1122-1160.

**Summary:** On page 1077 the author gives the following synopsis: "The basic principles of the duration [cumulative frequency] curve are reviewed in this paper in such a manner as to outline the various ways in which it is properly used in hydrologic analyses. Some of the details and differentiations discussed are of small significance compared to inaccuracies existing in the greater part of the hydrologic data available today. However, the proper use of the duration curve is essential to a fundamental understanding of statistical analysis and, since proper use is no more difficult than improper use, the differentiations are deemed relevant from an engineering standpoint." On page 1088 he states the following conclusions: "(a) In order to represent best a distribution curve using a finite number of occurrences, the occurrences should not be plotted with equal spacing; (b) Following from conclusion (a), the plotting of series of occurrences above a base value as a complete series is not mathematically rigorous; (c) The use of the Gaussian normal distribution curve has in no way been proved to be inapplicable to a properly chosen rainfall or runoff series; (d) The duration curve can be used to determine the occurrence that probably will not be exceeded within a given period as well as that occurrence which will be exceeded (on the average) once within a given period; and (e) The 'maximum probable' flood, adequately defined, can be determined by use of the duration curve." The discussion includes contributions by L. Standish Hall, H. Allen Foster, Ralph W. Powell, E. J. Gumbel and others. Much of the discussion deals with the choice of plotting positions. Gumbel's contribution (Vol. 69, pp. 995-1005) is especially significant. On page 995 he notes that the author has raised two important questions: "(a) What are the adequate statistical theories available for distributions observed in hydrology; and, especially, which theoretical distribution must be chosen for the largest observed values; and (b) How should the observed frequencies be corrected to account for the small number of observations." On page 1005 he summarizes his contribution as follows: "The correction of the observed frequencies is important for all observations if their number is small and, only for the largest observations, if the number of observations is large. A general correction, which holds for all distributions and for any number of observations, does not exist. To obtain the corrected frequencies, the theoretical distribution corresponding to the observations must be known. Assuming that the corrected frequency of the $m^{th}$ observations is the probability of the most probable $m^{th}$ value, the corrected frequencies and return periods for any distribution and for any number of observations are obtained from . . . [equations given on page 1000]. The corrections vary for different distributions. It is safe to assume that the theory of the largest value (and not the normal distribution) holds for the flood discharges. The return periods of the most probable largest flood discharges are the recurrence intervals."

**References:** Yule (1911) [1917], Foster (1924), Tippett (1925), Goodrich (1927), Hazen (1930), Foster (1934), Slade (1934) [1936], Foster (1936), Gumbel (1941a,b, 1942c), Kimball (1942), Powell (1943).

**Citations:** Gumbel (1945a), Lane & Lei (1949).

* Boldrini, Marcello (1942). *Statistica. Teoria e Metodi.* A. Giuffrè. Milan. (JFM 68, 286-287). Fifth edition, 1968.

**Summary:** This book is divided into two parts, the first theoretical and the second methodological. Chapter II (pp. 33-80) [see note below] is a brief history of probability and statistics. Mention is made of the following authors whose work is relevant to the present study: Galilei, Simpson, D. Bernoulli, Euler, Lagrange, Laplace, Legendre, Gauss, Bessel, Cauchy, Quetelet, Herschel, Todhunter, Lexis, Edgeworth, Fechner, Galton, Bertrand, K. Pearson, Sheppard, Bowley, Yule, M. Greenwood, Gini, Pietra, Galvani, "Student", R. A. Fisher, E. S. Pearson, Neyman, Hotelling, M. G. Kendall, Wald, Bartlett, Snedecor, and Yates. Chapters VIII and IX (pp. 301-382) deal with the method of least squares and the arithmetic mean. Chapter X (pp. 383-472) deals with empirical means, including the arithmetic mean, the mode, the median, the power means, the geometric mean, and the harmonic mean. The author discusses the median in Sec. 13 (pp. 433-434). He gives a proof by Bonferroni of the proposition that the sum of the absolute deviations of a set of values from their median is less than from any other value. In Sec. 14 (pp. 434-439), he discusses calculation of the median from grouped data, and in Sec. 15 (pp. 439-440), he compares it with the arithmetic mean and the mode. Other quantiles, specifically the quartiles, deciles, and centiles are discussed in Sec. 16 (pp. 440-444). In Chapters XI and XII (pp. 473-702), the author discusses measures of dispersion. In Ch. XI, Sec. 25 (pp. 525-527) he notes that (for samples from a normal population) the standard error of the median is 1.253 times that of the mean, and that of each of the quartiles is 1.363 times that of the mean. In Ch. XII, Secs. 3-7 (pp. 648-670), he treats the mean absolute deviation (from the mean and from the median), the range, the mean absolute difference of Gini, measures of relative dispersion, and measures of concentration. Analytical measures of concentration due to Gini and to Bonferroni are discussed in Sec. 13 (pp. 685-693). Further discussion of the method of least squares is given in Ch. XIV, Sec. 9 (pp. 848-852), but only brief mention is made of alternative methods — minimizing the sum of absolute deviations (proposed by Laplace and Edgeworth; no mention is made of Boscovich) and minimizing the maximum deviation (which the author attributes to Tchebycheff). On pp. 1324-1325, in Table I, the author gives values of the standard errors of the principal statistical measures, including the median $M_e$, the quartiles $Q_1$ and $Q_3$, and the mean difference $\Delta_r$. He gives both general formulas and specific ones for the case of samples from a normal distribution. For samples of size m from a population whose variance is $\sigma^2$, $\text{var}(\Delta_r) = (0.8068)^2\sigma^2/m$, $\text{var}(M_e) = 1/4mB^2$, and $\text{var}(Q_1) = \text{var}(Q_3) = 3\sigma^2/16mB^2$, where B is the ordinate at the corresponding population quantile (median or quartile). For a normal population, $\text{var}(M_e) = (1.2533)^2\sigma^2/m$ and $\text{var}(Q_1) = \text{var}(Q_3) = (1.3626)^2\sigma^2/m$.

**Note:** All references to numbers of specific chapters, sections and pages are to those in the fifth (1968) edition, the only one the compiler has seen.

**References:** Legendre (1805) [1806], Gauss (1809), Laplace (1812), Quetelet (1835) [1869], 1846), Todhunter (1865), Bertrand (1889) [1907], Pearson (1895), Pareto (1897), Bowley (1901) [1920], Charlier (1910) [1920], Yule (1911) {[Yule & Kendall (1944)]}, Galvani (1927), de Finetti (1931b), d'Addario (1934b), Dominedò (1934), Mortara (1934a,b), Pietra (1935a,b), Cisbani (1938), Gini (1938, 1939), Pizzetti (1940), de Vergottini (1940), Zappa (1940), Davis (1941), Pizzetti (1941c), {Kendall (1943) [1946], Cramér (1946), Kendall (1946) [1955]}.

* Dodd, Edward L. (1942). Certain tests for randomness applied to data grouped into small sets. *Econometrica* **10,** 249-257. (MR 4, 108).

**Summary:** The author presents four simple tests of randomness of digits which he calls Order Test No. 1, Order Test No. 2, Range Test, and Replication Test; only the third of these is relevant to our study. The range test is based on the range of a set of k integers, the range r being defined as the difference between the greatest and the least integers in the set. Under the hypothesis of randomness in a decimal system, the author tabulates the probabilities that $r = 0, 1, 2, \ldots, 9$ for $k = 2, 3, 4, 5, 6$. He suggests that the randomness of a group of k-digit numbers be tested by means of a chi-square test comparing theoretical

and observed frequencies of the various values of r. Since the probability of range zero is quite small when $k > 2$, he advocates combining the numbers of range zero with those of range one and possibly also with those of range two to form a single class for the chi-square test.

**Citation:** Kendall (1946).

* Dudding, B. P.; Jennett, W. J. (1942). *Quality Control Charts*. Part 1 of a revision of B. S. 600: 1935, *The Application of Statistical Methods to Industrial Standardisation and Quality Control*. B. S. 600 R. British Standards Institution, London.

**Summary:** In this partial revision of an earlier publication by Pearson (1935), the authors describe, in considerable detail, quality control charts for attributes and for variables. Our interest lies mainly in their treatment of the use of the sample range in control charts for variables. On pp. 30-31 the authors write: "In the following sections . . . the variation in the measures of variation is considered, and attention drawn to a relation between the standard deviation $\sigma$ and the average range ($\bar{w}$) of small samples. When circumstances are such that an accurate estimate of $\sigma$ can be made from $\bar{w}$ (for example when results are available for a large number of small samples and values of s are not required) then the limits for the control chart for averages can obviously be calculated from the average value of range ($\bar{w}$). . . . The limits may be expressed as

$$\text{Inner limits } \bar{\bar{X}} - A'_{0.025} \times \bar{w} \text{ and } \bar{\bar{X}} + A'_{0.025} \times \bar{w}$$
$$\text{Outer limits } \bar{\bar{X}} - A'_{0.001} \times \bar{w} \text{ and } \bar{\bar{X}} + A'_{0.001} \times \bar{w}$$

The factors $A'_{0.025}$ and $A'_{0.001}$ are . . . given in Table 10, Appendix C, page 64, for samples containing 2 to 12 individuals." On pp. 32-33, they write: "Whilst in general the standard deviation is the better measure for variability in a product when the latter has to be judged by the variability in the samples, cases do arise when the less sensitive measure 'range' is the more convenient. It has already been pointed out that the property which commends the use of range is the ease with which it can be evaluated. This may assume importance when large numbers of samples have to be handled. It is however necessary to emphasize that in addition to being less accurate than the standard deviation, range [1] becomes less reliable as the size of the sample increases (samples should not contain more than 12 individuals); [2] is less accurate when the frequency distribution departs from the Gaussian form. . . . When the circumstances are such that $\sigma$ would be estimated from $\bar{w}$ the control chart limits for range may be expressed as:

$$\text{Inner limits at } D'_{0.025} \times \bar{w} \text{ and } D'_{0.975} \times \bar{w}$$
$$\text{Outer limits at } D'_{0.001} \times \bar{w} \text{ and } D'_{0.999} \times \bar{w}$$

[Obvious typographical errors in the probability levels have been corrected by the compiler.] The factors D′ are . . . given in Table 13A, Appendix 3 [page 67]." On p. 36 they write: "Again, if the circumstances are such that the range (w) in the sample results is recorded, rather than the standard deviation, then $\sigma$ can be estimated from the average of the values of range for the samples ($\bar{w}$) by dividing by the appropriate factor $d_n$ given in Table 13, Appendix C, page 67, where n is the number of individuals in *each* sample." Examples of the use of the range are given on pages 39-40 and 69-73. The median, the mid-point [midrange] and the mean deviation [from the mean, not from the median] are defined on page 52 and arithmetical examples are given on pp. 55-56.

**References:** Davies & Pearson (1934), Pearson (1935), Pearson & Haines (1935), Pearson [& Hartley] (1942).

**Citations:** Dodge (1933), Pearson & Hartley (1942, 1943), Rodgers (1944), Grant (1946), Egudin (1947), Stevens (1948).

* Evans, W. Duane (1942). The standard error of percentiles. *Journal of the American Statistical Association* **37,** 367-376. (MR 4, 103).

**Summary:** The author states (p. 367): "One of the drawbacks to the use of percentiles has been the lack of some general method by which the probable error due to chance deviation in sampling might be investigated. With respect to the median, the rule that the standard error is about one and one-quarter times the easily estimated standard error of the mean is well known. However, this rule is applicable only when the population distribution approximates the normal form. It is not valid for the highly skewed and flattened distributions in which incomes usually fall. In fact, in such cases the standard error of the median is generally substantially less than that of the mean." He develops an approximate formula for the standard deviation of the estimate of any given percentile based on a sufficiently large stratified sample from any parent population. For the special case of the median of a non-stratified sample, his formula reduces to the usual one [see, for example, Yule & Kendall (1937)].

**Reference:** Yule (1911) [Yule & Kendall (1937)].

* Fréchet, Maurice (1942). La médiane d'un petit nombre de variables aléatoires et sa dispersion. (English summary). *Revue de l'Institut International de Statistique* **10,** 139-151. (JFM 68, 292).

**Summary:** In his introduction (p. 139), the author states that the practicing statistician often finds the median easier to deal with in numerical calculations than the arithmetic mean, even though the mathematical theory of the median is more difficult than that of the arithmetic mean. The latter fact is not very important, since the theory can be worked out once for all, a task which the author undertakes. The English summary (p. 151) reads as follows: "Writer shows first that if independent variates X, Y, Z, . . ., in a finite number, n, have at least one probable value in common, the probable segment of the median of X, Y, Z, . . ., (n being odd), or of at least one of their median values (n being even), coincides with the probable segment (eventually reduced to a point) common to X, Y, Z, . . . . In addition, in the case of n being even, if M′M″ is the median segment of X, Y, . . ., the probable segments of M′ and M″ contain entirely the probable segment common to X, Y, . . . . In case there is no probable value common to X, Y, Z, . . ., the probable segment of their median (or of one of their median values at least) is between the extreme probable values of X, Y, Z, . . . and this interval may even be restricted. Various writers have already studied the asymptotic behaviour, when n is very large, of the median $M_n$ of an odd number of values $X_1, \ldots, X_n$ of a variate X, and of the variance of $M_n$. Mr. Fréchet here studies the behaviour in this same case of $M_n$ and of its dispersion, but starting from the very first values of n (and even for n being even, taking in this case for $M_n$ a specially chosen median value [the midpoint of the median segment]). He points out that the [cumulative] frequency curve of $M_n$, which — as we knew already — tends towards zero on the left of the probable value (or of the probable segment) of X, and tends towards unity on the right, tends to these limits without increasing on the left, and without decreasing on the right. We know too that the dispersion of $M_n$ tends towards zero with increasing n, if X has but one probable value. From the nature of the variation of the frequency distribution of $M_n$ with n, as established by writer, this new property of the dispersion of $M_n$ may be deduced, that it can never increase with increasing n, when it is constantly measured by the average deviation (or constantly by half the quartile deviation [i.e., by the semi-interquartile range]). And this holds good whether X has a probable segment or only one probable value."

* Gaede, Kurt (1942). Anwendung statistischer Untersuchungen auf die Prüfung von Baustoffen. *Bauingenieur* **23,** 291-296.

**Summary:** The author discusses the use of statistical methods in the testing of materials. As measures of central tendency, he mentions the median C and the mode D in addition to the arithmetic mean M, but says M is the most useful to engineers and bases his discussion upon it. As a measure of dispersion, he mentions only the standard deviation $\mu = \sqrt{(x-M)^2/N}$, and as a measure of relative variability, the

coefficient of variation $V = \mu/M$. He raises the question: What is the expected value of the smallest of a sample of size N from a normal distribution with given mean M and standard deviation $\mu$? He asserts [incorrectly] that the expected deviation of the smallest value K from the mean is max v = M − K, where max v satisfies the equation $(2/\sqrt{\pi}) \int_0^{\max v/\mu\sqrt{2}} e^{-t^2}\,dt = 1 - 1/N$, so that the expected smallest value is $K = M(1 - V \max v/\mu)$ and its ratio to the arithmetic mean is $x = K/M = 1 - V \max v/\mu$. He tabulates $\max v/\mu$, as well as $x = K/M$ for $V = \mu/M = 0.1, 0.2, 0.3$, when N = 10, 20, 40, 100, 500, 1000, 5000, 10000.

**References:** Helmert (1872) [1907], Czuber (1891a, 1921 [1938]).

* Gumbel, E. J. (1942a). Simple tests for given hypotheses. *Biometrika* **32,** 317-333. (MR 4, 26).

**Summary:** Let x be a continuous random variable for which n observations have been made. Let $x_m$ be the observed values arranged in increasing order of magnitude, m = 1, 2, . . ., n being the serial number. In testing the hypothesis that the observations have been drawn from a population with c.d.f. W(x), the author points out that it is customary to compare the observations $x_m$, m with the cumulative frequency curve x, nW(x) [or $x_m$, m/n with the cumulative probability curve x, W(x)]. In a previous article the author [Gumbel (1935a)] has shown that for an ordinary unlimited distribution with n large and m of the order n/2, the cumulative probability of the $m^{th}$ observation converges toward a normal distribution with mean W(x) = m/n and standard deviation $\sigma = [1/w(x)]\sqrt{W(x)[1-W(x)]/n}$, where w(x) = dW(x)/dx is the p.d.f. The interval $x \mp \sigma$ is called the control interval and the two curves obtained by plotting $x \mp \sigma$, nW(x) are called control curves. For a given initial distribution the mean and standard deviation of the $m^{th}$ value may differ from those of the general solution, especially if n is small. The calculation of the probability W(x) and the control curves is simplified by application of the probability integral transform y = W(x), which has the uniform distribution in the interval (0,1). After considering classical tests based on the uniform distribution, the author turns to the question of the ideal *plotting positions* for n points distributed uniformly over the interval (0,1). Should the plotting position for the $m^{th}$ ordered value be the modal position $\tilde{y}_m = (m-1)/(n-1)$, the mean position $\bar{y}_m = m/(n+1)$, or the position $\acute{y}_m = (2m-1)/2n$ suggested by E. S. Pearson (1942)? (Other positions have been suggested, but the author considers only these three.) He chooses the mean position $\bar{y}_m$, with standard deviation $\sigma_m = \sqrt{m(n-m+1)/(n+1)^2(n+2)} = \sqrt{\bar{y}_m(1-\bar{y}_m)/(n+2)}$. As a statistic for testing the null hypotheses $H_0$ that a set of n ordered values have come from a population with c.d.f. $W_0(x)$, the author calculates $\mathfrak{S}^2 = [6(n+1)/n] \sum_{m=1}^{n} [y_m - m/(n+1)]^2$ where $y_m = W_0(x_m)$, which can assume values between 0 and 2n+2 and has mean value $\overline{\mathfrak{S}^2} = 1$. Critical values of $\mathfrak{S}^2$ are not given, but of two competing hypotheses, the one with the smaller value of $\mathfrak{S}^2$ is to be preferred. This test is similar to the Cramér-von Mises $\omega^2$-test, which would require the use of $\acute{y}_m$ instead of $\bar{y}_m$.

**References:** Cramér (1928), Neyman & Pearson (1928), von Mises (1931), Pearson & Pearson (1931, 1932), Gumbel (1935a, 1937b, c), Neyman (1937a), Gumbel (1939a), Pearson (1942).

**Citations:** Pearson (1942), Gumbel (1943a), Kendall (1946), Kimball (1947).

* Gumbel, E. J. (1942b). On the frequency distribution of extreme values in meteorological data. *Bulletin of the American Meteorological Society* **23,** 95-105. (MR 4, 29).

**Summary:** The author begins (p. 95) as follows: "According to the usual opinions, the extreme values, and especially those which occur in meteorology, are so irregular that no prediction can be made of the maximum precipitation, extreme temperature, maximum pressure, etc., that will occur in a given period of time. It is our aim to show that, in reality, they follow a general law which can be verified under the same conditions as any statistical law, namely that the number of observations must be sufficiently large. As Tolley [(1916)] has already stated, these values deserve a special treatment. Because their distributions are skew, he proposed the use of the different Pearsonian frequency curves. We believe that the *distribution of the largest values* [Gumbel (1935a)], recently found, furnishes the adequate formula." After deriving the first asymptotic distribution of extreme (largest or smallest) values, he fits

this distribution to four sets of observations: (1) largest monthly precipitation in Boston for the years 1818-1940; (2) maximum yearly temperature at Parc Saint-Maur, France, for the years 1851-1939; (3) maximum yearly pressure at Blue Hill Observatory (near Boston) for the years 1886-1940; and (4) average monthly minimum temperature in Boston for the winters 1872/73-1939/40. He applies four tests of goodness of fit, and concludes that there is a satisfactory fit between theory and observation, even though the number of years of record is not as large as might be desired and some of the measurements are somewhat imprecise. Another conclusion, stated on p. 105, is the following: "The return period can safely be used for obtaining the order of magnitude of future extreme values."

**References:** Spillman, Tolley & Reed (1916), Tolley (1916), Gumbel (1935a), Horton (1936), Gumbel (1937b, 1940, 1941a).

**Citation:** Gumbel (1943b).

* Gumbel, E. J. (1942c). Statistical control-curves for flood-discharges. *Transactions of the American Geophysical Union* **23,** 489-500; discussion, 501-509. (MR 4, 166—listing only).

**Summary:** On page 489 the author summarizes his results as follows: "For sufficiently large and accurate numbers of observations, the distribution of the flood-discharges, x, can be satisfactorily explained by the theory of largest values [Gumbel (1941a)]. Until now we have determined the two constants which occur in this theory by the method of moments. This implies the calculation of the arithmetic mean and of the standard deviation. The purpose of this article is to simplify and, at the same time, improve the determination of the constants, to establish confidence limits for the flood-discharges, and to indicate several methods of comparison between theory and observation. The probability $W^*(x)$ of a flood-discharge inferior to x is $W^*(x) = \exp[-\exp(-y)]$ and the distribution $w^*(x)$ of the flood-discharges is $w^*(x) = \alpha \exp[-y-\exp(-y)]$ where $y = \alpha(x-u)$ is the reduced variable. The numerical values of the probability $w^*(x)$ as a function of y are given . . . . The formulas contain two constants, u and $1/\alpha$ which have to be determined from the observations. In accordance with the usual practice, we shall express them by a mean [the mode u, for which $W^*(u) = 1/e$] and a measure of dispersion [the mean deviation $\theta = 0.98299/\alpha$]. . . ." The author compares his new method of determining the constants with the method of moments, and shows that the determination of $1/\alpha$ from the mean deviation is 25 per cent more precise than its determination from the standard deviation, while the mode is 98 per cent as precise as the arithmetic mean. The discussion includes contributions by B. F. Kimball and R. S. Goodridge and a reply by the author. Part of the discussion deals with the question of plotting positions.

**References:** Fisher & Tippett (1928), Hazen (1930), Gumbel (1935a), Slade (1936), Kimball (1938), Gumbel (1941a,b), Kimball (1942).

**Citations:** Beard (1942), Kimball (1942), Gumbel (1943a,b), Nordquist (1945), Kimball (1946a), Gumbel (1948), Press (1949.)

* Hartley, H. O. (1942). The range in random samples. *Biometrika* **32,** 334-348. (MR 4, 21).

**Summary:** The probability integral of the range in random samples of size n is written in the form $P_n(W) = n\int_{-\infty}^{\infty} f(\xi)[\int_{\xi}^{\xi+W} f(x)dx]^{n-1} d\xi$. For a normal parent population $f(x) = z(x) = (1/\sqrt{2\pi})e^{-x^2/2}$. The author performs certain mathematical manipulations and sets $u = \xi + W$, obtaining $P_n(W) = [\int_{-W/2}^{W/2} z(x)dx]^n + 2n\int_{W/2}^{\infty} z(u)[\int_{u-W}^{u} z(x)\,dx]^{n-1}du$. The first term on the right-hand side of this equation is the probability that all observations will lie between $-W/2$ and $+W/2$, in which case the range $\leq W$. For large W such samples constitute an ever-increasing proportion of the total number of samples with range $\leq W$. The second term is the probability that a sample is not contained in $(-W/2, +W/2)$, but still has range $\leq W$. This term cannot be ignored if high accuracy is required and if W is small or moderate. The numerical integration is best carried out simultaneously for values of n forming an arithmetic progression. For fixed u and W, the integrands then form a geometric progression with, say, $z(u)[\int_{u-W}^{u} z(x)dx]^2$ as first term and $\int_{u-W}^{u} z(x)dx$ as common ratio. This is the method used by Pearson & Hartley (1942). —

Other sections of the paper deal with the range of grouped samples, which is not directly relevant to the present study.

**References:** Tippett (1925), E. S. Pearson (1926), K. Pearson (1931), E. S. Pearson (1932), McKay & Pearson (1933), McKay (1935), Pearson & Haines (1935), Pearson & Hartley (1942).

**Citations:** Pearson & Hartley (1942), Kendall (1943), Pearson & Hartley (1943), Cramér (1946), Kendall (1946), Elfving (1947), Gumbel (1947), Lord (1947), Wilks (1948).

* Henry, Franklin (1942). The practice and fatigue effects in the Sargent test. *Research Quarterly of the American Association for Health, Physical Education, and Recreation* **13** (1), 16-29.

**Summary:** The author's summary and conclusions (p. 28) include the following statements: "A simplified and improved recording device was used to obtain the Sargent jump test scores of 61 male university students. The data obtained, when analyzed as to mean performance for each trial of a 10-trial series, are found to show a practice effect. There is also evidence of a fatigue effect. The serial occurrence of 'best' scores confirms these findings. . . . Scores based on the average of several trials appear to be more representative of individual ability in the Sargent test than 'best' scores; they also tend to be more valid. Correlations between data obtained by the method of 'best' scores cannot be interpreted at their face value, as they seem to contain a spurious factor. . . ." The author's conclusions are based entirely on empirical evidence, not on statistical theory.

* Kimball, Bradford F. (1942). Limiting type of primary probability distribution applied to annual maximum flood flows. *Annals of Mathematical Statistics* **13,** 318-325. (MR 4, 29—listing only).

**Summary:** The author begins with the following statement (p. 318): "There is no doubt that Gumbel's recent paper [Gumbel (1941a)] has supplied an admirably simple technique for engineers to use in approximating the trend of *return periods* of annual maximum flood flows for purposes of extrapolation. This treatment is scientifically of great interest because it introduces for the first time into a subject already treated at considerable length by engineers, the theory of the probability distribution of maximum values as developed by Fisher and Tippett, von Mises, and others. However, certain further observations should be made concerning the approach used by Gumbel." At the end of the paper, he states the following conclusion (pp. 324-325): "The writer has demonstrated in this paper that in fitting a theoretical probability distribution of maximum values to annual maxima of stream flows, the use of an upper bound for measures of stream flow by assumption of a primary probability distribution of the [*limited* Galton] type [$w(x) = K \exp(-u^2/2)$, where K is a constant and $u = k\{b - \log(a-x)\}$, $0 \le x \le a$] (1) is not inconsistent with the use of the Fisher-Tippett distribution of maxima, (2) has a reasonable logical basis from the point of view of the hydrologist, (3) may materially affect the estimation of return periods when extrapolation is involved, relative to results obtained when no upper bound is assumed. It has not been within the scope of this paper to discuss techniques for determining such an upper bound, nor to apply the theory to enough data series to draw conclusions concerning goodness of fit."

**Comments:** Gumbel (1958a) has pointed out that the author's suggested limited initial distribution leads to the third asymptotic distribution of largest values.

**References:** Fisher & Tippett (1928), Hazen (1930), von Mises (1936), Kimball (1938), Gumbel (1941a,b, 1942c).

**Citations:** Beard (1942), Gumbel (1942c, 1943a,b), Kimball (1946a).

* Livada, Gregorio (1942). Su alcune ordinate caratteristiche della curva di concentrazione. *Atti della III Riunione Scientifica della Società Italiana di Statistica*, pp. 25-34. (JFM 68, 290).

**Summary:** Given $a_1 \le a_2 \le a_3 \le \ldots \le a_n$; let $A_i = a_1 + a_2 + \ldots + a_i$ and $A_n = n \cdot A$, where A is the arithmetic mean of the $a_i$'s ($i = 1, 2, \ldots, n$). Then the concentration curve connects the points with coordinates $p_i = i/n$, $q_i = A_i/A_n = A_i/nA$. It lies below the straight line $p - q = 0$, from which the distances of successive points on the curve are $p_i - q_i$ ($i = 1, 2, \ldots, n$). The author proves the equality $\mathrm{Max}(p_i - q_i) = {}^1S_A{:}2A$, where the symbol ${}^1S_x$ denotes the expression $\Sigma_{i=1}^{n} |a_i - x|/n$. If one denotes by M the median of the sequence $a_1, a_2, \ldots, a_n$ then the author proves further that the distance ${}^1S_M{:}2A$ (from the line $p - q = 0$) corresponds to the value $p = 1/2$. The author gives various other properties of the concentration curve.

**References:** Pietra (1915), Gini (1939).

* Moore, H. F.; Morkovin, D. (1942-44). Progress reports on the effect of size of specimen on fatigue strength of three types of steel. *Proceedings of the American Society for Testing Materials* **42** (1942), 145-153; **43** (1943), 109-120; discussion, 121-124; **44** (1944), 137-155; discussion, 156-158.

**Summary:** The authors report the results of tests of fatigue strength on specimens of two carbon steels and one heat-treated chromium-molybdenum steel. Five different sizes of specimens were tested for each kind of steel, with minimum test diameters of 1/8, 1/4, 1/2, 7/8 or 1 in., and 1-1/2, 1-3/4 or 2 in., respectively. The results show a maximum size effect for unnotched specimens of about 30% of the endurance limit for 2-inch specimens of S.A.E. 1020 steel, with a tendency for the endurance limit to become constant for unnotched specimens of 1/2 in. minimum diameter and larger and for notched specimens 1 in. in diameter or larger. The authors compare their results with those of Faulhaber (1933) and Faulhaber, Buchholtz & Schulz (1933).

***Comments:*** *The results agree qualitatively with* what one would expect from the "weakest link" or "largest flaw" theory.

**References:** Faulhaber (1933), Faulhaber, Buchholtz & Schulz (1933).

**Citations:** Fowler (1945), van Meer & Plantema (1949).

* Nair, K. R.; Shrivastava, M. P. (1942). On a simple method of curve fitting. *Sankhyā* **6,** 121-132. (MR 4, 279).

**Summary:** The authors consider the problem of fitting a mathematical curve $y = f(x, \alpha, \beta, \gamma, \ldots)$ to the data points $(x_1,y_1)$, $(x_2,y_2)$, $\ldots$, $(x_n,y_n)$, where $x_1 \le x_2 \le \ldots \le x_n$. They discuss several methods of estimating the parameters $\alpha, \beta, \gamma, \ldots$. They start by forming the residuals $v_i = y_i - f(x, \alpha, \beta, \gamma, \ldots)$, $i = 1, 2, \ldots, n$. If the v's are normally distributed with zero means and the same standard deviation, the method of maximum likelihood is equivalent to the method of least squares; otherwise, minimizing $\Sigma v^2$ is only a matter of convenience. Next the authors consider the method of averages [see Euler (1749) and Mayer (1750)], in which the residual equations are divided into as many groups (m, say) as there are parameters to be estimated and the sum of the residuals in each group is equated to zero, yielding m linear equations in m unknowns, whose unique solution is easily obtained. Let $p_j$ ($j = 1, \ldots, m$) be the number of equations in the $j^{th}$ group. In the method of averages, $\Sigma_{j=1}^{m} p_j = n$. A modification proposed by the authors, called the method of group averages, allows the possibility of omitting some of the residual equations, so that we no longer have the equality $\Sigma_{j=1}^{m} p_n = n$, but the inequality $\Sigma_{j=1}^{m} p_j \le n$. The method of selected points ($p_j = 1, j = 1, \ldots, m$) and the method of averages ($\Sigma_{j=1}^{m} p_j = n$) are extreme cases of the method of group averages. The three methods proposed by Bose (1938) are special cases of either the method of averages or the method of group averages. The object of the present authors is to find the best values of the $p_j$'s so as to maximize the efficiencies of the estimates of $\alpha, \beta, \gamma, \ldots$ relative to those obtained by the method of least squares. Since it is impossible to maximize all these efficiencies simultaneously, they actually maximize the efficiency of the coefficient of the highest degree term in equations of straight lines, parabolas, and curves of higher degree, and make a detailed study of the efficiencies of the resulting estimates. In fitting a straight line $y = \alpha + \beta x$, for example, they find that the best estimate of $\beta$

(but not of $\alpha$) is obtained by assigning the first third (to the nearest integer) of the residual equations to the first group and the last third to the second group, then discarding the remaining ones in the middle. Similarly, for a parabola, they divide the whole set of residual equations into five equal parts and reject the second and fourth ones; for a curve of degree p, they divide the whole set of residuals into $2p + 1$ equal parts and reject all the even ordered ones.

**References:** Rietz (1924), Bose (1938).

**Citations:** Nair & Banerjee (1943), Kendall (1946), Mosteller (1946), Bartlett (1949.)

* Pearson, E. S. (1942). Notes on testing statistical hypotheses. *Biometrika* **32,** 311-316. (MR 4, 26).

**Summary:** The author examines two questions of general interest: (1) Is it necessary to specify an alternative hypothesis, or can a test be made to depend only on the form of the law associated with the null hypothesis? (2) Is the Neyman-Pearson theory of testing hypotheses applicable to testing the appropriateness of probability laws or only to testing hypotheses concerning numerical values of constants contained in those laws? More relevant to the present study is the author's proposal that, in testing the hypothesis that a set of n independent values of a random variable x have come from a population with c.d.f. $F(x)$, one should apply the probability integral transformation $y = F(x)$, and then test the hypothesis that the resulting sample of n independent random variables $y_1, y_2, \ldots, y_n (0 \leq y_i \leq 1)$ have come from the rectangular population for which $p(y) = 1, (0 \leq y \leq 1)$. This he proposes to do by comparing the set of $y_i$ (ordered in ascending magnitude) with the set $1/2n, 3/2n, 5/2n, \ldots, (2n - 1)/2n$. This, of course, corresponds to the use of plotting position $(2i - 1)/2n$, $i = 1, 2, \ldots, n$, for the $i^{th}$ order statistic of a sample of size n.

**References:** Neyman (1937a), Gumbel (1942a).

**Citations:** Gumbel (1942a), Kendall (1946).

* Pearson, E. S.; Hartley, H. O. (1942). The probability integral of the range in samples of n observations from a normal population. *Biometrika* **32,** 301-310. (MR 4, 19).

**Summary:** Let $x_1, x_2, \ldots, x_n$ be a random sample of size n, arranged in ascending order of magnitude, from a normal population with mean $\mu$ and standard deviation $\sigma$. Then the range of the sample is $x_n - x_1$ and its ratio to the standard deviation is $w = (x_n - x_1)/\sigma$. The probability integral $P_n(W) = \int_0^W f_n(w)dw$ is tabulated (4 DP) for n = 2(1)20 and for intervals of 0.05 in W. Linear interpolation is adequate except in the neighborhood of the upper and lower quartiles. In a separate table, the upper and lower percentage points at 0.1%, 0.5%, 1.0%, 2.5%, 5%, and 10% are tabulated (2 DP) for n = 2(1)12, along with the value of the factor $a_n$ (4 DP) by which the range (or mean range) in samples of n observations must be multiplied to obtain an unbiased estimator of $\sigma$. The foreword was written by the first author. In a separate section (pp. 309-310) by the second author, the numerical evaluation of the probability integral [see also Hartley (1942)] is discussed.

**References:** Tippett (1925), E. S. Pearson (1926), K. Pearson (1931), E. S. Pearson (1932), McKay & Pearson (1933), Davies & Pearson (1934), Pearson (1935), Dudding & Jennett (1942), Hartley (1942).

**Citations:** Dudding & Jennett (1942), Hartley (1942), Kendall (1943), Pearson & Hartley (1943), Gumbel (1944), Cramér (1946), Daly (1946), Grant (1946), Kendall (1946), Mosteller (1946), Eisenhart, Hastay & Wallis (1947), Elfving (1947), Grubbs & Weaver (1947), Gumbel (1947), Lord (1947), Nair (1947), Plackett (1947), Cox (1948), Stevens (1948), Wilks (1948), Cox (1949), Godwin (1949b), Howell (1949), Paulson (1949), Scheffé (1949).

* Roy, S. N. (1942a). The sampling distribution of p-statistics and certain allied statistics on the non-null hypothesis. *Sankhyā* **6,** 15-34. (MR 4, 106).

**Summary:** Let S′ and S″ be samples of sizes n′ and n″ drawn at random from two p-variate normal populations $\Sigma'$ and $\Sigma''$. Let $\alpha'_{ij}$, $\alpha''_{ij}$ (i, j = 1, 2, . . ., p) denote the sets of variances and covariances of $\Sigma'$ and $\Sigma''$ respectively, so that $\|\alpha'_{ij}\|$, $\|\alpha''_{ij}\|$ are the dispersion matrices for $\Sigma'$ and $\Sigma''$ respectively. Similarly, let $a'_{ij}$, $a''_{ij}$ (i,j = 1, 2, . . ., p) denote the sets of variances and covariances for the samples S′ and S″ respectively, so that $\|a'_{ij}\|$, $\|a''_{ij}\|$ respectively are the dispersion matrices for the samples. Let $\alpha'_i$, $\alpha''_i$ (i = 1, 2, . . ., p) denote the means of the different characters for the populations $\Sigma'$ and $\Sigma''$ respectively, and let $a'_i$, $a''_i$ (i = 1, 2, . . ., p) denote the corresponding sample means. The author derives the joint distribution of the ordered roots in the determinantal equation $|a'_{ij} - k^2 a''_{ij}| = 0$ in $k^2$. The roots of this determinantal equation are important in discriminating between multivariate normal populations with respect to covariances. Fisher (1939) and Hsu (1939) have previously determined the joint distribution of ordered roots under the null hypothesis $\|\alpha'_{ij}\| = \|\alpha''_{ij}\|$. The author extends their results by determining this distribution under the alternative hypothesis $\|\alpha'_{ij}\| \neq \|\alpha''_{ij}\|$.

**References:** Fisher (1939), Hsu (1939), Roy (1939, 1940a,b).

**Citations:** Roy (1942b, 1945), Kendall (1946), Roy (1946a,b), Bartlett (1937b).

* Roy, S. N. (1942b). Analysis of variance for the multivariate normal populations: the sampling distribution of the requisite p-statistics on the null and non-null hypotheses. *Sankhyā* **6,** 35-50. (MR 4, 106).

**Summary:** The author extends previous results of himself and other authors on the null and non-null distributions of ordered roots of determinantal equations arising in the case of two samples from p-variate normal populations to the case of $\ell$ samples. The results are useful in the analysis of variance for multivariate normal populations. Suppose we have $\ell$ samples S(1), S(2), . . ., S($\ell$) of sizes $n_1, n_2, \ldots, n_\ell$, mean values $\bar{x}(1,i), \bar{x}(2,i), \ldots, \bar{x}(\ell,i)$ and variances and covariances a(1,ij), a(2,ij), . . ., a($\ell$,ij) (i,j = 1, 2, . . ., p; the first argument referring to the sample and the next ones to the character) drawn at random from $\ell$ p-variate normal populations with mean values $\xi(1,i), \xi(2,i), \ldots, \xi(\ell,i)$ (i = 1, 2, . . ., p; the first argument referring to the population and the next one to the character) and a common set of variances and covariances $\alpha''(ij)$ (i,j = 1, 2, . . ., p). To test the null hypothesis $\xi(1,i) = \xi(2,i) = \ldots = \xi(\ell,i)$ (i = 1, 2, . . ., p), Fisher (1939) and Hsu (1939) constructed a set of statistics $t_1^2, t_2^2, \ldots, t_p^2$ which came out as the p roots of the determinantal equation $|a'(ij) - t^2 a''(ij)| = 0$ in $t^2$, where a′(ij) and a″(ij) are defined by $\Sigma_{r=1}^{\ell} (n_r - 1)\, a(r,ij) = (N - \ell)\, a''(ij)$, $\Sigma_{r=1}^{\ell} n_r [\bar{\alpha}(r,i) - \bar{\alpha}_i][\bar{\alpha}(r,j) - \bar{\alpha}_j] = (\ell - 1)\, a'(ij)$, N and $\bar{\alpha}_i$ (i = 1, 2, . . ., p) being given by $N = \Sigma_{r=1}^{\ell} n_r$; $\bar{\alpha}_i = \Sigma_{r=1}^{\ell} n_r \bar{\alpha}(r,i)/N$ with i,j = 1, 2, . . ., p. They found that the ordered $t_i$'s (i = 1, 2, . . ., p) have the same joint distribution under the null hypothesis as the set of $k_i$'s (i = 1, 2, . . ., p) in the determinantal equation $|a(1,ij) - k^2 a(2,ij)| = 0$ in $k^2$ have under the null hypothesis $\alpha(1,ij) = \alpha(2,ij)$. In the present paper, the author finds the joint distribution of the ordered $t_i$'s under the non-null hypothesis $\xi(1,i) \neq \xi(2,i) \neq \ldots \neq \xi(\ell,i)$ (i = 1, 2, . . ., p). The result may be viewed as a generalization both of the results of Fisher (1939) and Hsu (1939) and of the author's earlier results [Roy (1939, 1942a)].

**References:** Fisher (1939), Hsu (1939), Roy (1939, 1940a, 1942a).

**Citations:** Anderson (1945), Roy (1945), Anderson (1946), Kendall (1946), Roy (1946a,b), Bartlett (1947a,b).

* Wald, Abraham (1942). Setting of tolerance limits when the sample is large. *Annals of Mathematical Statistics* **13,** 389-399. (MR 4, 165).

**Summary:** The author considers the problems of setting tolerance limits on the population (or an infinitely large second sample) [Problem 1] or on a second sample of finite size [Problem 2] on the basis of a first sample of size n from a p-variate population. On pp. 389-390 he writes: "S. S. Wilks [(1941)] gave a solution of Problems 1 and 2 in the univariate case, i.e. if p = 1. It seems that Wilks' solution [based on order statistics] is the best possible one if nothing is known about the probability density function except

that it is continuous. However, if it is known a priori that the unknown density function is an element of a k-parameter family of functions, it will in general be possible to derive tolerance limits [not based on order statistics] which are considerably better than those proposed by Wilks. Wilks' results can easily be extended to the multivariate case provided the variates $x_1, \ldots, x_p$ are known to be independently distributed. [See Wilks (1942).] This is a serious restriction, since in many practical cases the independence of the variates $x_1, \ldots, x_p$ cannot be assumed. The case of dependent variates has not been treated by Wilks. In this paper we give a solution [based on normal theory, not on order statistics] of problems 1 and 2 when the size n of the [first] sample is large." The author considers both the univariate and the multivariate cases for dependent variates.

**References:** Wilks (1941, 1942).

**Citations:** Wilks (1942, 1943, 1948).

* Wilks, S. S. (1942). Statistical prediction with special reference to the problem of tolerance limits. *Annals of Mathematical Statistics* **13,** 400-409. (MR 4, 165).

**Summary:** In his introduction, the author writes (p. 401): ". . . We may define $100R_\alpha\%$ *tolerance limits* $L_1(x_1, x_2, \ldots, x_n)$ *and* $L_2(x_1, x_2, \ldots, x_n)$ *for probability level* $\alpha$ of a sample $S_1$ of size n from a population with distribution $f(x)dx$ as two functions of the X's in $S_1$ such that the probability is $\alpha$ that at least $100R_\alpha\%$ of the X's of a further indefinitely large sample $S_2$ (i.e. the population) will lie between $L_1$ and $L_2$. Or more briefly $P(\int_{L_1}^{L_2} f(x)dx \geq R_\alpha) = \alpha$. The same notion clearly applies if $S_2$ is a finite sample of size N, rather than an indefinitely large one. In this case we would be interested in the largest integer $N_\alpha$ such that the probability is at least $\alpha$ that at least $100\overline{R}_\alpha\%$ ($\overline{R}_\alpha = N_\alpha/N$) of the X's in $S_2$ would lie between $L_1$ and $L_2$. In most practical situations we are able to assume nothing more about f(x) than it is a probability density function. We make only that assumption here. The only functions of the values of X in $S_1$ that we shall consider here in setting tolerance limits are *order statistics*, i.e. the ordered values of X, because the results will then be fairly simple and independent of f(x)." At this stage the author derives a general probability formula. Let $X_1, X_2, \ldots, X_n$ be the n values of X in $S_1$ arranged in order of increasing magnitude. Let $r_1, r_2, \ldots, r_k$ be integers such that $1 \leq r_1 < r_2 < \ldots < r_k \leq n$. The author shows that the probability that $N_1, N_2, \ldots, N_{k+1}$ ($\sum_{i=1}^{k+1} N_i = N$) of the values of X in $S_2$ will fall in the intervals $(-\infty, X_{r_1})$, $(X_{r_1}, X_{r_2}), \ldots, (X_{r_k}, \infty)$ respectively is $[N!n!(N_1 + r_1 - 1)!(N_2 + r_2 - r_1 - 1)! \ldots (N_k + r_k - r_{k-1} - 1)!(N_{k+1} + n - r_k)!] / [(r_1 - 1)!(r_2 - r_1 - 1)! \ldots (r_k - r_{k-1} - 1)!(n - r_k)!(N + n)! N_1!N_2! \ldots N_{k+1}!]$, which is clearly independent of f(x). Further results are mentioned in the author's summary (pp. 408-409): ". . . Let $X_1$ be the smallest and $X_n$ the largest value of X which occurs in the initial sample. Now consider a further sample of size N. The following problems of prediction relating to the second sample from information yielded by the initial sample are considered: (1) What is the probability that at least $N_0$ values of X in the second sample will exceed the *tolerance limit* $X_1$ set by the first sample? (2) What is the probability that at least $N_0$ values of X in the second sample will lie between the two *tolerance limits* $X_1$ and $X_n$ set by the first sample? (3) For given values of n and N and $\alpha$ (e.g., .99 or .95), what is the largest integer $N_\alpha$ such that the probability is at least $\alpha$ that $N_0 \geq N_\alpha$? (4) What is the limiting value of $N_\alpha/N = \overline{R}_\alpha$ as N increases indefinitly? Tables of values of $N_\alpha$ and $\overline{R}_\alpha$ are given for each of the two problems (1) and (2), for several important combinations of values of n and N and for $\alpha = .99$ and .95. Problems similar to (1), (2) and (3) are discussed for the case in which tolerance limits are placed on two or more quality characteristics simultaneously. . . ."

**References:** Shewhart (1931), Wilks (1941), Wald (1942).

**Citations:** Wald (1942), Scheffé (1943), Wilks (1943), Grant (1946), Bickerstaff (1947), Murphy (1948), Wilks (1948), Wolfowitz (1949).

* Barnard, G. A. (1943). The Use of the Median in Place of the Mean in Quality Control Charts. Technical Report No. Q. C./R/2. (British) Ministry of Supply Advisory Service on Quality Control.

**Note:** This report is undoubtedly relevant to our subject, but the compiler has been unable to obtain a copy, and has no information as to its contents other than that given in the title.

* Barricelli, Nils Aall (1943). Les plus grands et les plus petits maxima ou minima annuels d'une variable climatique. *Archiv for Matematik og Naturvidenskab* (Oslo) **46** (6), 155-194.

**Summary:** "In this work the quantities, varying as a function of time, which are usually measured at a weather station in order to obtain a description of the local climate (as for example the air temperature, the atmospheric pressure, etc.) are called 'climatic variables'. I propose to set forth here a method of calculating theoretically the largest and the smallest annual maxima (or minima) of a climatic variable in a period of many years. In this search I base my results entirely on the work of E. J. Gumbel [(1934f, 1935d)] on the extremes of a statistical variable. . . ." (Compiler's translation of author's introduction, p. 155). The author finds excellent agreement between the largest and smallest annual maxima and minima of temperature and pressure over a period of many years, at various points in Norway, and the values predicted by Gumbel's theory of extreme values.

**References:** Fechner (1897), Birkeland (1928), Gumbel (1934f), Birkeland & Frogner (1935), Gumbel (1935d).

* Craig, Allen T. (1943). A note on the best linear estimate. *Annals of Mathematical Statistics* **14,** 88-90. (MR 4, 280).

**Summary:** The author writes (Introduction, p. 88): "Let the chance variable x be subject to the distribution function D(x) and as usual let E[g(x)] denote the mathematical expectation of the function g(x). If $x_1, x_2, \ldots, x_n$ constitute a sample [ordered so that $x_1 \leq x_2 \leq \ldots \leq x_n$] of n independent values of x, the function $y = c_1x_1 + c_2x_2 + \ldots + c_nx_n$ is frequently called the best linear estimate of E(x) when the c's are so chosen that E(y) = E(x) and $E[y - E(x)]^2 = \sigma_y^2$ is a minimum. It is the purpose of this note to give an example of an estimate y, best in the sense defined, yet such that, y′ being another estimate, $Pr[E(x) - \delta \leq y \leq E(x) + \delta] \leq Pr[E(x) - \delta \leq y' \leq E(x) + \delta]$, for every $\delta > 0$." The author's example involves estimation of the parameter $\theta$ of a uniform distribution of x in the interval zero to $2\theta > 0$. The best estimate of the class considered is $y = (n + 1)\, x_n/2n$. The author compares y with the estimate $\omega = (x_1 + x_n)/2$. He states that Pitman [1937c)] has proved that for every $\delta > 0$, $Pr[E(x) - \delta \leq \omega \leq E(x) + \delta]$ exceeds the probability that any other estimate, including y, will fall in this interval of length $2\delta$ about the mean a/2.

**Comments:** The reviewer (J. Neyman) notes that there seems to be some misunderstanding concerning this result, since, for the actual distributions of y and $\omega$ given by the author, it is easy to show that, contrary to the assertion, $P\{|\omega - \theta| < \delta\} < P\{|y - \theta| < \delta\}$ for all values of $\delta > 0$, except those within an interval $0 < \delta < a < \theta/n$. The author's incorrect result is based on a misinterpretation of a result given by Pitman (1937c), who showed that the midrange $(x_1 + x_n)/2$ is the closest estimate of the parameter a of the rectangular population $f(x) = 1/c$, $a - c/2 \leq x \leq a + c/2$, even if c is known [but not if c is a known multiple of a (Craig's example)].

**References:** Fisher (1922), Pitman [(1937c)].

* Curtiss, J. H. (1943). Convergent sequences of probability distributions. *American Mathematical Monthly* **50,** 94-105. (MR 4, 248).

**Summary:** The author proves a convergence test, related to the central limit theorem, for sequences of probability distributions. He applies his results to binomial, Poisson, Gamma and Fisher z distributions and to distributions of linear forms and of positional means. The final section, on the distribution of positional means, deals specifically with the median, but the author points out that it is equally

applicable to other quantiles. He writes (p. 105): "... The median in large samples drawn by simple random sampling from a continuous parent distribution is approximately normally distributed with mean $\tilde{\mu}$ and variance $1/\{4[F'(\tilde{\mu})]^2(2n+1)\}$ [where $\tilde{\mu}$ is the population median and $2n+1$ is the sample size]. If applied to the first and third quartiles of a sample of $4n+1$ observations, our method yields the results that these statistics in large samples from a continuous parent distribution are approximately normally distributed with respective means $q_k$ and variances $3/\{64n+16)[F'(q_k)]^2\}$, $k = 1, 3$, where $q_1$ and $q_3$ are the first and third quartiles of the [parent] distribution. ... If Z has the Fisher z-distribution ..., and if m and n are even integers, then an integration by parts will show that $P(a \leq e^{2Z} < \infty)$ is equal to the sum of the first m/2 terms, and $P(-\infty < e^{2Z} \leq a)$ is equal to the sum of the last n/2 terms, of the binomial expansion of $(q+p)^{(m+n-2)/2}$, with $p = ma/(n+ma)$, $q = 1 - p$. The distribution of $[F =]e^{2Z}$ has been tabulated by Snedecor . . ., and this provides a means of determining the points of the distribution of $W_r$ [the sample median] in terms of those of F(w). For example, in the case of the median of a sample of $N = 23$, we find that if $H_{12}(w) = .95$, then $F(w) = .66$; that is, the 5 percent point of the exact distribution of the median is the 34 percent point of the distribution of the variates $X_j$ [the parent distribution]."

**Reference:** Fisher & Yates (1938).

**Citation:** Eisenhart, Deming & Martin (1948a).

* Deming, W. Edwards (1943). *Statistical Adjustment of Data*. John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London.

**Summary:** The author writes (Preface, p. iv): "Different kinds of problems of adjustment (e.g., geodesy on the one hand and curve fitting on the other) are here unified and brought under one general principle and one solution. The distinctions between different kinds of problems are left where they belong, namely, in the conditions that the adjusted values are subjected to. . . . Unfortunately and inadvertently, intellectual gulfs have grown up between writers in statistics, least squares, and curve fitting. Each of the three groups has gone its own way, rediscovering developments long since discovered by the others, or—what is worse—not rediscovering them. Here the reader will find contributions from all three groups, and he will perceive that they are complementary." Besides giving a thorough discussion of the method of least squares, not only when only one variable is subject to error and all observations are equally weighted but when one of these conditions is violated, he discusses briefly various other methods which can be used in place of the method of least squares or to find a preliminary approximate curve. Let $F(x,y; a,b,c) = 0$ be the function to be fitted, and let $F_0 = F(X,Y; a_0,b_0,c_0)$, where $a_0$, $b_0$, $c_0$ are approximate values of a, b, and c and $F_0$ is a small quantity. The author writes (pp. 137-138): "In regard to the matter of arriving at the approximations $a_0$, $b_0$, $c_0$, it should be made clear at the outset that in practice this is usually not difficult. Often one will have good enough approximations simply from previous experience. There are graphical methods, by which one draws in a curve free-han. . . . There is the 'method of averages', called by Norman Campbell [(1920)] the 'method of zero sum', by which one finds what values of a, b, and c will force the calculated curve to average out correctly over groups of points (three groups if there are three parameters). Then there is Cauchy's method, which has much to recommend it. Lastly, there is the method of selected points . . . . By this method one simply selects three points—usually two end points and a middle point, if there are three parameters—and solves three simultaneous equations to find what values of a, b, and c force the calculated curve to pass through these points. If there are two parameters, two points are selected, and so on. . . . The method of selected points is much used and easily justified on grounds of simplicity, yet it is about the worst conceivable method of curve fitting. If the computer is not careful to select representative points, he throws away practically all the information contained in the rest of the points. . . ." A brief discussion of rejection of observations is given.

**References:** Mayer (1750), Gauss (1809, 1823), Czuber (1891a), Pizzetti (1892), Campbell (1920), Keynes (1921), Whittaker & Robinson (1924), Shewhart (1931), Wald (1940).

**Citations:** Snedecor (1937) [1946], Stevens (1948).

* Gnedenko, B. (1943). Sur la distribution limite du terme maximum d'une série aléatoire. *Annals of Mathematics* (2) **44** (3), 423-453. (MR 5, 41).

**Summary:** Let $x_1, x_2, \ldots$ be a sequence of independent random variables all subject to the same c.d.f. $F(x)$ and let $\xi_n = \max(x_1, x_2, \ldots, x_n)$. One can easily see that the c.d.f. of $\xi_n$ is $F_n(x) = P\{\xi_n < x\} = F^n(x)$. The author notes that much work has been done on the limiting distributions of $F_n(x)$ for large values of n (asymptotic distributions of extreme values). In particular, Fréchet (1927) found laws which can be limits for $F_n(a_n x)$ for a convenient choice of positive constants $a_n$, viz. $\Phi_\alpha(x) = 0, x \leq 0, \Phi_\alpha(x) = \exp(-x^{-\alpha})$, $x > 0$ and $\Psi_\alpha(x) = \exp[-(-x)^\alpha], x \leq 0, \Psi_\alpha(x) = 1, x > 0$, where $\alpha$ denotes a positive constant. Fisher & Tippett (1928) showed that limiting laws for $F_n(a_n x + b_n)$, where $a_n > 0$ and $b_n$ are conveniently chosen real constants, reduce to laws of types $\Phi_\alpha(x), \Psi_\alpha(x)$ and the law $\Lambda(x) = \exp(-e^{-x})$, and von Mises (1936) found sufficient conditions for the convergence of the laws $F_n(a_n x + b_n)$ to each of the three types, for a certain fixed choice of the constants $a_n > 0$ and $b_n$. The author points out, however, that these authors have not completely solved the fundamental problems concerning the limiting distributions of the maximum of a random series. He proceeds to prove that the class of limit laws for the maxima contains only the three indicated types. He also gives necessary and sufficient conditions for the domains of attraction of each of the limiting types. These conditions are definitive for the laws $\Phi_\alpha(x)$ and $\Psi_\alpha(x)$, but not for the law $\Lambda(x)$. In addition, he gives necessary and sufficient conditions for the law of large numbers and for the relative stability of maxima. He points out that the results for iimiting distributions of the maximum of a random series can easily be extended to the minimum, since if $\eta_n = \min(x_1, x_2, \ldots, x_n)$, then $-\eta_n = \max(-x_1, -x_2, \ldots, -x_n)$.

**References:** Fréchet (1927), Fisher & Tippett (1928), de Finetti (1932), von Mises (1936).

**Citations:** Gartštein (1948), Juncosa (1949), Meĭzler (1949), Smirnov (1949b).

* Gumbel, E. J. (1943a). On serial numbers. *Annals of Mathematical Statistics* **14,** 163-178. (MR 4, 280).

**Summary:** The author points out that, for an ordered sample of size n the cumulative distribution may be represented either by the step function $(m, x_m)$ or the step function $(m - 1, x_m)$, $m = 1, 2, \ldots, n$, depending upon whether the observations are arranged in order of increasing or decreasing magnitude. He shows that the differences between the two ways of plotting play an important role in certain graphical methods used by engineers. In plotting on probability paper, one uses the reduced ranks m/n instead of the ranks m. If the plotting position of the $m^{th}$ point is taken to be m/n, the last point (corresponding to the largest observation) cannot be plotted; if it is taken as $(m - 1)/n$, the first point cannot be plotted. To avoid these difficulties, the compromise position $(m - 1/2)/n$ has been widely used. This may be acceptable for comparison between the sample c.d.f. and the theoretical population c.d.f. {the continuous curve $[F(x), x]$}, but it leads to serious difficulties for the return period. The theoretical return period of a value greater than or equal to x is defined as $T(x) = 1/[1 - F(x)]$. Corresponding to the step function $(m, x_m)$, we have the exceedance interval $'T(x_m) = n/(n - m)$, $m = 1, 2, \ldots, n - 1$ and corresponding to the step function $(m - 1, x_m)$, the recurrence interval $''T(x_m) = n/(n - m + 1)$, $m = 1, 2, \ldots, n$. Corresponding to the compromise position, we have the corrected return period $T(x_m) = n/(n - m + 1/2)$ proposed by Hazen (1930) and used by Kimball (1942). According to this formula, the highest of n observations has return period 2n, which does not seem to be sound. To obtain one and only one series of observations, the author adjusts the cumulative frequencies, the corrections depending upon the theoretical distribution which is adequate for the observations. Later he deals with the relation between serial numbers and grades, where a *grade* is the statistical variate corresponding to cumulative probability $F(x)$, with $F(x) = 1/4, 1/2, 3/4$, for the quartiles, $F(x) = 1/10, \ldots, 9/10$ for the deciles, etc. For the median, $x_0$, the most probable serial number $\tilde{m}_0$ is $\tilde{m}_0 = (n + 1)/2 - f'(x_0)/4f^2(x_0)$, where $f(x)$ is the theoretical p.d.f. The most probable serial number for the median of a symmetrical distribution is $\tilde{m}_0 = (n + 1)/2$, which gives the usual estimate of the median for any distribution. For the two quartiles $q_1$ and $q_2$ the most probable serial numbers $\tilde{m}(q_1)$ and $\tilde{m}(q_2)$ are $\tilde{m}(q_1) = (n + 3)/4 - 3f'(q_1)/16f^2(q_1)$ and $\tilde{m}(q_2) = (3n + 1)/4 - 3f'(q_2)/16f^2(q_2)$. For the uniform, normal and exponential distributions, respectively, we

have $\tilde{m}(q_1) = (n + 3)/4$, $\tilde{m}(q_2) = (3n + 1)/4$; $\tilde{m}(q_1) = n/4 + .352$, $\tilde{m}(q_2) = 3n/4 + .648$; $\tilde{m}(q_1) = n/4 + 1$, $\tilde{m}(q_2) = 3n/4 + 1$. Finally, the author determines standard errors of the estimates and constructs confidence bands for the comparison between theory and observations.

**References:** Fisher & Tippett (1928), Hazen (1930), Pearson & Pearson (1931, 1932), Gumbel (1935a, 1939a, 1941a, 1942a,c), Kimball (1942).

**Citations:** Kendall (1946), Gumbel (1947).

* Gumbel, E. J. (1943b). On the plotting of flood-discharges. *Transactions of the American Geophysical Union* **24,** 699-716; discussion, 716-719. (MR 5, 211—listing only).

**Summary:** The author writes (Introduction, p. 699): "We ask: Which frequency shall we attribute to the m'th observation of a statistical variate, and especially to the m'th flood discharge? Let $x_m$ [m = 1, 2, . . ., n−1, n] be n observations arranged according to increasing magnitude. The question is, shall we use (m/n) or [(m−1)/n] as the observed cumulative frequency? . . . Hazen [(1930)] proposed a compromise, a corrected frequency $W(x_m) = [(m-1/2)/n]$ which has been accepted widely. . . . For the largest among n observations this method leads to a return-period 2n. But no statistical device can increase the number of observations beyond n. Furthermore, it does not seem safe to attribute a return-period 2n to an event which has already happened within n years. Therefore, Hazen's proposal does not solve the problem." The author summarizes his proposed general solution of the problem as follows (Conclusion, p. 715): "There are two ways of plotting the observed cumulative frequencies, and two ways of plotting the observed return-periods. Both ways are justified. However, it is a logical postulate to compare one and only one observed series with the theoretical series. Therefore, we attribute to the m'th observation $x_m$ an adjusted cumulative frequency $[W(\tilde{x}_m)]$, which is the probability of the most probable m'th value . . . . To facilitate this procedure, we give tables of the difference $[m/n - \tilde{W}]$ for the normal distribution and the distribution of the largest value. If the theoretical distribution of the observed sample is unknown, the problem of the correction has no exact solution. However, under certain analytical conditions, which are not very restrictive, we obtain an approximate solution . . . $\{\overline{W}(x_m) = (m/n) - [(m-1)/n(n-1)]\}$. For large numbers of observations, the approximate solution may be substituted for the adjusted frequencies. . . ." On page 702, the author notes that another general approximation is $\overline{W}'(x_m) = [m/(n+1)]$. The discussion includes comments by Bradford F. Kimball, who accepts the author's proposed correction, and by R. S. Goodridge.

**References:** Hazen (1930), Horton (1936), Gumbel (1941a,b, 1942b,c), Kimball (1942), Powell (1943).

**Citations:** Gumbel (1945a), Nordquist (1945), Kimball (1946b).

* Gumbel, E. J. (1943c). La durée de retour. *La Météorologie* (3) no. 42, 71-98. (MR 7, 319).

**Summary:** The author reviews the theory of extreme values which he has developed in earlier papers [e.g., Gumbel (1935a)], with special emphasis on the return period, which he applies to meteorological and hydrological data. He writes (Conclusion, pp. 97-98, as translated by the compiler): "Our method consists in this: For a statistical variable of which the observations are mutually independent and occur in time, we first trace the observed return periods. For this purpose, we arrange the observations in a nondecreasing order, and we attribute to the $m^{th}$ value, then the $(m-1)^{st}$ value, the observed period log $T(x_m) = \log [n/n-m)]$. In order to represent these observations one chooses a distribution determined with regard to the problem itself. One calculates the constants by the method of moments and traces the theoretical curve $x = f(\log T)$. In order to control the agreement between the theory and the observations one also traces the two control curves $x \pm s_x$, log T. With the necessary precautions one can decide whether the choice of the theoretical distribution is good, and determine whether the assumed independence exists, or whether certain values are closer to each other than the theory predicts. Thus the return period gives a new criterion for comparing observations with a theory and allows an interpretation of the range of the observations."

**References:** Galton (1875), Bertrand (1889) [1907], Fuller (1914), Fisher & Tippett (1928), Gibrat (1932b), Gumbel (1935a), von Mises (1936), Coutagne (1937), Gumbel (1937b, 1939c,e).

* Kendall, Maurice G. (1943). *The Advanced Theory of Statistics*, Volume I. Charles Griffin and Company Limited, London. (MR 6, 89).

**Summary:** In Chapter 2 (pp. 29-48), the author discusses measures of location and dispersion. Measures of location considered include the arithmetic, geometric and harmonic means, the median and the mode (all of which are measures of central tendency) and the quantiles (in addition to the median, the quartiles and the deciles are specifically mentioned). The author notes that the (arithmetic) mean, the median and the mode are the same for symmetric unimodal distributions, but differ for skew distributions. He gives the empirical relation Mean-Mode = 3 (Mean-Median), which he says holds approximately for unimodal distributions of moderate asymmetry, and mentions the advantages and disadvantages of the median as compared with the mean. He notes that measures of dispersion fall into three groups: (a) Measures of the distance between certain representative values, such as the range, the interdecile range and the interquartile range; (b) Measures compiled from the deviations of all of the observations from some central value, such as the mean deviation from the mean (or from the median) and the standard deviation; (c) Measures compiled from the deviations of all of the observations from each other, such as the mean difference. He also discusses measures of relative dispersion, such as Pearson's coefficient of variation and Gini's coefficient (ratio) of concentration. In Chapters 5 and 6 (pp. 116-163), he discusses standard distributions, including binomial, Poisson, hypergeometric and normal, also the Pearson system, Gram-Charlier series and Edgeworth series. In Chapter 7 (pp. 164-185), he deals with probability and likelihood, and in Chapter 8 (pp. 186-203), with random sampling. In Chapter 9 (pp. 204-230), he gives formulas, many of them only approximate, for the standard errors of sample statistics, including the mean and other moments (and functions of moments), the median and other quantiles, the mean deviation, the mean difference, the semi-interquartile range and the range. He also finds the covariance of quantiles, and studies the distributions of $m^{th}$ largest and $m^{th}$ smallest values in a sample of size n.

**References:** Lagrange (1774), Venn (1866) [1888], Helmert (1876a), Pearson (1895), Edgeworth (1905), Charlier (1906), Gini (1912), Fisher (1920), Pearson (1920), Jackson (1921), Keynes (1921), Morant (1921), Fisher (1922), Lévy (1925), Tippett (1925), Pearson (1926), Jordan (1927), Bowley (1928), Cramér (1928), Fisher & Tippett (1928), Pearson & Adyanthāya (1928), Gini & Galvani (1929), de Finetti & Paciello (1930), Steffensen (1930), Hojo (1931), Pearson & Pearson (1931), Pearson (1932), Davies & Pearson (1934), Gumbel (1935a), Pearson & Haines (1935), Nair (1936), Neyman (1937b), Jeffreys (1939), Hartley (1942), Pearson & Hartley (1942).

**Citations:** Snedecor (1937) [1946], Arley & Buch (1940) [1950], Boldrini (1942) [1968], Cramér (1946), Gumbel (1946a), Kendall (1946), Mosteller (1946), Winsten (1946), Gumbel (1947), Purcell (1947), Huzurbazar (1948), Benson (1949), Tukey (1949b).

* Knudsen, Lila F. (1943). A method for determining the significance of a shortage. *Journal of the American Statistical Association* **38,** 466-470.

**Summary:** The author writes (p. 466): "The method given here for making quick decisions with regard to shortages in weight utilizes the range of a sample. There are occasions confronting the practical person where a choice has to be made between a rapid utilization of statistical methods or none at all. The method suggested here attempts to fill this need for simplicity and rapidity of application and can be used in any question of shortage—not necessarily weight only—but any suspected deviation from a standard or specification. This is a criterion for the average. . . . Rule: Take the shortage in weight [the declared net weight D minus the average net weight $\bar{x}$ of the sample], and the range between the smallest and largest weights in the sample tested, and perform the simple division: (shortage in

weight)/(range in sample). Compare this ratio with the figures in the table corresponding to the number of units weighed (size of sample). (i) If the calculated value of this fraction is less than the value in Column A, the presumption is that no shortage exists. (ii) If this fraction lies between the entries shown in the two columns A and B of the table, shortage is suspected and additional units should be weighed. (iii) If the fraction is greater than the value in column B, the presumption is that a definite shortage exists." The table (p. 468) gives .05 (Column A) and .001 (Column B) significance levels of the ratio (shortage/range) for samples of size N = 3(1)25, 30, 40, 60, 120, 500. These are approximate significance levels obtained from the corresponding significance levels of Student's t by means of the formula $t/(S\sqrt{N})$, where S is a constant such that the average range in a sample of size N (from a normal population) is $S\sigma$.

**References:** Tippett (1925), Student (1927), Pearson (1931), Shewhart (1931), Davies & Pearson (1934), Pearson (1935), Simon (1941).

**Citations:** Daly (1946), Grant (1946).

* Kontorova, T. A. (1943). Concerning one of the applications of the statistical theory of the scale factor (problem concerning the strength of a cable). (Russian). *Zhurnal Tekhnicheskoi Fiziki* [*Journal of Technical Physics* (USSR)] **13** (6), 296-308.

**Summary:** The statistical theory previously developed by the author, partially in collaboration with Frenkel, was directed to the solution of problems involving the rupture of specimens in which the elements were effectively in series (as the links in a chain). The author proposes now to adapt this theory to specimens in which the elements are effectively in parallel (as the wires in a cable). The problems studied include (1) the rupture of the first wire; (2) the rupture of all the wires; and (3) the degree of rupture for given stress. It is assumed that the strength of each wire is uniquely determined by the most dangerous defect in it. The probability of encountering a wire having the strength $\sigma$, as given by Kontorova & Frenkel (1941), is given by $W(\sigma)d\sigma = C\bar{n}\omega_0 e^{-\alpha(\sigma_0-\sigma)^2}[1 - e^{-\alpha(\sigma_0-\sigma)^2}/2\sqrt{\pi\alpha}(\sigma_0 - \sigma)]^{\bar{n}\omega_0} d\sigma$ (1), where $\omega_0$ is the size of the wire, $\bar{n}$ is the mean number of defects occurring per unit volume, $\sigma_0$ is the most probable value of the strength of the wire, and C and $\alpha$ are constants, with $C = \sqrt{\alpha/\pi}$. The equation (1) is a valid approximation under the assumption that the difference $\sigma_0 - \sigma$ is not too small, which can safely be made if the volume of each wire is not too small. The probability that one wire (the weakest one) has strength $\sigma$ is given by $P_1 = N \cdot W(\sigma)[\int_\sigma^\infty W(\sigma)\, d\sigma]^{N-1} d\sigma$ (2), where N is the number of wires. By substituting (1) and (2) and making various approximations involving the neglect of $\sigma$ as compared with $\sigma_0$, the author arrives at the final result $P_1 \cong CN\bar{n}\omega_0 e^{-\alpha(\sigma_0-\sigma)^2}[1 - e^{-\alpha(\sigma_0-\sigma)^2}/2\sqrt{\pi\alpha}\,\sigma_0]^{\bar{n}\omega_0}\{[1 - e^{-\alpha(\sigma_0-\sigma)^2}/2\sqrt{\pi\alpha}\,\sigma_0]^{\bar{n}\omega_0}\{1 - 1/2\sqrt{\pi\alpha}\,\sigma_0\}^{\bar{n}\omega_0}\}^{N-1} d\sigma$ (7). The probability that the strongest of the N wires has strength $\sigma$ is given by $P_N = N \cdot W(\sigma)[\int_0^\sigma W(\sigma)\, d\sigma]^{N-1} d\sigma$ (9). The author assumes that the number of wires is sufficiently great so that the strength $\sigma$ of the strongest wire is much greater than the mean strength $\sigma_0$, that is, $\sigma >> \sigma_0$. In this case, equation (1) must be replaced by $W(\sigma)\, d\sigma \cong C\bar{n}\omega_0\{e^{-\bar{n}\omega_0\alpha(\sigma-\sigma_0)^2}/[2\sqrt{\pi\alpha}(\sigma - \sigma_0)]^{\bar{n}\omega_0}\} d\sigma$ (1'). A comparison of (1) and (1') shows that a positive deviation of a given magnitude from the mean strength is much less probable than a negative deviation of the same magnitude. The probability $P_N$ that the strongest wire has strength $\sigma$ is given by $P_N \cong CN\bar{n}\omega_0\{e^{-\bar{n}\omega_0\alpha(F-\sigma_0)^2}/[2\sqrt{\pi\alpha}(F - \sigma_0)]^{\bar{n}\omega_0}\{[J(0)]^{\bar{n}\omega_0} - e^{-\bar{n}\omega_0\alpha(F-\sigma_0)^2}/[2\sqrt{\pi\alpha}(F - \sigma_0)]^{\bar{n}\omega_0}\}^{N-1} d\sigma$ (15), where $J(0) = 1 - e^{-\alpha\sigma_0^2}/2\sqrt{\pi\alpha}\,\sigma_0$, F being the whole force applied to the cable, so that the stress applied to one remaining wire is $\sigma = F$ as compared with $\sigma = F/N$ originally. Let $\sigma^*$ be the most probable value of the rupturing tension applied to the first wire and $F^*$ be the most probable value of the rupturing tension bringing about the rupture of the last wire of the cable. Then $\sigma^* \cong \sigma_0 - \sqrt{A \lg N \omega_0 + B}$ and $F^* \cong \sigma_0 + \sqrt{A \lg N \bar{n} \omega_0}$, where $A = 1/\alpha$ and $B = (1/\alpha)\lg(\bar{n}/2\sqrt{\pi\alpha}\,\sigma_0)$. Next the author considers the degree of rupture of the cable by an applied F. The ratio of the number of wires $\nu$ already broken to the original number N is called $\gamma$, so that $\gamma = \nu/N$. If, at a given stage, the force on each of the remaining wires is $\sigma'$, this means that the strength of each of these wires exceeds $\sigma'$. Under the assumption $\sigma << \sigma_0$, corresponding to the case of not too small values of N, the number of remaining wires at this stage is $N\int_{\sigma'}^{\sigma_0} W(\sigma)\, d\sigma$, and the number of broken wires is $\nu$

$= N\,[1 - \int_{\sigma'}^{\sigma_0} W(\sigma)\,d\sigma]$ (24). For applied force F, the stress on each wire is then $\sigma' = F/N \int_{\sigma'}^{\sigma_0} W(\sigma)\,d\sigma$ (25). To find $\gamma$ as a function of F and N, we must solve simultaneously Equations (24) and (25). Using Equations (1) and (7), and approximating a binomial expression by its first two terms the author finds $\nu \cong N\{1 - (\bar{n}\omega_0 / 2\sqrt{\pi\alpha}\sigma_0)[1 - e^{-\alpha(\sigma_0 - \sigma')^2}]\}$ and $\gamma = \nu/N \cong a[e^{-\alpha(\sigma_0 - \sigma')^2} + b]$, where the parameters $a = \bar{n}\omega_0 / 2\sqrt{\pi\alpha}\,\sigma_0$ and $b = (1 - a)/a$ are constants of the material; also $F \cong (N\bar{n}\omega_0 / 2\sqrt{\pi\alpha}\,\sigma_0)\sigma'\,[1 - e^{-\alpha(\sigma_0 - \sigma')^2}]$, which can best be solved graphically.

**Note:** It appears that the differential should be deleted on the right-hand side (or inserted on the left-hand side) of Equations (2), (7), (9) and (15).

**References:** Kontorova (1940), Kontorova & Frenkel (1941).

**Citation:** Kontorova & Timoshenko (1949).

* Mathisen, Harold C. (1943). A method of testing the hypothesis that two samples are from the same population. *Annals of Mathematical Statistics* **14,** 188-194. (MR 5, 128).

**Summary:** The author writes (Summary, p. 194): "The problem considered here is that of devising a simple method of testing the hypothesis that two samples are from identical populations having continuous distribution functions. It may be summarized briefly as follows. The first sample is used to establish any desired number of intervals into which the observations of the second sample may fall. A test criterion is proposed which is based on the deviations of the numbers of elements of the second sample which fall in the intervals from the expected values of the respective numbers. Two cases are discussed, that of two intervals and that of four intervals, making use of a median and quartiles in the first sample to determine the intervals. Tables of 1% and 5% points for several sample sizes of both cases are given." The author makes use of the facts that, for first sample of size $2n + 1$, the median $x_{n+1}$ divides the range of x into two intervals into which, under the null hypothesis, it is equally likely that an observation from the second sample will fall; for first sample of size $4n + 3$, the quartiles $x_{n+1}$, $x_{2n+2}$ and $x_{3n+3}$ divide the range of x into four such intervals.

**Note:** Median tests and quartile tests, such as those considered in this paper, depend only on the ranks of the observations, and hence this paper is relevant to a study of order statistics only because it deals incidentally with the problem of determining sample quartiles (and estimating population quartiles). The author's solution to this problem agrees with that advocated by the compiler [see comments on Hojo (1931)].

**Reference:** Thompson (1941).

**Citations:** Scheffé (1943), Bowker (1944), Kendall (1946), Wilks (1948), Wolfowitz (1949).

* Nair, K. R.; Banerjee, K. S. (1943). A note on fitting of straight lines if both variables are subject to error. *Sankhyā* **6,** 331.

**Summary:** The authors write: "Wald [(1940)] has given a simple method of fitting a straight line to a set of n observed pairs of values $(x_i, y_i)$ when both x and y are subject to mutually uncorrelated errors. The method which Wald has advocated can be easily recognized as the Method of Averages for fitting curves. Nair and Shrivastava (1942) have recently studied the efficiency of this method, in comparison to the method of least squares, for fitting polynomials in x, when y alone is subject to error. They have found that a slightly modified method which they called the 'Method of Group Averages' will give more efficient results than the 'Method of Averages'. For fitting a straight line they found that by plotting the points of mean values of x and y for the first one-third and the last one-third of the whole set of observations, arranged in order of magnitude of x, and by joining these mean points we get a better estimate of the straight line than any other two group means. By model sampling we have collected evidence that when both x and y are subject to error, then also the method of Group Averages gives better

estimates for a and b, of the line $y = ax + b$, than the method of averages which Wald has put forward."

**References:** Wald (1940), Nair & Shrivastava (1942).

**Citations:** Kendall (1946), Bartlett (1949).

* Pearson, E. S.; Hartley, H. O. (1943). Tables of the probability integral of the studentized range. *Biometrika* **33,** 89-99. (MR 6, 92).

**Summary:** The authors write (Introduction, p. 89): "Denote by $x_1, \ldots, x_n$ a random sample of n observations arranged in ascending order of magnitude and drawn from a normal population with standard deviation $\sigma$. The range, or spread, in the sample is then $x_n - x_1$ and may be expressed in units of the standard deviation by the ratio $w = (x_n - x_1)/\sigma$. Tables of the probability integral of w were given . . . [by Pearson & Hartley (1942)], together with tables of its percentage points. . . . The probability distribution of the difference $x_n - x_1$ depends on the standard deviation of the sampled population. In many cases this will be unknown, but can be estimated from a second independent sample $X_1, \ldots, X_{\nu+1}$, drawn from the same population. If we denote by $s^2 = \Sigma_1^{\nu+1}(X - \overline{X})^2/\nu$, the estimate of $\sigma^2$ derived therefrom, it is easy to see that the probability law of the ratio $q = (x_n - x_1)/s$ does not depend on $\sigma$. This idea of 'studentizing' the range seems to have occurred first to W. S. Gosset himself (see letter of 29 January 1932 quoted by E. S. Pearson . . . [(1939)], p. 245). Following this suggestion, D. Newman (1939) calculated a number of percentage points for q. Newman's table was obtained by quadrature from E. S. Pearson's approximate probability law of w. Since exact tables of this latter integral are now available, it appeared appropriate to revise and amplify the 'studentized' distribution law resulting from it. . . ." They describe the new tables as follows (pp. 89-90): "The probability law of q depends both on the size n of the first sample (from which the range $x_n - x_1$ is determined) and on the degrees of freedom $\nu$ of the standard deviation estimated from the second sample. The probability integral may be denoted by ${}_\nu P_n(Q)$ and represents the chance that the ratio q does not exceed the limit Q. As $\nu \to \infty$ and $s^2 \to \sigma^2$, ${}_\nu P_n(Q)$ will tend to the probability integral $P_n(W)$ of the ratio $w = (x_n - x_1)/\sigma$, taken at $W = Q$. It can now be proved that to an accuracy sufficient for practical purposes and for $\nu \geq 10$, ${}_\nu P_n(Q)$ may be represented as a quadratic in $1/\nu$, viz. ${}_\nu P_n(Q) = P_n(Q) + a_n(Q)/\nu + b_n(Q)/\nu^2$. The coefficients $P_n(Q)$, $a_n(Q)$ and $b_n(Q)$ are given in Table 1 [for Q = 0 (0.25) 6.50 and n = 3 (1) 20] . . . . For certain applications it is necessary to know the limits Q corresponding to standard probability levels. Table 2 gives these percentage points for four levels, viz. the upper and lower 5 and 1% points. The lower percentage points are given to two places of decimals, for degrees of freedom $\nu$ [ = 10(1)20, 24, 30, 40, 60, 120, $\infty$ ] . . . and for sample sizes n [ = 2(1)20] . . . . The upper percentage points are given for the same values of $\nu$ and n and to the same accuracy, except that for $10 \leq \nu < 20$ only one place of decimals is given. . . ." Applications and calculation of the tables are also discussed.

**References:** Student (1927), Pearson (1935), Newman (1939), Pearson (1939), Simon (1941), Dudding & Jennett (1942), Hartley (1942), Pearson & Hartley (1942).

**Citations:** Hartley (1944), Baker (1946), Kendall (1946), Duncan (1947), Nair (1948a,b), Wilks (1948), Paulson (1949).

* Powell, Ralph W. (1943). A simple method of estimating flood frequencies. *Civil Engineering* **13** (2), 105-107.

**Summary:** The author describes a special plotting paper, to be used with the method of Gumbel (1941a), on which the relationship between the T-year flood and T itself will plot as a straight line. Gumbel's method is based upon the assumption that the data have come from a Type I asymptotic distribution of largest values, for which he showed that $Q_m = Q_a + S(0.7797y - 0.45005)$ and $y = -\ln[-\ln(1 - 1/T)]$, where $Q_m$ = the flood equaled or exceeded on the average once in T years, $Q_a$ = average of annual floods,

S = standard deviation of annual floods = $\sqrt{n(\Sigma Q^2/n - Q_a^2)/(n-1)}$, y = function of stream flow defined by the preceding equation, n = number of years of record, and $\Sigma Q^2$ = sum of the squares of the observed annual floods. The author tabulates y and T for 100/T = 0.05, 0.1 (0.1) 0.5, 1(1)5, 10(5)40, 42.96, 45(5)60, 70(10)90, 95, 99 and uses this table to prepare a special plotting paper on which the horizontal lines are spaced uniformly and the distance between the vertical lines is made proportional to the values of y, so that the relationship between $Q_a$ and T will plot as a straight line. The vertical scale is the flow (in units of $10^3 ft^3/sec$), values of 100/T are shown at the bottom, and values of T (the return-period) are shown at the top. The author lists three ways in which this paper can be used: 1. Compute the values of the mean annual flood, $Q_a$, and of the standard deviation, S, from the observed data. Plot the points T = 2,000, $Q_m = Q_a + 5.476S$ and T = 2.376 [sic; should be 2.3276] (or 100/T = 42.96), $Q_m = Q_a$. Then draw a straight line through these two points, which will give the value of $Q_m$ for any value of T up to 2,000 years. As a check, the observed values of all the annual floods may be plotted (based on Horton's recurrence interval rather than his exceedance interval or Hazen's compromise). In n years of record, the highest is taken as having 100/T = 100/n, the next highest as having 100/T = 200/n, and the next 300/n, etc. The lowest, 100/T = 100, cannot be plotted. 2. Simply plot the observed data as just described and fit a straight line by eye, without computing $Q_a$ or S. 3. Compute $Q_a$ but not S, and draw the line through T = 2.3276, $Q_m = Q_a$ as near to the other points as possible. The author states that with a record of 50 years or more, all three methods give practically the same results.

**Reference:** Gumbel (1941a).

**Citations:** Beard (1942) [1943], Gumbel (1943b), Price (1943), Gumbel (1945b), Lane & Lei (1949).

* Price, Reginald C. (1943). The Gumbel method of estimating flood frequencies. *Civil Engineering* **13** (6), 285.

**Summary:** The author questions the validity of a criterion used by Powell (1943) to justify the use of the method of Gumbel (1941a), viz. "of two methods that may seem equally sound, the one that tends to give the larger values [for the flood to be equaled or exceeded in a given period] would appear to be the safer". He says that the acceptability of such a criterion depends upon the use to be made of the results—if they are used for economic computations rather than for design purposes, middle-of-the-road values rather than the maximum envelope values indicated by Powell's criterion should be used. The author compares results given by the Slade, Gumbel and Fuller methods [see Slade (1934), Gumbel (1941a) and Fuller (1914)] for the 1875-1931 record for the Tennessee River at Chattanooga. He writes: "The Gumbel curve follows the Fuller method curve very closely in shape, but with slightly lower values. Both the Gumbel curve and the Fuller method curve give values decidedly above those given by the Slade curve, and both are asymptotic to a markedly less degree than the Slade curve, thus raising a question as to whether the Gumbel curve is sensitive to a probable physical limit of flood magnitude. The Gumbel curve method appears to possess the following advantages: (1) Computations are relatively easy and few; (2) it requires no arbitrary assumptions of limits as do some other methods; and (3) the straight-line relationships discussed by Professor Powell lend themselves to easy graphical solution of data. Its disadvantages include the fact that the Gumbel method curve does not approach a limit nearly as definitely as does the Slade curve. Also, the results it gives are apparently high as compared to the Slade method, and the 'larger values' are undesirable for economic studies."

**Reference:** Powell (1943).

* Rajalakshman, D.V. (1943). On the interval between the ranked individuals of samples taken from a rectangular population. *Journal of Madras University B* **15**, 31-44. (MR 6, 235).

**Summary:** The author writes (Introduction, pp. 31-32): "J. Neyman and E. S. Pearson [(1928)] in establishing the criteria for testing the hypothesis for a sample that has been drawn at random from a

specified population, obtained the chance of drawing a sample with range in the limits $W \pm dW/2$ from a rectangular population, where W is the range $(x_n - x_1)$ of a sample of size n, the characters of the n observations of the sample being $x_1, x_2, \ldots, x_n$ such that $x_1 \leq x_2 \leq \ldots \leq x_n$ and measured from one end of the range w $[=\omega]$ of the infinite population. Also Karl Pearson [& Margaret V. Pearson (1931)] derived an expression for the $s^{th}$ moment of the interval between the $q^{th}$ and $q'^{th}$ ranked individuals to be $\mu'_s = \omega^s (q' - q - 1 + s)!/(n + s)!(q' - q - 1)!$ about one end of the range w of the rectangular population. By taking the first four moments in particular, he obtained the distribution of the interval x to be $y = (N/\omega) \cdot [n!/(q' - q - 1)!(n - q' + q)!][(q' - q - 1)/(n - 1) + x/\omega]^{q'-q-1}[(n - q' + q)(n - 1) - x/\omega]^{n-q'+q}$ where the origin is the mode of the distribution when N samples of size n are taken, from the specified population. In this paper the distribution of the interval between the $q^{th}$ and $q'^{th}$ ranked individuals is obtained directly by adopting a method [using hypergeometric series] similar to that of Neyman and Pearson, and also the $m^{th}$ moment of this interval about the arithmetic mean of the intervals of samples is obtained and some special features of this distribution are discussed."

**References:** Neyman & Pearson (1928), Pearson & Pearson (1931, 1932), Sastry (1935), Rajalakshman (1941).

* Scheffé, Henry (1943). Statistical inference in the non-parametric case. *Annals of Mathematical Statistics* **14,** 305-332. (MR 5, 211).

**Summary:** After an introduction to non-parametric statistical inference, the author deals in turn with non-parametric tests, non-parametric estimation, and a general theory. Most of the tests considered, including the Kolmogoroff-Smirnoff test in the two-sample case, depend only upon ranks, not order statistics. An exception is the test proposed by Nair (1940a) for the analysis of randomized blocks with two columns. Let $D_i = X_{i1} - X_{i2}$. Fisher (1935) proposed employing as a test statistic the absolute value of the mean of the $D_i$, but the exact distribution of this statistic is laborious to calculate, and Nair proposed using instead the median of the $D_i$, of which the distribution can be easily calculated. The author notes that Nair also showed that the distributions of the range and of the midrange of the $D_i$ are very simple. Order statistics play a large role in non-parametric interval estimation. Examples given by the author include the use of order statistics in determining (1) confidence intervals for an unknown median, or the difference of two medians or other quantiles, based on the work of Thompson (1936) and Nair (1940b), (2) confidence limits for an unknown (cumulative) distribution function, based on the one-sample Kolmogoroff-Smirnoff statistic [see Kolmogoroff (1933), Smirnoff (1939b)], and (3) tolerance limits for individual observations, based on the work of Wilks (1941) and Wald (1943). In the last section, the author discusses Wald's formulation of the general problem of statistical inference (decision theory), which encompasses testing and estimation as well as problems which are neither of these, solutions to some of which make use of order statistics.

**References:** Kolmogoroff (1933), Fisher (1935), Gumbel (1935a), Smirnoff (1935), Thompson (1936), Savur (1937b), Smirnoff (1937b), Thompson (1938), Smirnoff [Smirnov] (1939b), Wald & Wolfowitz (1939), Nair (1940a,b), Kolmogoroff (1941), Wald & Wolfowitz (1941), Wilks (1941, 1942), Mathisen (1943), Wald (1943), Wilks (1943).

**Citations:** Scheffé & Tukey (1944, 1945), Kendall (1946), Wilks (1948), Wolfowitz (1949).

* Wald, Abraham (1943). An extension of Wilks' method for setting tolerance limits. *Annals of Mathematical Statistics* **14,** 45-55. (MR 4, 222).

**Summary:** The author writes (pp. 54-55): "Consider a pair of random variables x and y. Denote by f(x,y) dx dy the joint probability density function of x and y and suppose that nothing is known about f(x,y) except that it is continuous. A sample of n pairs of independent observations $(x_1, y_1), \ldots, (x_n, y_n)$ is drawn from this bivariate population. The sample can be represented by n points $p_1, \ldots, p_n$ in the plane (x, y), $p_i$

being the point with the coordinates $x_i$ and $y_i$. . . . We have dealt with the problem of finding a rectangle T in the plane (x, y), called tolerance region, such that we can state with high probability, say . . . .98 or .99, that the proportion Q of the bivariate population included in the rectangle T is not less than a given number b, say not less than .98 or .99. The rectangle T is constructed as follows: Suppose that the points $p_1, \ldots, p_n$ are arranged in order of increasing magnitude of their abscissa values, i.e. $x_1 < x_2 < \ldots < x_n$. We draw a vertical line $V_{r_1}$ through the point $p_{r_1}$ and a vertical line $V_{s_1}$ through $p_{s_1}$ where $r_1$ and $s_1$ are positive integers such that $1 \leq r_1$, $r_1 \leq s_1 - 3$ and $s_1 \leq n$. We consider the set S consisting of the points $p_{r_1+1}$, $\ldots$, $p_{s_1-1}$ which lie between the vertical lines $V_{r_1}$ and $V_{s_1}$. We draw a horizontal line $H_{r_2}$ through the point of S which has the $r_2$-th smallest ordinate in S. Finally a horizontal line $H_{s_2}$ is drawn through the point of S which has the $s_2$-th the smallest ordinate in S. The values $r_2$ and $s_2$ are positive integers for which $r_2 < s_2$. The probability p that at least the proportion b$(0<b<1)$ of the universe is included in T is given by $p = \int_b^1 [\Gamma(n+1)/\Gamma(s_2 - r_2)\,\Gamma(n - s_2 + r_2 + 1)]\, Q^{s_2 - r_2 - 1}(1 - Q)^{n - s_2 + r_2}\, dP \ldots$ [which] is equal to the probability that $F \leq [2(s_2 - r_2) / 2(n - s_2 + r_2 + 1)][(1-b)/b]$ where F has the analysis of variance distribution with $2(n - s_2 + r_2 + 1)$ and $2(s_2 - r_2)$ degrees of freedom. . . . Tolerance regions are considered which are composed of several rectangles. Such a tolerance region may be more favorable than a single rectangle if x and y are highly correlated."

**Reference:** Wilks (1941).

**Citations:** Scheffé (1943), Kendall (1946), Tukey (1947), Murphy (1948), Wilks (1948), Wolfowitz (1949).

* Wilks, S.S. (1943). *Mathematical Statistics.* Princeton University Press, Princeton, N.J. (MR 5, 41). Revised and enlarged edition, John Wiley & Sons, Inc., New York, 1962.

**Summary:** Chapter IV (pp. 79-97) deals with sampling theory. Section 4.5 (pp. 89-94) deals with the sampling theory of order statistics. The simultaneous distribution of any k order statistics, the distribution of the largest (or smallest) variate, the distribution of the median, and the distribution of the sample range are given in Subsections 4.51-4.54 (pp. 89-93), while Subsection 4.55 (pp. 93-94) deals with tolerance limits based on order statistics. Chapters VI and VII are devoted to the theories of estimation and of testing statistical hypotheses. The former contains a discussion of both point and interval estimation, and includes Section 6.2 (pp. 133-142) and Section 6.3 (pp. 142-145) on point estimation and on tolerance limits, respectively. In an example on page 135, the author shows that the sample mean $\bar{x}$ is more efficient than the sample median $\tilde{x}$ as an estimator of the central value a (which is both the population mean and the population median) of a normal population $N(a, \sigma^2)$. More generally, he shows that for populations symmetric about a and continuous at a, $\bar{x}$ is a more efficient estimator of a than $\tilde{x}$ if $\sigma < [2f(a)]^{-1}$, but $\tilde{x}$ is more efficient if $\sigma > [2f(a)]^{-1}$. An introduction to multivariate statistical analysis is given in Chapter XI (pp. 226-270). Section 11.9 (pp. 252-257) on principal components deals with ordered eigenvalues of a matrix and is based on the work of Hotelling (1933). Section 11.11 (pp. 260-270) on the sampling theory of (ordered) roots of determinantal equations is parallel to the work of Fisher (1939), Girshick (1939) and Hsu (1939), which the author mentions, and that of Roy (1939), which he does not, but his derivations are based on work performed by A. M. Mood in 1939 but not published until 1951.

**Note:** All references in the above summary to specific chapters, sections and pages are to those in the 1947 printing of the original (1943) edition.

**References:** Yule (1911) [1936], Keynes (1921), Fisher (1922), Whittaker & Robinson (1924), Neyman & Pearson (1928), von Mises (1931), Shewhart (1931), Craig (1932a), Fisher (1933), Hotelling (1933), Fisher (1935), Gumbel (1935a), Hotelling (1936b), Neyman & Pearson (1936), Neyman (1937b), Fisher & Yates (1938), Fisher (1939), Girshick (1939), Hsu (1939), Smirnov (1939a), Wald (1939), Wilks (1941), Wald (1942), Wilks (1942).

**Citations:** Snedecor (1937) [1946], Scheffé (1943), Anderson (1945, 1946), Grant (1946), Hoel (1946), Kendall (1946), Kimball (1946a,b), Mosteller (1946), Tintner (1946), Walsh (1946a,b), Bartlett (1947b), Bickerstaff (1947), Duncan (1947), Grubbs & Weaver (1947), Jacobson (1947), Bartlett (1948), Schützenberger (1948), Wilks (1948), Birnbaum & Zuckerman (1949), Howell (1949), Kimball (1949), Walsh (1949d).

* Barnard, G. A. (1944). An Analogue of Tchebycheff's Inequality in Terms of Range. Technical Report No. Q. C./R/11. (British) Ministry of Supply Advisory Service on Quality Control.

**Note:** This report is undoubtedly relevant to our subject, but the compiler has been unable to obtain a copy, and has no information as to its contents other than that given in the title.

* Bowker, Albert H. (1944). Note on the consistency of a proposed test for the problem of two samples. *Annals of Mathematical Statistics* **15,** 98-101.

**Summary:** The author writes (p. 99): "A test is called consistent if the probability of rejecting the null hypothesis when it is false (the power of the test) approaches one as the sample number approaches infinity. . . . It will be shown that a test recently proposed by Mathisen [(1943)] is not consistent with respect to certain alternatives. The test proposed by Mathisen . . . may be described briefly as follows: Given two samples [from populations with c.d.f.'s $F(x)$ and $G(y)$], observe the number (m) of elements of the second sample whose values are less than the median of the first sample. The distribution of m is independent of the population distribution under the null hypothesis [$F(x) \equiv G(x)$]. Let $P\{m<a\}$ denote the probability of the relation in braces under the null hypothesis. If $m_1$ and $m_2$ are significance points ($m_1 > m_2$) such that $P\{m>m_1\}=\beta_1, P\{m<m_2\}=\beta_2$, $\beta_1+\beta_2=\beta<1$, the statistic m can be used to test the hypothesis at the significance level $\beta$. This is the case of two intervals. The method is extended by using the two quartiles and the median of the first sample to define four intervals into which the elements of the second sample may fall." After giving proofs (too long to reproduce here) of the inconsistency of Mathisen's test, first in the two-interval case and then in the four-interval case, the author continues (p. 101): "Returning to the case of two intervals, if the samples are from different populations and if their cumulative distribution functions are identical in the neighborhood of their medians, the test is not consistent. If such a possibility is excluded from the class of admissible alternatives, we may expect that the test will be consistent. For example, if the class of alternatives is limited to those where $G(x) \equiv F(x+c)$, c a constant, the test will be consistent. A similar remark holds for the case of four intervals or for any fixed finite number of intervals."

**Reference:** Mathisen (1943).

**Citation:** Wolfowitz (1949).

* Geary, R.C. (1944). Comparison of the concepts of efficiency and closeness for consistent estimates of a parameter. *Biometrika* **33,** 123-128. (MR 6, 10).

**Summary:** The author writes (p. 123): "Having given two estimates X and Y of an unknown parameter $\theta$, E. J. G. Pitman (1937) has suggested that X should be regarded as a 'closer' estimate of $\theta$ if the probability that $|X-\theta|<|Y-\theta|$ is greater than 1/2. . . . The object of this communication is to compare 'closeness', as defined, with the familiar concept of relative 'efficiency' (Fisher, 1922) as determined by the variances of the two estimates. Continuous variation is assumed throughout." The author compares the concepts of closeness and efficiency for mean, median and midrange as estimates of the location parameter of a rectangular population with known range 1. He writes (Summary and Conclusion, p. 128): "The criterion of *efficiency* as determined by a comparison of the variances of two estimates of an unknown statistic [sic; parameter] is identical with the criterion of *closeness* when the joint distribution

of the estimates is normal and the criteria will not yield significantly different results in practice when the estimates are estimated from large samples and are consistent. Study of some particular examples suggests that, even when the distribution of the estimates is very different from normal [e.g., rectangular] the value of the probability associated with the criterion of closeness may not be very different from what its value would be if normality of the joint distribution were assumed. An application of Pitman's theory of closeness [to use of the highest car number observed as an estimate of the number of cars in a town, known to be numbered consecutively from 1] is discussed."

**Reference:** Fisher (1922), Neyman & Pearson (1928), Pitman (1937).

**Citations:** Cramér (1946), Kendall (1946).

* Gumbel, E. J. (1944). Ranges and midranges. *Annals of Mathematical Statistics* **15,** 414-422. (MR 6, 162).

**Summary:** The author writes (Introduction, p.414): "A large sample of size n is considered to be drawn from an unlimited symmetrical continuous distribution with zero mean. The difference and the sum of the largest and of the smallest observation, the extremes, are called *range* and *midrange*. [See Note (1) below.] . . . The concepts 'extremes', 'range' and 'midrange' allow a simple generalization. Let $_mx$ and $x_m$ be the $m^{th}$ observation in increasing and in decreasing magnitude, henceforth called $m^{th}$ value 'from below' and 'from above'. As long as the index m is small compared to the sample size n, the $m^{th}$ values under consideration are extremes. The difference and the sum of the $m^{th}$ extreme observations are called the *$m^{th}$ range* and the *$m^{th}$ midrange*. [See Note (2) below.] We will investigate the asymptotic distributions of the $m^{th}$ extremes, of the $m^{th}$ range and of the $m^{th}$ midrange. Assuming that the number of observations is large, the correlation between the largest and the smallest observation may be neglected. Then the $m^{th}$ range and the $m^{th}$ midrange are the difference and the sum of two independent variates, the $m^{th}$ extremes. It was found that the distribution of the $m^{th}$ range is skew and the distribution of the $m^{th}$ midrange is of the generalized logistic type, which is symmetrical. For m increasing the distributions of the $m^{th}$ extremes, the $m^{th}$ ranges, and the $m^{th}$ midranges converge toward normality.

**Notes:** (1) The midrange, as defined by the author, is twice the usual midrange. (2) The $m^{th}$ range and half the $m^{th}$ midrange (as defined by the author) are also called the $m^{th}$ quasi-range and the $m^{th}$ quasi-midrange.

**References:** Tippett (1925), Fisher & Tippett (1928), Gumbel (1935a), von Mises (1936), Pearson & Hartley (1942).

**Citations:** Gumbel (1946a, 1947), Wilks (1948), Gumbel (1949).

* Hartley, H.O. (1944). Studentization, or the elimination of the standard deviation of the parent population from the random sample-distribution of statistics. *Biometrika* **33,** 173-180. (MR 6, 10).

**Summary:** The author gives a mathematical solution of the problem of "Studentization" in statistical distribution. Consider a normal distribution with mean $\mu$ and standard deviation $\sigma$. Let $x_1, x_2, \ldots, x_n$ be a sample of n observations and let $X_1, X_2, \ldots, X_m$ be a second independent sample of m observations, both drawn from the specified population. Let $\bar{x}$ and $\bar{X}$ be the means of the two samples and let $s = \sqrt{\Sigma(x-\bar{x})^2/(n-1)}$ and $s^* = \sqrt{\Sigma(X-\bar{X})^2/(m-1)}$ be estimates of $\sigma$. The original statistics $\bar{x}-\mu$ and $s^*$, as well as s, are proportional to $\sigma$, but the studentized statistics $t=(\bar{x}-\mu)\sqrt{n}/s$ and $\sqrt{F} = s^*/s$ are independent of $\sigma$ and are distributed as Student's t and a transformation of Fisher's z, respectively. The author writes (pp. 173-174): "This simple studentization process of dividing a statistic by an independent estimate s of the parental standard deviation $\sigma$ may be applied to a number of other useful statistics which are also proportional to $\sigma$. . . We may mention here the range in a sample, the extreme observation, the extreme value in a sample of random standard deviations, but there are many others. It is the object of this paper to derive a general

formula for the studentized distribution law of such statistics. The accuracy of the formula will be demonstrated with a numerical example . . . . . A second application of the method may be found in a paper by Pearson & Hartley (1943), and it is hoped to publish others. Finally, computational methods for the systematic tabulation of studentized distribution laws are briefly outlined."

**References:** Newman (1939), Simon (1941), Thompson (1941), Pearson & Hartley (1943).

**Citations:** Baker (1946), Kendall (1946), Nair (1948a, 1948b), Hartley (1949).

* Reiersöl, Olav (1944). Measures of departure from symmetry. *Skandinavisk Aktuarietidskrift* **27,** 229-234. (MR 7, 211).

**Summary:** The author notes that many different measures of skewness have been proposed, all of which are zero when the distribution is symmetric, but many of which may also be zero when the distribution is unsymmetric. He presents some measures of skewness which are zero when and only when the distribution is symmetric. He writes (pp. 229-233): "Let us first consider a statistical series. Let $X_1, X_2, \ldots, X_n$ denote the individual values of this series in ascending order of magnitude (1) $X_1 \leq X_2 \leq \ldots X_n$ and let $Y_1, Y_2, \ldots, Y_n$ denote the same series in descending order of magnitude (2) $Y_1 \geq Y_2 \geq \ldots \geq Y_n$. Then we have (3) $Y_i = X_{n-i+1}$ $(i = 1, 2, \ldots, n)$. A necessary and sufficient condition for symmetry is (4) $X_i + X_{n-i+1} = 2c$ $(i = 1, 2, \ldots, n)$ where c denotes the centre of symmetry of the series $X_1, X_2, \ldots, X_n$. On account of (3) equation (4) may also be written (5) $X_i + Y_i = 2c$ $(i = 1, 2, \ldots, n)$. . . . Proposition 1: A necessary and sufficient condition for symmetry is that the dispersion of the variable X + Y is zero. . . . Definition 1: As a measure of departure from symmetry we shall choose the ratio between a measure of dispersion of the variable X + Y and a measure of dispersion of the variable X, possibly multiplied by a constant independent of the actual distribution. . . . Proposition 2: Any measure of departure from symmetry given by Definition 1 is zero when and only when the distribution is symmetric. *Proposition 3*: Suppose that any of the measures of dispersion used is independent of the origin of the variable and that if the variable is multiplied by a constant, the measure of dispersion is multiplied by the same constant. Then the measure of departure from symmetry is independent of the origin and the scale of the variable. The conditions of Proposition 3 are for instance fulfilled for the standard deviation, the mean deviation and the quartile deviation. . . . Definition 2. Consider the ratio between the standard deviation of the variable X + Y and the standard deviation of the variable X. As a measure S of departure from symmetry we shall use this ratio multiplied by $1/\sqrt{2}$ . . . . Proposition 4. The measure of departure from symmetry given by Definition 2 is always nonnegative and less than unity. It may take any value between zero and unity.

* Robbins, Herbert (1944a). On distribution-free tolerance limits in random sampling. *Annals of Mathematical Statistics* **15,** 214-216. (MR 6, 9).

**Summary:** The author writes (pp. 214-216): "Let $X_1, \ldots, X_n$ be independent random variables each with the continuous and differentiable cumulative distribution function $\sigma(x) = Pr(X_i < X)$. A continuous function $f(x_1, \ldots, x_n)$ with the property that the random variable $Y = \sigma(f(X_1, \ldots, X_n))$ has a probability distribution which is independent of $\sigma(x)$ will be called a *distribution-free upper tolerance limit* (d.f.u.t.l.). We shall prove Theorem 1. A necessary and sufficient condition that the continuous function $f(x_1, \ldots, x_n)$ be a d.f.u.t.l. is that the function $\bar{f}(x_1, \ldots, x_n) = \prod_{i=1}^{n} \{f(x_1, \ldots, x_n) - x_i\}$ be identically zero. . . . Let $O_r(x_1, \ldots, x_n)$ be the function whose value is the $r^{th}$ term when the numbers $x_1, \ldots, x_n$ are arranged in non-decreasing order of magnitude [the $r^{th}$ order statistic of the sample $x_1, \ldots, x_n$ of size n]. . . . A function $f(x_1, \ldots, x_n)$ is *symmetric* if its value is unchanged by any permutation of its arguments. It is clear that the only continuous and symmetric functions f which satisfy the identity $\bar{f} \equiv 0$ are the n functions $O_r(x_1, \ldots, x_n)$. Hence we can state Theorem 2. The only symmetric d.f.u.t.l.'s are the n functions $O_r(x_1, \ldots, x_n)$ $(r = 1, \ldots, n)$."

**Reference:** Wilks (1941).

**Citations:** Scheffé & Tukey (1945), Wilks (1948).

* Robbins, Herbert (1944b). On the expected values of two statistics. *Annals of Mathematical Statistics* **15,** 321-323. (MR 6, 162).

**Summary:** Let Y be a real random variable with c.d.f. $\sigma(y)$, so that for every y, $\Pr(Y < y) = \sigma(y)$. Let $Y_1, \ldots, Y_n$ be n independent variables, each with the distribution of Y. Let $R = B - A$ and $F = \sigma(B) - \sigma(A)$, where $A = \min(Y_1, \ldots, Y_n)$ and $B = \max(Y_1, \ldots, Y_n)$. The author shows that $E(F) = (n-1)/(n+1)$ and $E(R) = \int_{-\infty}^{\infty} \{1 - \sigma^n(t) - (1 - \sigma(t))^n\}\, dt$. He states that the former is a known result but that the latter is believed to be new. He uses the latter to show that if the random variable has the p.d.f. $\phi(y) = e^y/(1+e^y)^2$ and hence the c.d.f. $\sigma(y) = e^y/(1+e^y)$, $E(R) = 2, 3, 11/3$ for samples of size $n = 2, 3, 4$ respectively. He points out that it is always true that the expected value of the range for $n = 3$ is three-halves that for $n = 2$, which follows from the algebraic identity $\{1 - \sigma^3 - (1 - \sigma)^3\} = (3/2)\{1 - \sigma^2 - (1 - \sigma)^2\}$.

* Rodgers, J. W. (1944). Associated statistical techniques. *Proceedings of Symposium on Statistical Quality Control,* pp. 67-74. Ministry of Production, Birmingham, England.

**Summary:** The author discusses the use of the mean range $\bar{w}$ of a sample to estimate the population standard deviation $\sigma$, with applications to the use of the range instead of the standard deviation in the analysis of variance. He notes that $\sigma$ may be estimated by the relation $\sigma = \bar{w}/d_n$, where $d_n$ is the factor given by Dudding & Jennett (1942), provided the parent population is not too far from normal and the sample is not too large (not more than 10 to 15). He points out that tables of the standard error of the range have been drawn up and incorporated in the factors $D_{.999}$ and $D_{.975}$, etc., given by Dudding and Jennett. He considers the simple one-way analysis of variance of S samples each of size n. Let $w_1, w_2, \ldots, w_S$ be the ranges of the S samples and $\bar{w} = \sum_{i=1}^{S} w_i/S$ be the mean range. Then the estimate $\sigma_0$ of the within sample standard deviation is given by $\sigma_0 = \bar{w}/d_n$. The estimate of between sample standard deviation is more difficult to obtain. One could take the range of the S means and apply a formula similar to the one used for the within sample standard deviation. But this gives only one estimate of the range of a sample of size S, which may be large (say, 50). This difficulty can be overcome by dividing the S samples into S/5 groups of 5, preferably at random, rather than as they come in time, so as to minimize the effects of tool wear, setting variations with time, etc. This having been done, the between sample standard deviation is then estimated by $\sigma_{\text{between}} = \bar{w}/d_n$, where $\bar{w}$ is now the mean range of the groups of five sample means. The estimates $\sigma_0$ and $\sigma_{\text{between}}$ are then squared to obtain the corresponding estimates of variance, and the analysis of variance is carried out in the usual manner from that point. The author cautions that while this method is simple and convenient, especially when mean ranges have already been calculated for use in plotting control charts, it depends upon normality in a way that the conventional analysis of variance does not, since the factors $d_n$ were computed under the normal assumption.

**Reference:** Dudding & Jennett (1942).

* Scheffé, H.; Tukey, J. W. (1944). A formula for sample sizes for population tolerance limits. *Annals of Mathematical Statistics* **15,** 217. (MR 6, 9).

**Summary:** The authors write (p. 217): "An interval I is said to cover a proportion $\pi$ of a univariate population with cumulative distribution function F(x) if $\int_I dF = \pi$. Let $X_1, X_2, \ldots, X_n$ be a random sample from the population and $Z_1 \le Z_2 \le \ldots \le Z_n$ be a rearrangement of $X_1, X_2, \ldots, X_n$. Define $Z_0 = -\infty$, $Z_{n+1} = +\infty$, and consider the proportion B of the population covered by the random interval $(Z_k, Z_{n-m+1})$. Then $\Pr\{B \ge b\}$ is independent of F(x) if F(x) is continuous, and equals $1 - I_b(n-r+1, r)$, where $r = k + m$ and

$I_x(p, q)$ is K. Pearson's notation for the incomplete Beta function. Choose a confidence coefficient $1 - \alpha$, a pair of positive integers k, m, and a fraction b. The sample size n for which we can make the statement 'the probability is $1 - \alpha$ that the random interval $(Z_k, Z_{n-m+1})$ cover a proportion b or more of the population' is then determined by the equation (1.1) $I_b(n-r+1, r) = \alpha$, where $r = k + m$. Our approximate solution is (1.2) $n \doteq (1/4)\,\chi^2_\alpha\,(1+b)/(1-b) + (1/2)(r-1)$, where $\chi^2_\alpha$ is the 100 $\alpha$ percent point on the $\chi^2$ – distribution with 2r degrees of freedom. . . . For $\alpha = .1, .05, .025, .01, .005$ . . . and for $b \geq .9$, extensive numerical calculations indicate that the error of (1.2) is less than one tenth of one percent, and is always positive, that is, n is slightly overestimated by (1.2). We have not yet obtained an analytic proof of this statement, which refers to the difference from the exact (and, in general, non-integral) solution of (1.1)."

**Reference:** Scheffé (1943).

**Citations:** Wilks (1948), Birnbaum & Zuckerman (1949).

* Smirnov, N. V. (1944). Approximate laws of distribution of random variables from empirical data. (Russian). *Uspekhi Matematicheskikh Nauk* **10,** 179-206. (MR 7, 19).

**Summary:** The author obtains limit theorems concerning the relations between a theoretical cumulative distribution function (c.d.f.) and a sample c.d.f. and between two sample c.d.f.'s. On page 203, he states the following theorem concerning the convergence of the empirical c.d.f.'s $S_{n_1}(x)$ and $S_{n_2}(x)$ of two independent samples: If the sizes of the samples $n_1$ and $n_2$ increase without limit so that the ratio $n_2/n_1 = \tau$ remains constant whatever may be the continuous law of distribution F(x) of the quantity $\xi$, then $P(n_1, n_2) = P\{D(n_1, n_2)\,[= \sup_{-\infty<x<+\infty} |S_{n_1}(x) - S_{n_2}(x)|] \leq \lambda/\sqrt{n}\} \rightrightarrows \Phi(\lambda)$, $(n_1 \to \infty, n_2 \to \infty)$, where $\lambda > 0$, $n = n_1 n_2/(n_1 + n_2) = n_2/(1+\tau)$ and $\Phi(\lambda) = \sum_{k=-\infty}^{+\infty} (-1)^k e^{-2k^2\lambda^2}$. He points out that the limiting law $\Phi(\lambda)$ here is the same as for the upper limit of the deviation $|S_n(x) - F(x)|$ (the Kolmogorov-Smirnov test statistic). Concerning the distribution of the $\omega^2$ – criterion of von Mises, he writes [pp. 203-204 (translation)]: "$\omega_n^2 = n \int_{-\infty}^{\infty} g(F(x))\,[S_n(x) - F(x)]^2 dF(x)$, where g(t) is a positive weight function. The limiting distribution of $\omega_n^2$ as $n \to \infty$ does not depend on the form of the distribution F(x) and approaches the limiting law $\phi(\omega) = 1 - (1/\pi) \sum_{k=1}^{\infty} \int_{\lambda_{2k-1}}^{\lambda_{2k}} [e^{-\lambda\omega/2} / \sqrt{-D(\lambda)}]\,(d\lambda/\lambda)$, where $D(\lambda) = \prod_{k=1}^{\infty} (1 - \lambda/\lambda_k)$ is the determinant of the Fredholm positive symmetric kernel $K(x, y) = \sqrt{g(x)g(y)}\,x(1-y)$ for $x \leq y$, $K(x, y) = \sqrt{g(x)g(y)}\,y(1-x)$ for $x \geq y$, $(0 \leq x, y \leq 1)$ and $\lambda_k$ $(k = 1, 2, \ldots)$ are the characteristic values of the kernel K(x, y). It is assumed that the function $g(t)t(1-t)$ has continuous derivative in [0, 1]. In particular, taking $g(x) \equiv 1$, we will have the following limit law of the distribution: $\phi(x) = 1 - (2/\pi) \sum_{k=1}^{\infty} \int_{(2k-1)\pi}^{2k\pi} (e^{-\lambda^2\omega/2} / \sqrt{-\lambda \sin \lambda})\,d\lambda$."

**References:** von Mises (1931), Glivenko (1933), Kolmogoroff (1933), Smirnov [Smirnoff] (1935, 1937a,b, 1939a,b).

* Wing, S. P.; Price, Walter H.; Douglass, Clemeunt T. (1944). Precision indices for compression tests of companion concrete cylinders. *Proceedings of the American Society for Testing Materials* **44,** 839-851; discussion, 852-869.

**Summary:** The authors make extensive use of the range as a substitute for the standard deviation in connection with a study of the effect, on the mean and the coefficient of variation of the strength of concrete cylinders, of the ratio of cylinder diameter to maximum aggregate diameter. They give an approximate formula $(\overline{R}/d_2)\,](D_4 - 1)/3](1/\sqrt{m})$ for the standard deviation of a coefficient of variation determined from the ranges of m batches of n samples each, where $\overline{R}$ is the mean range and $d_2$ and $D_4$ are given in Table I of the *ASTM Manual on Presentation of Data* [Dodge et al. (1933)]. Portions of the discussion deal with the rejection of observations and with the distribution of ranges and the effects of grouping. The authors refer to the work of Rider (1933) on the former and that of Shewhart (1931) on the latter.

**References:** Shewhart (1931), Dodge et al. (1933), Rider (1933).

* Anderson, Theodore W., Jr. (1945). The Non-central Wishart Distribution and its Application to Problems in Multivariate Statistics. Doctoral dissertation, Princeton University, Princeton, N. J. Publication No. 2907, University Microfilms, Ann Arbor, Michigan.

**Summary:** The author derives the non-central Wishart distribution for the rank one and rank two cases. From the non-central Wishart distribution he derives the distribution of the ordered roots of certain determinantal equations. He develops likelihood ratio criteria for testing dimensionality of population means. The distribution of the roots involved in the test of dimensionality is derived in the cases of the population means being on a line or a plane. For the linear case, the distribution is given explicitly; for the planar case, the distribution is indicated as a certain definite integral. The author notes that Roy (1942b) obtained essentially the same distribution for the linear case (rank h = 1) by geometric methods. Roy claimed that his result holds for the general case, that is, for h K's [roots] different from 0, with $1 \leq h \leq p$, where p is the number of variables. Roy's distribution has only one parameter, namely, $\Sigma_{i=1}^{h} K_i^2$. The author states that it can easily be shown that Roy's result does not hold in the generality he claimed. He points out that for $h \geq 2$ Roy's distribution does not give proper limiting results, which shows that Roy's results hold only for h = 1. However, he notes that Hsu (1941a) obtained the limiting distribution in the planar case (h = 2), as well as in general.

**References:** Hotelling (1936b), Fisher (1938, 1939), Hsu (1939, 1941a,b), Roy (1942b), Wilks (1943).

**Citations:** Anderson (1946, 1948).

* Daniels, H. E. (1945). The statistical theory of the strength of bundles of threads. I. *Proceedings of the Royal Society of London A* **183,** 405-435. (MR 7, 19).

**Summary:** The author writes (Abstract, p. 405): "A group of parallel threads of equal length, clamped at each end so that all threads extend equally under tension, is called a bundle, and the maximum load which the bundle can support is called its strength. The object of the work is to study the probability distribution of the strength of bundles whose constituent threads are sampled randomly from an infinite population of threads in which the probability distribution of strength is known. The relation between the strength of a bundle and the strengths of its constituent threads is first discussed, and results are stated for bundles so large that the proportions of threads of different strengths approach their expectations. The properties of the probability distribution of bundle strength are next developed in detail, attention being confined in the present paper to the case where all threads have the same load-extension curve up to breaking point. Finally, the asymptotic behavior of the distribution for large numbers of threads is studied, and it is shown that in the commonest cases the distribution tends to assume the normal form." The author reviews the work of Peirce (1926) on the strength of bundles, and notes that Weibull (1939b) considered the strength of materials composed of independent parallel elements. On the relation between a bundle and its constituent threads, he writes (p. 406): "It is assumed that for each thread there is a definite extension and load at which it breaks, so that threads which yield rather than snap are excluded from the discussion. Whatever assumptions are made about the elastic properties of the threads, the following sequence of events occurs when a given load is applied. Suppose the load is increased gradually from zero to its final value. At first it is distributed in some way between the n threads, and equilibrium may be reached without any thread giving way. If the load is great enough, however, one of the threads breaks at some stage and the load is redistributed among the remaining n − 1 threads, each bearing a somewhat larger share of the load than before and so being more likely to give way. Similarly, a number of threads may break successively until either a point is reached where the remaining threads have sufficient strength to maintain between them the final load, or no such point is reached and all the threads ultimately give way, in which case, of course, the bundle is broken. To formulate the problem more precisely, it is necessary to know how the load distributes itself between the individual threads of the bundles, and this depends on the elastic properties of the threads. . . ."

**Note:** The simplest case is that in which the unbroken threads share the load equally, so that the weakest thread breaks first, then the second weakest, and so on, in order of increasing strength.

**References:** Peirce (1926), Weibull (1939b).

**Citations:** Epstein (1948a,b), Wilks (1948).

* Fowler, F. H., Jr. (1945). On fatigue failure under triaxial static and fluctuating stresses and a statistical explanation of size effect. *Transactions of the American Society of Mechanical Engineers* **67,** 213-215; discussion, 215-216.

**Summary:** The author offers the following statistical explanation of size effect (pp. 214-215): "Let $P(\sigma_p)d\sigma_p$ = probability of a yield point after pth period between $\sigma_p$ and $\sigma_p + d\sigma_p$, . . . $P(\sigma_f)d\sigma_f$ = probability of a true ultimate strength between $\sigma_f$ and $\sigma_f + d\sigma_f$. . . . . Now the probability that the true ultimate strength is less than $\sigma_f$ is given by the formula $P_f(\sigma_f) = \int_0^{\sigma_f} P(\sigma_f)d\sigma_f$. The probability that the final yield point lies between $\sigma_p$ and $\sigma_p + d\sigma_p$, and that such a final yield point will result in failure, is given by the expression $P_f(\sigma_p)P(\sigma_p)d\sigma_p$ and the probability of failure in a unit volume is given by the formula $P_v = \int_0^x P_f(\sigma_p)P(\sigma_p)d\sigma_p$. The next step is to consider the final problem of whether there will be a failure in the unit. $1 - P_v$ = probability of no failure in a unit volume; q = probability of no failure in a volume dv, or the probability of no failure at a given point; 1/dv = number of volumes dv per unit volume. As the probability of no failure in a unit volume is the probability of no failure in all the volumes dv comprising it, or the product of the probabilities of no failure in all of these elementary volumes $(q)^{1/dv}dv = 1 - P_v$; $q = (1 - P_v)^{dv}$. Therefore $(1 - P_v)^{dv}$ = probability of no failure at a given point having a volume dv. As $P_v$ is now considered known for every point in the solid, the probability of there being a point in this solid at which a failure will occur can be calculated as follows: Let P = probability of a failure in the entire unit; then $1 - P = \prod_{i=1}^{v/dv}(1 - P_v)^{dv}$, and $\ln(1 - P) = \Sigma \ln(1 - P_v)^{dv} = \Sigma \ln(1 - P_v)\,dv$; $P = 1 - e^{-\int \ln (1 - P_v)dv}$ [6]. This statistical procedure has been used and checked experimentally by Weibull [(1939b)] for brittle materials. In place of Equation [6], Weibull uses the approximate formula $1 - e^{-\int P_v dv}$. . . . If two pieces of different size are geometrically similar and subjected to the same stress distributions, the probability of failure is changed. In Equation [6] let $B_0 = e^{\int \ln(1 - P_v)dV_0}$, where $V_0$ = volume of small piece, for the small piece. Note that $1 - P_v$ is always less than 1, hence $\ln(1 - P_v)$ is always negative and $B_0$ is never more than 1. Let $B_1 = e^{-\int \ln(1 - P_v)dV_1} = B_0^{V_1/V_0}$, where $V_1$ = volume of large piece; as $V_1 > V_0$, $B_1 < B_0$. Hence, for the large piece, P is always greater than for the small piece unless $B_0$ is 0 or 1. If the endurance limit is regarded as the fluctuating stress for which a fatigue failure and no fatigue failure are equiprobable, then the endurance limit becomes, for geometrically similar pieces, a function of the size of the piece."

**Note:** If the stress is uniform throughout the piece, the piece fails if and only if the strength of the weakest elementary volume is less than the stress, in accordance with the "weakest link" theory.

**References:** Peterson (1930), Weibull (1939b), Moore & Morkovin (1942-44).

**Citation:** Peterson (1949a).

* Gumbel, E. J. (1945a). Simplified plotting of statistical observations. *Transactions of the American Geophysical Union* **26,** 69-82. (MR 7, 133—listing only).

**Summary:** The author writes (pp. 70-71): "The distributions $w(x_n, n)$ and $w(x_1, n)$ of the largest and smallest value [of a sample of size n from a population with p.d.f. f(x) and c.d.f. F(x)] are $w(x_n, n) = n F(x)^{n-1} f(x)$ and $w(x_1, n) = n[1 - F(x)]^{n-1} f(x)$ (5). If the distributions (5) possess modes $\tilde{x}_n$ and $\tilde{x}_1$ they are the solution of . . . $(n - 1) = -[f'(\tilde{x}_n)F(\tilde{x}_n)]/f^2(\tilde{x}_n)$ or $(n - 1) = [f'(\tilde{x}_1)\,[1 - F(\tilde{x}_1)]/f^2(\tilde{x}_1)$ (7). The probabilities $F(\tilde{x}_n)$ and $F(\tilde{x}_1)$ obtained from these equations will be called *adjusted frequencies* and will

be used for plotting the extreme members of a sample. Finally $T/\tilde{x}_n) = 1/[1 - F(\tilde{x}_n)]$ (8) is the occurrence-interval. This solution holds for any value of n." He states the following conclusions (p. 81): "The observed return-period of a value equal to, or larger than, the largest among n observations, called the occurrence-interval, is a function of n, and ought to be calculated from the most probable largest value, that is, from the formulas (7) and (8). For the usual distributions, the occurrence-interval $T(\tilde{x}_n)$ converges toward n. This would not hold if we determine the occurrence-interval from the median or from the mean of the largest value. . . . The adjusted frequencies $F(\hat{x}_m)$ of the m'th observation are obtained by linear interpolation between the probability $F(\tilde{x}_1)$ of the most probable smallest observation and the probability $F(\tilde{x}_n)$ of the most probable largest observation. This method . . . leads to one, and only one, series of observations to be plotted on probability paper and eliminates the use of the exceedance- and recurrence-intervals. . . . If the number of observations is at least 100, it is sufficient to use the asymptotic formulas (34) and (35) $\{F(\hat{x}_m) = (m/n) - [m - 1]/[n(n - 1)] = [m - 1)/n] + [n - m]/[n(n - 1)]$ and $T/\hat{x}_m) = n/[n - m + (m - 1)/(n - 1)]$, after correction of an obvious omission in formula (35)}."

**References:** von Mises (1923), Tippett (1925), Fisher & Tippett (1928), Hazen (1930), Gumbel (1937b, 1941a), Beard (1942) [1943], Gumbel (1943b).

**Citations:** Nordquist (1945), Kimball (1946b), Potter (1949a,b), Press (1949).

* Gumbel, E. J. (1945b). Floods estimated by probability method. *Engineering News Record* **134,** 833-837.

**Summary:** The author writes (Contents in brief, p. 833): "Defining a flood for mathematical purposes as the largest daily discharge observed in a 365-day period and considering flood as an unlimited statistical variable, the author utilizes the theory of probability to estimate future floods. The probability of a flood equal to or less than a certain magnitude is expressed by a formula, but to facilitate the application of the theory a special probability plotting paper is used. However, since there is no plotting position on probability paper for zero or one, in order to plot the entire series of observations it is suggested that the plotting positions be based on the probabilities of the most probable floods. The procedure is demonstrated for the floods of the Mississippi River as recorded at Vicksburg from 1890 to 1939. . . ." The author uses extreme-value probability paper with plotting positions $F_1 = 1/n$, $F_n = (n - 1)/n$, and $F_m = F_1 + (m - 1)(F_n - F_1)/(n - 1)$, $m = 2, \ldots, n - 1$ (linear interpolation between $F_1$ and $F_n$).

**References:** Fuller (1914), Hazen (1930), Gumbel (1941a,b), Powell (1943).

**Citations:** Gumbel (1948), Lane & Lei (1949).

* Gurney, C. (1945). Effect of length on tensile strength. *Nature* **155,** 273-274.

**Summary:** The author writes (p. 273): "It is of interest to derive the distribution of strength of rods the length of which differs from those tested. Consider first the strength of rods which are n times as long as those tested. . . . If $F_1(x)$ and $F_n(x)$ are the probability integrals for the strength of test pieces of length 1 and n units respectively, the distribution of the strength of the long test pieces is obtained by calculating the chance that n unit test pieces chosen at random shall all be stronger than a given strength; for this is the chance that the weakest of n test pieces shall exceed the given strength. Thus $1 - F_n = (1 - F_1)^n$. This result was given by Pierce [sic; Peirce (1926)]." The author shows that this result holds not only for n an integer, but also for $n = 1/m$, where m is an integer. Assuming that it holds for all values of n and that the strength is N(0, 1) [normally distributed with zero mean and unit standard deviation] when $n = 1$, he plots the mean $\mu$, the standard deviation $\sigma$, and the skewness $\sqrt{\beta_1}$ of the distribution of strength as functions of $\log_{10}n$. His graph indicates that $\mu$, $\sigma$ and $\sqrt{\beta_1}$ all decrease as $\log_{10}n$ (and hence n) increases; however, $\sqrt{\beta_1}$ actually increases as n increases. The author has taken the wrong sign for $\sqrt{\beta_1}$ throughout, an error which probably would not have occurred if he had denoted the skewness by $\alpha_3$ instead of $\sqrt{\beta_1}$.

**References:** Tippett (1925), Peirce (1926).

**Citations:** Gensamer, Saibel & Lowrie (1947), Gurney (1947), Gurney & Pearson (1947).

* Nordquist, John M. (1945). Theory of largest values applied to earthquake magnitudes. *Transactions of the American Geophysical Union* **26** (1), 29-31.

**Summary:** The author writes (pp. 29-30): "The theory of largest values [see Gumbel (1942c, 1943b)] requiring data on only the largest event in each of a number of equal time-intervals, recommended itself to the author as a possible way of describing the earthquake activity of a region. A preliminary analysis using the calendar month as the interval having demonstrated the applicability of the theory in this field, the author has investigated the feasibility of using still shorter intervals . . . . In the present paper the interval used is one-third of a month — the first day through the tenth, the eleventh through the twentieth, and the twenty-first to the end of the month. This interval therefore corresponds to that of one year used in flood-analyses. A compensating change has been made in the return-period scale . . . . Incomplete results of further analysis indicate that it is feasible to use a time-interval as short as five days. . . . The materials used include (a) a list . . . of all 'normal' earthquakes of magnitude 7-3/4 and over, for the years 1904-39, (b) a list . . . of all 'normal' earthquakes of magnitude 7 and over from January, 1926, through March, 1934, (c) a list of the earthquakes in a region of about 100,000 square miles, including most of southern California and part of Lower California, from January, 1934, through September, 1942, and (d) a selection from (c) of the earthquakes in a region of about 1300 square miles, east of Los Angeles. . . . Table 1 . . . [not reproduced here] shows the reduction [see Gumbel (1945a)] of the data from the corresponding lists to the variable $F(\bar{x}_m)$ plotted on GUMBEL'S probability paper in Figure 2 [not reproduced here]. It will be seen from Figure 2 that all the points plotted for the earthquakes of the entire world can be represented fairly well by a straight line. The same is true for the plotted points for each smaller region separately, indicating that in each case the observed distribution of the magnitude of the largest earthquake is in good agreement with the theory of largest values even when so short a time-interval as ten days is used as a unit."

**References:** Gumbel (1942c, 1943b, 1945a).

* Roy, Samarendra Nath (1945). The individual sampling distribution of the maximum, the minimum and any intermediate of the p-statistics on the null hypothesis. *Sankhyā* **7,** 133-158. (MR 7, 317).

**Summary:** The author writes (Introduction, p. 133): "Given two random samples S′ and S″ of sizes n′ and n″, and variance and covariance matrices $||a'_{ij}||$ and $||a''_{ij}||$ drawn from two p-variate normal populations $\Sigma'$ and $\Sigma''$ with variance and covariance matrices $||\alpha'_{ij}||$ and $||\alpha''_{ij}||$, it was shown earlier by the author [Roy (1939)] and also by others [Fisher (1939), Hsu (1939)] that the null hypothesis (for the populations) $||\alpha'_{ij}|| = ||\alpha''_{ij}||$ could be appropriately tested by a set of p-statistics given by the p roots $k_1^2, k_2^2, \ldots, k_p^2$ (all positive in this particular case) of the determinantal equation in $k^2 \rightarrow |a'_{ij} - k^2 a''_{ij}| = 0$. The joint sampling distribution of these p-statistics on the null hypothesis $||\alpha'_{ij}|| = ||\alpha''_{ij}||$ was at the same time obtained by the author [Roy (1939)] and by others [Fisher (1939), Hsu (1939)]. The joint distribution of these $k_i$'s ($i = 1, 2, \ldots, p$) on the non-null hypothesis (that is, when $||\alpha'_{ij}|| \neq ||\alpha'_{ij}||$) was also given by the author a little later [Roy (1942a, b)] and it was found that this involved as population parameters p quantities $\kappa_1^2, \kappa_2^2, \ldots, \kappa_p^2$ which are all unity when and only when $||\alpha'_{ij}|| = ||\alpha''_{ij}||$ and which come out as the p roots (all positive in this case) of the determinantal equation in $\kappa^2 \rightarrow |\alpha'_{ij} - \kappa^2 \alpha''_{ij}| = 0 \ldots$." In the present paper the author derives the distributions of the maximum, the minimum and any intermediate root under the null hypothesis. These distributions can be expressed as p-fold integrals which are evaluated with the aid of a recursion formula reducing them step by step to single integrals. The author applies the results to the multivariate analysis of variance.

**References:** Fisher (1939), Hsu (1939), Roy (1939, 1942a,b).

**Citations:** Roy (1946a,b), Bartlett (1947b), Nanda (1948a,b).

* Scheffé, H.; Tukey, J. W. (1945). Non-parametric estimation. I. Validation of order statistics. *Annals of Mathematical Statistics* **16,** 187-192. (MR 7, 21).

**Summary:** The authors write (p. 187): "Previous work on non-parametric estimation has concerned three problems: (i) confidence intervals for an unknown quantile [Thompson (1936), Nair (1940b)], (ii) population tolerance limits [Wilks (1941, 1942)], (iii) confidence bands for an unknown cumulative distribution function (*cdf*) [Kolmogoroff (1933), Wald & Wolfowitz (1939, 1941)]. For problem (iii) a solution has been available which is valid for any *cdf* whatever, but for (i) and (ii) it has heretofore been assumed that the population has a continuous probability density. This paper validates the existing solutions of (i) and (ii) assuming only a continuous *cdf*. It then modifies these solutions so that they are valid for any *cdf* whatever. . . . $E(X_1, \ldots, X_n)$ will denote a random sample from a population with *cdf* $F(x)$, whereas $e = (x_1, \ldots, x_n)$ will denote a point in the sample space $R_n$. If t is a function of e only, $t = \phi(e)$, then the random variable $T = \phi(E)$ is a statistic. The *order statistics* of the sample E are defined to be $-\infty$, $Z_1, \ldots, Z_n, +\infty$, where $z_1 \leq z_2 \leq \ldots \leq z_n$ is a rearrangement of $x_1, x_2, \ldots, x_n$. We shall write $Z_0 = -\infty, Z_{n+1} = +\infty$. The device of including $+\infty$ and $-\infty$ among the order statistics will enable us to avoid special statements to cover the case of one-sided estimation. . . ." The authors state and prove the following theorem (p. 190): "A sufficient condition for the joint distribution of $F(T_1)$, $F(T_2)$, . . ., $F(T_r)$ to be independent of F in $\Omega_2$ [the class of all continuous cdf's] is that the $\{T_j\}$ be a subset of the order statistics $\{Z_i\}$ of the sample." They modify the solution to make it valid for any cdf.

**References:** Scheffé (1943), Robbins (1944a).

**Citations:** Kimball (1947), Tukey (1947), Noether (1948), Tukey (1948a), Wilks (1948), Noether (1949), Walsh (1949a).

* Tintner, Gerhard (1945). A note on rank, multicollinearity and multiple regression. *Annals of Mathematical Statistics* **16,** 304-308. (MR 7, 132-133).

**Summary:** The author writes (pp. 304-305): "Let $X_{it}$ (i = 1, 2, . . ., M) be [a] set of M random variables, each being observed at t = 1, 2, . . ., N. $X_{it} = M_{it} + y_{it}$. . . . The systematic part of our variables $M_{it} = E\,X_{it}$. The $y_{it}$ are normally distributed with means zero. Their variances and covariances are independent of t. The $M_{it}$ and $y_{it}$ are independent of each other. Define $\overline{X}_i = \Sigma_t X_{it}/N$ the arithmetic mean of $X_{it}$ and $x_{it} = X_{it} - \overline{X}_{it}$ the deviation from the mean. Then $a_{ij} = \Sigma_t x_{it} x_{jt}/(N - 1)$ gives the variances and covariances of the observations. We want to determine the rank of the matrix of the variances and covariances of $M_{it}$. Now assume that $\| V_{ij} \|$ is an estimate of the variance-covariance matrix of the error terms or 'disturbances' $y_{it}$. The elements of this matrix are distributed according to the Wishart distribution and are independent of the $M_{it}$. . . . Assume that the estimate is based upon N′ observations. Form the determinantal equation: (1) $|a_{ij} - \lambda V_{ij}| - 0$. Apart from sampling fluctuations there should be r solutions $\lambda = 1$ of equation (1) if there are r independent linear relationships between the $M_{it}$. The rank of the variance-covariance matrix of $M_{it}$ is then M − r. Following a suggestion of P. L. Hsu [(1941b)] made on the basis of the earlier work of R. A. Fisher [(1938)] we form the test function (2) $\Lambda_r = (N - 1)(\lambda_1 + \lambda_2 + \ldots + \lambda_r)$, where $\lambda_1$ is the smallest root of (1), $\lambda_2$ the next smallest, etc. Hence (2) is [(N − 1) times] the sum of the r smallest roots of equation (1). The hypothesis to be tested is that there are exactly r independent linear relationships between the systematic parts of our variables in the population. This quantity (2) is distributed like $\chi^2$ with r(N − M − 1 + r) degrees of freedom for large samples, i.e. if N′ becomes large. It can be used for forming an opinion about the number of independent relationships existing among the systematic parts of our variables ($M_{it}$)." On page 307, the author gives an expression which can be used to find the R smallest latent roots $\lambda_\nu$ and the corresponding characteristic vectors by the methods of Hotelling (1936a).

**References:** Hotelling (1936a), Fisher (1938), Wald (1940), Hsu (1941b).

**Citations:** Tintner (1946), Anderson (1948), Geary (1948).

* Tucker, John, Jr. (1945a). Effect of dimensions of specimens upon the precision of strength data. *Proceedings of the American Society for Testing Materials* **45,** 952-959; discussion, 960.

**Summary:** The author writes (Synopsis, p. 952): "Statistical theories on the strength of materials are applied in estimating the effects of change in a dimension of the specimen on the mean strength and on the dispersion of the strength within a group of like specimens. The weakest-link theory [see Weibull (1939a)] predicts the effects of an increase in the length of specimens subjected to tension, compression, flexure, or torsion and in the depth of specimens subjected to flexure. The strength-summation theory [see Tucker (1941)] appears to predict reasonably well the effects of an increase in the cross-sectional area of specimens subjected to tension or compression and of an increase in width of the flexural specimen. . . ."

**Note:** The weakest-link theory depends on order statistics (extreme values), but the strength-summation theory does not.

**References:** Tippett (1925), Tucker (1927), Weibull (1939a), Tucker (1941).

**Citations:** Tucker (1945b), Frankel (1948).

* Tucker, John, Jr. (1945b). The maximum stresses present at failure of brittle materials. *Proceedings of the American Society for Testing Materials* **45,** 961-973; discussion, 974-975.

**Summary:** The author writes (p. 962): "It is not apparent which particular statistical theory [see Tucker (1941)] will apply to different materials, and which should be used to predict the changes in strength and in scatter in strength, concomitant with change in dimensions. In our present analysis, the tensile strength characteristics of elementary fibers, of which a tension test cylinder is assumed to be composed, must be determined from the strength characteristics of the cylinder. Test data, for example on concrete [see Tucker (1927)]. . . indicated that for a change in cross-sectional area, the summation theory applied. If we apply this theory to our analysis in the present paper, the predicted modulus of rupture is but little more than the tensile strength of the materials, a result discordant with fact. If we apply the weakest link theory [see Tucker (1941)] we obtain a predicted modulus of rupture greater than the tensile strength, but not so great as experiment shows. A combination theory is offered, which has the merit of predicting values for the modulus of rupture which are adequately large.

**References:** Tippett (1925), Tucker (1927), Weibull (1939a), Tucker (1941, 1945a).

**Citation:** Frankel (1948).

* Anderson, T. W. (1946). The non-central Wishart distribution and certain problems of multivariate statistics. *Annals of Mathematical Statistics* **17,** 409-431. (MR 8, 394).

**Summary:** The author states (in a footnote on p. 409) that this paper is part of his doctoral dissertation [Anderson (1945)]. He again points out (p. 429), as in the dissertation, that the distribution of the (ordered) roots of the determinantal equation $|a_{ij} - \lambda b_{ij}| = 0$ in the linear case was given by Roy [(1942b)] for $a_{ij}$ of dimensionality p, that Roy erroneously claimed that his distribution is valid for the planar case and higher rank, and that the correct distribution in the planar case was given in the author's dissertation [Anderson (1945)].

**References:** Roy (1942b), Wilks (1943), Anderson (1945).

**Citations:** Bartlett (1947a,b).

* Baker, G. A. (1946). Distribution of the ratio of sample range to sample standard deviation for normal and combinations of normal distributions. *Annals of Mathematical Statistics* **17,** 366-369. (MR 8, 43—listing only).

**Summary:** Previous studies have been made of the distribution of the ratios of the sample range to the population standard deviation and to an independent estimate of the population standard deviation. Here a study is made of the distribution of the ratio of the range to the standard deviation when both are computed from the same sample. The mean, standard deviation, $g_1 = \sqrt{\beta_1}$, $\sigma_{g_1}$, $g_2 = \beta_2 - 3$, and $\sigma_{g_2}$ of the distribution of the ratio of range to standard deviation for samples of size $n = 2, 4, 16, 36, 64, 100$ are tabulated for a normal parent [N(0,1)], a symmetrical distinctly bimodal parent (A) and a weakly bimodal but strongly skewed parent (B). The results are quite different from the corresponding distributions of range in terms of the population standard deviation [see Pearson (1931)]. The standard deviation and the range of the same sample are correlated. This correlation is negligible for $n \geq 100$ for samples from the standard normal population N, but not for the populations A and B, for which it is of the order of 0.5 for samples of 100.

**References:** K. Pearson (1931), Newman (1939), Pearson & Hartley (1943), Hartley (1944).

* Brown, George W.; Tukey, John W. (1946). Some distributions of sample means. *Annals of Mathematical Statistics* **17,** 1-12. (MR 7, 463).

**Summary:** The authors summarize their results as follows (p. 1): "It is shown that certain monomials in normally distributed quantities have stable distributions [see Note below] with index $2^{-k}$. This provides, for $k > 1$, simple examples where the mean of a sample has a distribution equivalent to that of a fixed, arbitrarily large multiple of a single observation. These examples include distributions symmetrical about zero, and positive distributions. Using these examples, it is shown that any distribution with a very long tail (of average order $\geq x^{-3/2}$)has the distributions of its sample means grow flatter and flatter as the sample size increases. Thus the sample mean provides *less* information than a single value. Stronger results are proved for still longer tails." They write (pp. 2-3): "Section 10 presents a simple example of a distribution symmetric about zero with such long tails that (i) the distribution of the sample mean spreads out faster than any power of n [the sample size], (ii) the median of a sample of any size fails to have finite moments of any positive order, integral or fractional. . . . The basic consequences of these results for applied statistics can be summarized in the following statements. (a) The positions that the Cauchy distribution is an isolated case, or else an extreme example of pathology, are now untenable. (b) The use of the mean of a sample as a measure of location (or, when dealing with positive distributions fixed at zero, as a measure of scale) implies a belief that the tails of the underlying distribution are not too long. (c) It is probable that the relative efficiencies of mean and median are greatly affected by the length of the tail."
**Note:** The authors write (p. 4): "A distribution is stable if whenever k and $\ell$ are positive and A and B are independent chance quantities distributed according to the same law, then $kA + \ell B$ is distributed like a fixed multiple of A. It is known . . . that any stable distribution has a characteristic function of the form $e^{-(\alpha + i\beta \operatorname{sgn} u)|u|^\lambda}$, where $0 < \lambda \leq 2$, $\alpha > 0$, and $|\beta| \leq |\alpha \tan \frac{1}{2}\pi\lambda|$. Each stable distribution thus has an index $\lambda$ such that $kA + \ell B$ and $(k^\lambda + \ell^\lambda)^{1/\lambda} A$ have the same distribution when A and B are a sample of two from the given distribution."

**Citation:** Mosteller (1946).

* Carlton, A. George (1946). Estimating the parameters of a rectangular distribution. *Annals of Mathematical Statistics* **17,** 355-358. (MR 8, 41—listing only).

**Summary:** The author writes (Introduction, p. 355): "In this note, the range and midrange of the sample are shown to be a pair of sufficient statistics, and maximum likelihood estimates, for the true range and true mean of a rectangular distribution; exact and limiting distribution of midrange, range, and their ratio are derived; the 'efficiencies' of the sample mean and median as estimates of the true mean are calculated; and the limiting distribution of the difference between two sample midranges is derived. All

the limiting distributions are non-normal, and the error of estimate is of order $n^{-1}$ rather than the customary $n^{-1/2}$. The limiting distribution of midrange, and the limiting ratio of variances of the midrange and the sample mean were given by Fisher [(1922)]." For the one-parameter rectangular distribution $f(x) = 1/\lambda$, $0 \le x \le \lambda$, the author points out (p. 358) that "v [the largest observation] is a sufficient statistic and is evidently the maximum likelihood estimate of $\lambda$."

**Reference:** Fisher (1922).

**Citations:** Gumbel (1947), Wilks (1948).

* Cramér, Harald (1946). *Mathematical Methods of Statistics*. Princeton University Press, Princeton, NJ. (MR 8, 39).

**Summary:** This book is divided into three parts, devoted to mathematical introduction, random variables and probability distributions, and statistical inference. Only the portions directly relevant to our subject will be summarized here. In the second part, Chapter 15 (pp. 166-192) deals with general properties of univariate distributions. In Section 15.5 (pp. 177-179), the author discusses measures of location, including the median, which he defines as a value which divides the whole mass of the distribution into two equal parts, each containing the mass 1/2. The median thus defined may or may not be unique. In either case, the median has the property that the first absolute moment is a minimum when taken about the median. Section 15.6 (pp. 179-182) deals with measures of dispersion, including the mean deviation (about the median rather than the mean), the range, and the semi-interquartile range. The author also defines the quantiles in general, and specifically mentions the quartiles and the deciles. All of these definitions and results are for population measures, but analogous ones hold for sample measures, which are discussed in the third part, especially in Chapter 28 (pp. 363-378), which deals with asymptotic properties of sampling distributions. In Section 28.5 (pp. 367-370), the author discusses the sample quantiles, which he defines as follows (p. 368): "If np is not an integer, and if we arrange the sample values in ascending order of magnitude: $x_1 \le x_2 \ldots \le x_n$, there is a unique quantile $z_p$ equal to the sample value $x_{\mu+1}$, where $\mu = [np]$ denotes the greatest integer $\le np$. If np is an integer, . . . $z_p$ may be any value in the interval $(x_{np}, x_{np+1})$. In order to avoid trivial complications, we assume . . . that np is not an integer." If the frequency function [p.d.f.] $f(x)$ is continuous and has a continuous derivative $f'(x)$, the author shows that the sample quantile $z_p$ is asymptotically normal $(\zeta, \sqrt{pq/n}/f(\zeta))$, where $\zeta = \zeta_p$ is the corresponding quantile of the population, and in particular the median of the sample is asymptotically normal $(\zeta, 1/2f(\zeta)\sqrt{n})$, where $\zeta = \zeta_{1/2}$ is the median of the population. For a normal distribution with parameters m and $\sigma$, the median is m, and $f(m) = 1/\sigma\sqrt{2\pi}$. Thus the median z of a sample of n from this distribution is asymptotically normal $(m, \sigma\sqrt{\pi/2n})$, whereas the mean $\bar{x}$ is exactly normal $(m, \sigma/\sqrt{n})$, so that the standard deviation of z is $\sqrt{\pi/2} \doteq 1.2533$ times that of $\bar{x}$. The author also shows that the joint distribution of two sample quantiles z′ and z″ is asymptotically normal. If $\zeta'$ and $\zeta''$ are the lower and upper quartiles of the population, the semi-interquartile range of the sample, $(z'' - z')/2$, is asymptotically normal with mean $(\zeta'' - \zeta')/2$ and standard deviation $\sqrt{3f^2(\zeta') - 2f(\zeta')f(\zeta'') + 3f^2(\zeta'')}/8\sqrt{n}$. Thus, for a normal $(m, \sigma)$ population, the mean and standard deviation of the semi-interquartile range of a sample of size n are $0.6745\,\sigma$ and $0.7867\,\sigma/\sqrt{n}$, respectively. In Section 28.6 (pp. 370-378), the author considers some properties of the sampling distributions of the extreme values, the $r$:th values [order statistics], the range, and other similar characteristics of the sample for continuous distributions in general and the rectangular, (isosceles) triangular, Cauchy, Laplace (double exponential), and normal distributions in particular, along with the limiting forms of extreme-value distributions [see Fisher & Tippett (1928) and Gumbel (1935a)]. Chapters 32-34 (pp. 473-524) deal with the theory of estimation, and in them the author gives several examples in which he compares the efficiency of estimators based on order statistics (median, smallest value, or largest value) or functions of order statistics (range, midrange, or quasi-midranges) with that of estimators based on the mean and the standard deviation. In Example 4 (p. 485) of Chapter 32, he shows that in some cases of nonregular estimation, estimates based on order statistics have variances which are smaller than the Cramér-Rao lower bound.

**References:** Gauss (1809), Laplace (1812), Gauss (1816, 1823) [1880], Helmert (1876b), Pearson (1895), Poincaré (1896) [1912], Sheppard (1899a), Kapteyn (1903) [Kapteyn & van Uven (1916)], Edgeworth (1905), Charlier (1906, 1910 [1931]), Yule (1911) [Yule & Kendall (1940)], Fisher (1912), Keynes (1921), Fisher (1922), Irwin (1925a), Lévy (1925), Tippett (1925), Cramér (1928), Fisher & Tippett (1928), Neyman & Pearson (1928), Pearson [& Adyanthāya] (1928), von Mises (1931), Pearson (1931), Fisher (1933), Davies & Pearson (1934), Fisher (1935) [1937], Gumbel (1935a), Thompson (1935), von Mises (1936), Neyman & Pearson (1936), Smirnoff (1936), Neyman (1937 a,b), Pitman (1937), Snedecor (1937), Fisher & Yates (1938) [1943], Wilks (1938), Jeffreys (1939), Pitman (1939), Arley (1940), Hartley (1942), Pearson & Hartley (1942), Kendall (1943), Geary (1944).

**Citations:** Arley & Buch (1940) [1950], Boldrini (1942) [1968], Elfving (1947), Lehmann (1947), Epstein (1948a,b), Epstein & Brooks (1948), Hoeffding (1948), Huzurbazar (1948), Jones (1948), Wilks (1948), Benson (1949), Epstein (1949a), Noether (1949), Smirnov (1949b), Walsh (1949d).

* Daly, Joseph F. (1946). On the use of the sample range in an analogue of Student's t-test. *Annals of Mathematical Statistics* **17,** 71-74. (MR 7, 464).

**Summary:** The author writes (p. 71): "Let $x_1, \ldots, x_n$ represent independent observations on a variate x which is normally distributed with mean $\mu$ and variance $\sigma^2$. Assuming no prior information about the value of either parameter, let $H_0$ be the hypothesis that $\mu$ is equal to or less than a specified quantity $\mu_0$. . . . Dodge [(1932)] and Knudsen [(1943)] among others have proposed tests of $H_0$ based on a statistic of the form $G = (\bar{x} - \mu_0)/w$ where w is the sample range. It is the object of this note to show how the probability distribution of G can be obtained with the aid of the distribution law of w tabulated by Pearson and Hartley [(1942)], and to present some numerical results which indicate that the power of the resulting test is the same for all practical purposes as that of 'Student's' t-test for sample size $N \leq 10$. The calculation of the percent points of the G distribution is greatly facilitated by the following result, which does not appear to be generally known: LEMMA: If $\bar{x}$ and w represent respectively the average and the range of a sample of N independent observations on a normally distributed variate x, then $\bar{x}$ and w are statistically independent." The author tabulates estimated upper 5% points (3DP) for the distribution of G for N = 3(1)10.

**References:** Tippett (1925), Dodge (1932), Pearson & Hartley (1942), Knudsen (1943).

**Citations:** Lord (1947), Walsh (1947), Wilks (1948), Howell (1949), Tukey (1949b), Walsh (1949b).

* Freudenthal, A. M. (1946). The statistical aspect of fatigue of materials. *Proceedings of the Royal Society* (London) *A* **187,** 416-429.

**Summary:** The author writes (Abstract, p. 416): "The phenomenon which is usually described as 'fatigue' of materials, but for which the term 'progressive failure' would be more adequate, is the expression, on a macroscopic scale, of the progressive destruction of the cohesive bonds as a result of the repetitive action of an external load. It has the typical features of a mass phenomenon; both the cohesive bonds and the load repetitions are collectives in a statistical sense [von Mises (1931)]. By applying the fundamental rules of the theory of probability many of the experimentally established relations between the principal variables can be theoretically deduced from the purely formal assumption of the existence of a statistical distribution function of the separation-strength of cohesive bonds." Concerning the size effect, he writes (p. 422): "For samples of different size and, consequently, of different number of bonds a definite size effect must be expected; as the probability of rupture under a definite load amplitude S increases with increasing number of bonds, the fatigue strength is bound to decrease with increasing sample size [weakest link theory], a conclusion which is supported by experimental evidence. . . . The size

effect in the stage of the initiation of fatigue cracks is modified by an opposing size effect which pertains to the stage of crack propagation: at equal rate of propagation the small cross-section will be destroyed more rapidly. The resulting effect depends, therefore, on the relative importance of the stages of crack initiation and of crack propagation in the process of rupture in fatigue."

**Reference:** von Mises (1931).

**Citations:** Epstein (1948a), Peterson (1949a), Weibull (1949).

* Grant, Eugene L. (1946). *Statistical Quality Control.* McGraw-Hill Book Co., Inc., New York-London.

**Summary:** Part One of this pioneering textbook contains two chapters dealing with the objectives of statistical quality control and some representative applications. Part Two deals with Shewhart control charts for variables ($\overline{X}$ and R charts). Chapter III (pp. 47-84) introduces some fundamental statistical concepts. In particular, Section 51 (pp. 58-60) deals with averages (arithmetic mean, median, and mode) and measures of dispersion (standard deviation and range). The arithmetic mean (average) $\overline{X}$ is used throughout the book, and the median (like the mode) is referred to only occasionally. The range R is used extensively instead of the standard deviation $\sigma$. Chapters IV (pp. 85-123) and V (pp. 124-147) explain why the control chart works. In Section 69 (pp. 93-95), the author discusses the relationship between $\sigma'$ (the true standard deviation) and $\overline{R}$ (the average range of subgroups of size n). He compares the estimates of $\sigma'$ from $\bar{\sigma}$ and $\overline{R}$, which are $\bar{\sigma}/c_2$ and $\overline{R}/d_2$, respectively, where $c_2$ and $d_2$ are tabulated (to 4 significant figures) in Table B (p. 536) of Appendix III for n = 2(1)25(5)100. He writes (p. 95): "A practical point is that it is much easier to compute R for a subgroup by making a simple subtraction than it is to compute $\sigma$ by calculating several squares and a square root. In control-chart work, this ease of calculation of R is usually much more important than any slight theoretical advantages that might come from the use of $\sigma$ as a measure of dispersion of subgroups. However, in some cases where the measurements themselves are costly (for example, destructive tests of valuable items) and it is necessary that the inferences from a limited number of tests be as reliable as possible, the extra cost of calculating standard deviations of subgroups is justified." In Section 79 (pp. 115-116), the author discusses the distribution of the range. He notes that Pearson & Hartley (1942) have tabulated the probability integral of the range in samples from a normal population. The upper and lower control limits on a control chart for R are $D_4\overline{R}$ and $D_3\overline{R}$, respectively, where $D_3$ and $D_4$ are tabulated in Table C (p. 537) of Appendix III. If the control limits are to be calculated from a known or assumed value of $\sigma'$, they are $D_2\sigma'$ and $D_1\sigma'$, respectively; where $D_1$ and $D_2$ are tabulated in Table E (p. 539). In Chapter VI (pp. 145-180), the author gives directions for simple $\overline{X}$ and R charts. The central line for the $\overline{X}$ chart is either the grand mean $\overline{\overline{X}}$ or a standard or aimed-at value $\overline{X}'$. The control limits lie at a distance $A_2\overline{R}$ on either side of the central line, at $\overline{\overline{X}} \pm A_2\overline{R}$ or $\overline{X}' \pm A_2\overline{R}$, where $A_2$ is tabulated in Table C of Appendix III. The values of $A_2$, $D_3$ and $D_4$ in Table C and of $D_1$ and $D_2$ in Table E are given to two decimal places for n = 2(1)20; for $n \leq 6$, the lower control limit for the R chart is zero, so $D_1 = D_3 = 0$. Chapter VII (pp. 181-198) deals with the selection of rational subgroups and Chapter VIII (pp. 199-220) with different adaptations of the control chart for variables (including variable subgroup size, control limits for moving averages, probability limits, warning limits, etc.). Part Three discusses other Shewhart control charts (for fraction defective and for number of defects), which do not make use of order statistics. Part Four gives a statistical approach to acceptance procedures, and includes Chapter XII (pp. 315-338) on some statistical aspects of tolerance [Wilks (1941, 1942) and Wald (1943)] and Chapter XVI (pp. 419-446) on acceptance procedures based on the control chart for variables. Part Five is concerned with making statistical quality control work.

**References:** Shewhart (1931), Dodge (1932), Dodge et al. (1933), Fisher (1935) [1937], Pearson (1935), Simon (1941), Wilks (1941), Dudding & Jennett (1942), Pearson & Hartley (1942), Wilks (1942), Knudsen (1943), Wilks (1943), Eisenhart, Hastay & Wallis (1947).

**Citation:** Dodge (1933) [1951].

* Greenwood, Major (1946). The statistical study of infectious diseases. *Journal of the Royal Statistical Society A* **109,** 85-103; discussion, 103-110. (MR 8, 591).

**Summary:** The only parts of this paper and the ensuing discussion which are relevant to our subject are Appendix 1 (pp. 97-102) and M. G. Kendall's contributions to the discussion (pp. 103-105), which deal with the problem of the random division of an interval. In Appendix 1, the author considers the problem of n points taken at random on an interval of length L. We have seen [comments on the work of Whitworth (1867)] that this has an obvious interpretation in terms of the order statistics of a sample of size n from a uniform (rectangular) distribution, the lengths of the partial intervals between successive points being the differences between successive order statistics. After discussing the contributions of Whitworth and later ones by von Bortkiewicz (1915) and by Morant (1921), the author finds the first four moments of the distribution of the length of the partial intervals. The n points divide the interval of length L into n+1 partial intervals whose mean length is $L/(n+1)$, with $\mu_2 = L^2 n/(n+1)^2(n+2)$, $\mu_3 = 2L^3 n(n-1)/(n+1)^3(n+2)(n+3)$ and $\mu_4 = 3L^4 n(3n^2-n+2)/(n+1)^4(n+2)(n+3)(n+4)$, from which it follows that $\beta_1 = 4(n-1)^2(n+2)/n(n+3)^2$ and $\beta_2 = 3(n+2)(3n^2-n+2)/n(n+3)(n+4)$. The author also derives the distribution of the sum of the squares of the n+1 partial intervals. Kendall writes (p. 104: "As he [Greenwood] says, it was Whitworth who seems to have been the first to consider this problem in any sort of generality, but it may be of interest to point out (what is not generally realized) that the essential mathematics of Whitworth's theorem were carried out by Dirichlet [no reference given] earlier in the nineteenth century. Whitworth himself proves his result by a long inductive chain of argument, whereas it follows almost immediately from Dirichlet's integrals. More recently the problem has been studied from the point of view of finding the distribution of the greatest part into which the magnitude is divided. Whitworth himself solved this problem, but it has been re-discovered and applied, for example, by Fisher [(1929)] to harmonic analysis, several writers to the test of significance of the variance ratio in the analysis of variance, and by Garwood [(1940)] and others in problems arising out of the control of traffic and telephone switchboards. More recently still Stevens [(1939)] solved what is essentially the same problem in connection with the distribution of points on a circle, a form which I think was suggested by biological considerations. Nearly all this work has concerned the distribution of the *greatest* of the fragments into which the random interval is divided. Professor Greenwood has, however, put forward rather a different form of the problem, since he is more interested in the closeness of equality of the parts, and he has accordingly suggested as a test of departure from randomness the distribution of the variance of the parts into which the interval is divided. I do not remember seeing this version before. . . ."

**References:** Whitworth (1867) [1887], von Bortkiewicz (1915), Morant (1921), Fisher (1929), Stevens (1939), Garwood (1940).

**Citation:** Moran (1947).

* Gumbel, E. J. (1946a). On the independence of the extremes in a sample. *Annals of Mathematical Statistics* **17,** 78-81. (MR 7, 464).

**Summary:** Let x be a continuous unlimited variable with probability density function (p.d.f.) $\phi(x)$ and cumulative distribution function (c.d.f.) $\Phi(x)$, and let ${}_m x$ be the $m^{th}$ observation from the bottom and $x_k$ the $k^{th}$ observation from the top in a sample of size n. The author shows that the joint p.d.f. of ${}_m x$ and $x_k$ is equal to the product of the p.d.f.'s of ${}_m x$ and $x_k$, and hence that ${}_m x$ and $x_k$ are independent, provided (1) n is large, (2) the ranks m and k are small, and (3) the initial distribution $\phi(x)$ is of the exponential type subject to L'Hospital's rules $\lim_{x=-\infty}[\phi'(x)/\phi(x)] = \lim_{x=-\infty}[\phi(x)/\Phi(x)]$ and $\lim_{x=\infty}[\phi'(x)/\phi(x)] = -\lim_{x=\infty}\{\phi(x)/[1-\Phi(x)]\}$.

**References:** Gumbel (1935a), Kendall (1943) [1945], Gumbel (1944).

**Citations:** Gumbel (1947), Wilks (1948), Gumbel (1949).

* Gumbel, Émile-J. (1946b). Détermination commune des constantes dans les distributions des plus grandes valeurs. *Comptes Rendus de l'Academie des Sciences* (Paris) **222**, 34-36. (MR 7, 464).

**Summary:** The author writes (pp. 34-36, compiler's translation): "M. Fréchet [(1927)] and R. A. Fisher [& L.H.C. Tippett (1928)] have established two stable asymptotic distributions of largest values. The two probabilities $F_1(z)$ and $F_2(x)$ that a largest value is less than or equal to z (or to x) can be written in a uniform way, (1a) $F_1(z) = \exp[(-v/z)^k]$, (1b) $F_2(x) = \exp[-e^{-\alpha(x-u)}]$, where exp denotes the exponential function. The parameters v (and u) are the largest values such that their probability is the reciprocal of the number e. The first distribution does not possess moments of order greater than or equal to k. For the second distribution, all the moments exist. An estimate of the constants v and k can be made with the aid of the inverse moments $m_{-\ell}$ of order $\ell$ defined by $m_{-\ell} = \int_0^\infty z^{-\ell}\,dF(z)$. The calculation leads to $m_{-\ell} = v^{-\ell}\Gamma(1 + \ell/k)$, from which follows, for $\ell = 1$, (2) $m_{-1} = v^{-1}\Gamma(1 + 1/k)$ and, for $\ell = 2$, (3) $m_{-2}/(m_{-1})^2 = \Gamma(1 + 2/k)/\Gamma^2(1 + 1/k)$. We substitute for the first members of equations (2) and (3) the numerical values obtained from the observations; then equation (3) leads to the knowledge of the parameter k and equation (2) gives v. These formulas can provide a comparison between an observed distribution of largest values and the formula (1a). It is usual to determine the constants $\alpha$ and u of formula (1b) by the method of moments, which leads to (4) $\bar{x} = u + \gamma/\alpha$, $\alpha^2\sigma^2 = \pi^2/6$, where $\bar{x}$ denotes the arithmetic mean, $\sigma$ the standard deviation, and $\gamma$ Euler's constant. An analogous method can be used for the formula (1a). We note first that the two formulas are related by a logarithmic transformation. We set $x = \log z$ in (1b) and we obtain (5) $F(z) = \exp[(-e^u/z)^\alpha]$, that is to say the formula (1a), v being replaced by $e^u$ and k by $\alpha$. That is why we calculate the geometric moments for the first distribution. The geometric mean $\bar{g}$ and the geometric mean square $\overline{g^2}$ are defined by (6) $n \log_e \bar{g} = \Sigma_1^n \log_e x_\nu$, $n \log_e \overline{g^2} = \Sigma_1^n \log_e^2 x_\nu$, where n is the number of observations. Since the derivatives $f_1(z)$ and $f_2(x)$ of the probabilities (1a) and (1b) are related by $f_1(z)\,dz = f_2(x)\,dx$, the geometric moments of order $\ell$ of the first distribution (7) $\log_e \overline{g^\ell} = \int_0^\infty \log_e^\ell z\, f_1(z)\,dz$ are equal to the ordinary moments $\overline{x^\ell}$ of the second distribution $\overline{x^\ell} = \int_{-\infty}^{\infty} x^\ell f_2(x)\,dx$. Therefore, for $\ell = 1, 2$, (8) $\log_e \bar{g} = \log_e v + \gamma/k$, $\log_e \overline{g^2} = \pi^2/6k^2 + (\log_e \bar{g})^2$. The geometric dispersion $\sigma_g^2$, defined by $\sigma_g^2 = \log_e \overline{g^2} - (\log_e \bar{g})^2$, gives (9) $k^2\sigma_g^2 = \pi^2/6$. By replacing the first members of equations (8) and (9) by the observed values obtained from (6), one obtains an estimate of the constants in the formula (1a) and this method is entirely analogous to equations (4) valid for the formula (1b)."

**Note:** Three obvious typographical errors were corrected during translation.

**References:** Fréchet (1927), Fisher [& Tippett] (1928).

* Hoel, Paul G. (1946). The efficiency of the mean moving range. *Annals of Mathematical Statistics* **17**, 475-482. (MR 9, 151).

**Summary:** The mean moving range $w = \Sigma_{i=1}^{n-1}|x_{i+1} - x_i|/(n-1)$ of n observations defined by the author is simply the mean successive difference defined by Bréger (1881), which does not require ordering the observations, and hence is not directly relevant to our study. However, the author also defines a generalized mean moving range. He writes (p. 479): "Let $r_i$ deonte the range of the [k] observations from $x_i$ to $x_{i-k+1}$. Then the variable $W = (r_1 + r_2 + \ldots + r_{n-k+1})/(n - k + 1)$ will represent a generalized mean moving range, of which w will be a special case when k = 2." The author shows that $[W - E(W)]/\sigma_w$ is asymptotically normally distributed with zero mean and unit variance, provided only that x possesses a continuous distribution function for which the third absolute moment exists. He notes that the existence of the third absolute moment for x insures the existence of the same moment for $r_1$."

**Reference:** Wilks (1943).

**Citation:** Dodge (1933)[1951].

* Jones, A. E. (1946). A useful method for the routine estimation of dispersion from large samples. *Biometrika* **33,** 274-282. (MR 8, 42).

**Summary:** The author writes (Introduction, p. 274): "It is often possible, in certain types of mass production, to use a large sample of articles for sample routine inspection and to find with ease the articles with more extreme values of the characteristic measures. . . . In these cases, a great deal of labour can be saved, if the dispersion is estimated from these extreme values, which may comprise only about 5% of the total. Such an estimate of dispersion may be used in controlling variability by specifying limits for this estimate. One method of specifying the variability, which avoids the complication of subdividing the sample, is to lay down limits for the difference between the sum of the r highest and r lowest values in the sample. In this paper it will be shown how the mean, variance and also higher moments of this difference can be found. Approximate formulae, which are reasonably easy to calculate, are given for the mean and variance of the difference. These should be satisfactory for most practical purposes. In Table 1 [p. 280] are given exact values [to 3 significant digits] of the mean and variance of the difference in the case when the parent population is Gaussian (normal) for selected sample sizes [n] and values of r [n = 100, r = 5, 7; n = 200, r = 5, 7, 10, 12; n = 400(200)1000, r = 5, 7, 10, 12 , 16, 20]. The mean and variance with other parent populations [which decrease exponentially in their tails] may be calculated by applying equations . . . [and tables given by the author]."

**Note:** The difference $S = (x_n + x_{n-1} + \ldots + x_{n-r+1}) - (x_1 + x_2 + \ldots x_r)$ [not the author's notation] satisfies the equation $S = \Sigma_{i=0}^{r-1} w_i$, where $w_i = x_{n-i} - x_{i+1}$ is the $i^{th}$ quasi-range of the sample of size n.

**Citations:** Nair (1947, 1948b), Cole (1949).

* Kendall, Maurice G. (1946). *The Advanced Theory of Statistics*, Volume II. Charles Griffin & Company Limited, London. (MR 8, 473).

**Summary:** The author writes (Preface, p. v): "This volume falls into five sections. The first, comprising chapters 17 to 20, deals with Estimation. The second, comprising chapters 21, 23, 24 and 26 to 28, covers the Theory of Statistical Tests, including the Analysis of Variance and Multivariate Analysis. The third, consisting of Chapter 22, deals with Regression Analysis and completes the account of statistical relationship begun in chapters 13 to 16 of Volume I [Kendall (1943)]. In the fourth, chapter 25, I have tried to give an introductory account of the reaction of theoretical considerations on the Design of Statistical Inquiries. Finally, the fifth, comprising chapters 29 and 30, deals with the Analysis of Time-Series. The literature of statistical theory is now so vast that it seemed worth while devoting considerable space to a bibliography, which is given in Appendix B. . . ." Material in the various sections which is relevant to order statistics will be summarized, and the pertinent references will be listed. Chapter 17 (pp. 1-49) begins with a discussion of estimation and some desirable properties of estimators [consistence (consistency), unbiasedness, efficiency and sufficiency] and continues with a treatment of maximum likelihood estimators. The author (pp. 5-6) compares the sample mean and the sample median as estimators of the center of a symmetric population. For a normal population both are consistent and unbiased estimators of the population mean (median). The efficiency of the sample median relative to the sample mean is always $\leq 1$, with equality holding only for sample size $n = 2$, in which case the two are identical, and asymptotically (as $n \to \infty$) approaches $2/\pi \doteq 0.637$. For a Cauchy distribution, however, the sample median is a consistent estimator, but the sample mean is not. The asymptotic variance of the former is $\pi^2/4n$, while that of the latter is infinite. In Example 17.5 (pp. 6-7), the author shows that the efficiency of the sample mean deviation (multiplied by $\sqrt{\pi/2}$ to make it asymptotically unbiased), relative to that of the sample standard deviation, as an estimator of the standard deviation of a normal population with unknown mean is asymptotically $1/(\pi - 2) \doteq 0.876$. He does not state whether the mean deviation is taken about the sample median or about the sample mean; asymptotically, of course, it makes no difference. On pp. 27-28, he discusses sufficient estimators when the range (of the population)

depends on the parameter(s). If the range of the frequency function is from $\theta$ to b (a to $\theta$), where b(a) is fixed, a sufficient estimator (if one exists) for $\theta$ must be a function of the smallest (largest) sample member $x_1(x_n)$. Modification is necessary when both extremes of the range depend on $\theta$. In Example 17.15 (p. 28), the author considers the rectangular distribution $dF = dx/2\theta$, $-\theta \le x \le \theta$. He states (incorrectly) that we should take as our estimate of $\theta$ the smaller of $x_1$ and $-x_n$; actually, we should take the larger of $-x_1$ and $x_n$. Among the exercises at the end of Chapter 17 (pp. 45-49), Exercises 17.3, 17.4 and 17.6 involve use of the median, Exercise 17.11 the largest member of the sample, and Exercise 17.16 the midrange, as estimators. Chapter 18 deals with miscellaneous methods of estimation, including minimum variance, minimum $\chi^2$ and least squares, with no specific mention of order statistics. Chapter 19 is concerned with confidence intervals. On pp. 81 and 83, the author mentions the tables of confidence intervals (between pairs of order statistics) for the median and other quantiles in samples from any (continuous) population given by Nair (1940b)—see Exercise 19.5 (p. 84) and also Thompson (1936) and Savur (1937b). Exercises 19.1 and 19.3 (p. 83) involve use of the sample range or largest value to estimate $\theta$ for the rectangular population $dF = dx/\theta$, $0 \le x \le \theta$. Chapter 20 deals with fiducial inference, with no specific mention of order statistics. Chapter 21 treats some common tests of significance, including tests of goodness of fit [the $\chi^2$-test, which is not based on order statistics, and the $\omega^2$-test (Cramér-von Mises test) with test statistic $\omega^2 = \int_{-\infty}^{\infty} \{\bar{F}(x) - F(x)\}^2 \, dx$, where $\bar{F}(x)$ is the observed cumulative distribution function (c.d.f.) and $F(x)$ is the hypothetical c.d.f., and a modification by Smirnoff (1936) with test statistic $\omega_n^2 = \int_{-\infty}^{\infty} (\bar{F} - F)^2 \, dF$, both of which do involve order statistics]. It is pointed out on p. 108 that $E(\omega^2) = \Delta_1/2n$, where $\Delta_1$ is Gini's mean difference. The Kolmogorov-Smirnov test is not mentioned. Chapter 22 covers regression (linear and curvilinear, simple and multiple), but only the method of least squares is presented, not any of the alternatives based on order statistics. Chapters 23 and 24 on the analysis of variance and Chapter 25 on the design of sampling inquiries contain nothing relevant to our subject. In Chapters 26-28 on the further theory of statistical tests, the only explicit mention of order statistics is in Exercise 27.3 (p. 327), which involves the distribution of the range of samples of 3 from a normal population. In his discussion of the distribution of canonical correlations in Chapter 28, the author alludes (p. 357) to more general results (on the distribution of the ordered roots of random matrices) by Fisher (1939), Hsu (1939) and Roy (1939) [see also Girshick (1939)]. Chapters 29 and 30 deal with the analysis of time series, and the only mention of order statistics occurs on p. 434, where the author discusses both the approximate test of Walker (1914) and the exact test of Fisher (1929) for the largest intensity in a periodogram. He points out that Davis (1941) tabulated probabilities for both tests and that Stevens (1939) [see also Fisher (1940) and Finney (1941)] extended Fisher's results.

**References:** Laplace (1812) [1818], Todhunter (1865), Venn (1866) [1888], Whitworth (1867) [1901], Helmert (1875b, 1876a,b), Bertrand (1889), Pearson (1895), Fechner (1897), Sheppard (1899a), Galton (1902), Pearson (1902), Czuber (1903) [1921], Kapteyn (1903), Edgeworth (1905), Charlier (1906), Keynes (1911), Fisher (1912), Gini (1912), Walker (1914), von Bortkiewicz (1915), Gini (1916b), Brunt (1917), Fisher (1920), Pearson (1920), Czuber (1921), Jackson (1921), Keynes (1921), Morant (1921), Fisher (1922), Dodd (1923), Rietz (1924), Whittaker & Robinson (1924) [1940], Irwin (1925a,b), Lévy (1925), Pietra (1925), Tippett (1925), Pearson (1926), Jordan (1927), Student (1927), Bowley (1928), Cramér (1928), Fisher & Tippett (1928), Neyman & Pearson (1928), Pearson & Adyanthāya (1928), Sophister (1928), Fisher (1929), Gini & Galvani (1929), Pearson & Adyanthāya (1929), Rider (1929), de Finetti & Paciello (1930), Gini (1930), Steffensen (1930), Edgett (1931), de Finetti (1931a), Galvani (1931), Hojo (1931), von Mises (1931), Pearson (1931), Pearson & Pearson (1931), Shewhart (1931), Student (1931), de Finetti (1932), Galvani (1932), Gini (1932), Pearson (1932), Pearson & Pearson (1932), Pietra (1932a), Bartlett (1933), Cantelli (1933), Castellano (1933a,b), de Finetti (1933), Fisher (1933), Glivenko (1933), Hojo (1933), Kolmogoroff (1933), McKay & Pearson (1933), Rider (1933), Romanovsky (1933), Watkins (1933), Davies & Pearson (1934), Fisher (1934), Mortara (1934b), Pollard (1934), Anderson (1935), Castellano (1935), Fisher (1935), Gumbel (1935a,b,c), McKay (1935), Olds (1935), Pearson & Haines (1935), Smirnoff (1935), Thompson (1935), Wold (1935), Girshick (1936), Nair (1936), Neyman & Pearson (1936), Pearson & Chandra Sekar (1936), Pitman (1936), Smirnoff (1936), Sukhatme (1936), Thompson (1936), Ville (1936), Aitken (1937), Camp (1937), Gumbel (1937a, 1937-38),

Madow (1937), Neyman (1937a,b), Pitman (1937), Savur (1937b), Sukhatme (1937), Bartlett (1938), Bose (1938), Bruen (1938), Cisbani (1938), Fisher (1938), Fisher & Yates (1938), Gini (1938), Gumbel (1938a), Jeffreys (1938), Thompson (1938), Wilks (1938), Wishart (1938), Eisenhart (1939), Fisher (1939), Girshick (1939), Gumbel (1939a), Hsu (1939), Jeffreys (1939), Lévy (1939) Newman (1939), Pitman (1939), Roy (1939), Stevens (1939), Wald (1939), Wald & Wolfowitz (1939), Fisher (1940), Garwood (1940), Nair (1940b), Wald (1940), Chandra Sekar & Francis (1941), Cochran (1941), Daniels (1941), Davis (1941), Finney (1941), Gumbel (1941a), Hsu (1941a,b), Kolmogoroff (1941), Paulson (1941), Simon (1941), Thompson (1941), Wald & Wolfowitz (1941), Wilks (1941), Dodd (1942), Gumbel (1942a), Hartley (1942), Nair & Shrivastava (1942), Pearson (1942), Pearson & Hartley (1942), Roy (1942a,b), Gumbel (1943a), Kendall (1943), Mathisen (1943), Nair & Banerjee (1943), Pearson & Hartley (1943), Scheffé (1943), Wald (1943), Wilks (1943), Geary (1944), Hartley (1944).

**Citations:** Snedecor (1937)[1956], Boldrini (1942)[1968], Bartlett (1947b), Hartley (1949), Press (1949).

* Kimball, Bradford F. (1946a). Sufficient statistical estimation functions for the parameters of the distribution of maximum values. *Annals of Mathematical Statistics* **17,** 299-309. (MR 8, 475).

**Summary:** The author considers the problem of estimating from a sample a confidence region for the parameters $\alpha$ and u of the distribution of maximum values with p.d.f. $f(x) = \alpha \exp(-t-e^{-t})$, $t = \alpha(x-u)$, $-\infty < x < +\infty$. He sets up what he calls "statistical estimation functions" suggested by the functional form of the probability distribution of the sample, and finds the moment generating function of these estimation functions. He also treats such an estimation by the method of maximum likelihood. As a numerical illustration, he discusses four methods for setting up an approximate 95% confidence interval for the 99% point of x for the distribution of maximum values, using data taken from the 57-year record of annual maximum flood flows on the Tennessee River at Chattanooga, 1875-1931 [see Kimball (1938, 1942)]. The result sought is a 95% confidence interval for the 100-year flood. Method 1 is based on the statistical estimation functions developed by the author, Method 2 on maximum likelihood statistical estimation functions, Method 3 on the limiting distribution of the maximum likelihood estimates $\hat{u}$ and $\hat{\alpha}$, with variances unknown, and Method 4 on the limiting distribution of $\hat{u}$ and $\hat{\alpha}$, with variances estimated by taking $\alpha = \hat{\alpha}$ as estimated from the sample. The author recommends Methods 1 and 4.

**References:** Fisher & Tippett (1928), Wald (1940), Gumbel (1941a, 1942c), Kimball (1942), Wilks (1943).

**Citations:** Kimball (1946b, 1949), Press (1949).

* Kimball, Bradford F. (1946b). Assignment of frequencies to a completely ordered set of data. *Transactions of the American Geophysical Union* **27,** 843-846; discussion, **28** (1947), 951-953.

**Summary:** The author writes (Abstract, p. 843) ". . . If, for purposes of graphical fitting, it is found desirable to assign a frequency to each sample value, the writer, in general, recommends the mean frequency of the $m^{th}$ ordered observation: $\hat{F}_m = m/(n+1)$. In certain cases it may be possible to estimate the frequencies of the $m^{th}$ values by $\hat{F}_m = F[E(x_m)]$ independently of the unknown parameters, and such points might prove more desirable to use for graphical fitting near the extremes, if errors along the x direction are balanced out visually, relative to these points." He also writes (pp. 844-845): "In a broad sense the estimation of the parameters (Problem A) does not involve assignment of frequencies to observed data but rather furnishes an estimate of the complete frequency function—under a subjective assertion as to the form of function that should be used. It is when one wishes to test the validity of the above assertion (Problem B) that it may be found necessary to assign frequencies to observed data. . . . Let fitted cumulative frequency curve be denoted by $G(x)$ . . . . The test of fit which the writer is working on would involve the squares of the deviations of $G(x_m)$ from its expected value $E(F_m) = m/(n+1)$, $m = 1, 2, \ldots n$. Perhaps for this reason the author is especially inclined to favor the use of $\hat{F}_m = E(F_m) = m/(n+1)$ as a reference point for plotting . . . ." E. J. Gumbel writes (Discussion, pp. 951-952): "Probability

coordinate papers are constructed in order to obtain straight lines for the observed frequencies which allow a simple forecast. Kimball's question, which frequency to assign to the m'th value, is important for the use of such papers designed for an unlimited variate since this frequency is the plotting point for the m'th observation. In the writer's opinion, the following prerequisities must be met: (1) The plotting position must be such that all observations can be plotted. This condition is not fulfilled for the plotting positions m/n and (m−1)/n. . . . (2) The return period of a value equal to or larger than the largest observation, and the return period of a value equal to or smaller than the smallest observation should converge toward n, the number of observations. This condition is not fulfilled in Hazen's method since the choice of (m−1/2)/n as plotting position leads to a return period of 2n for the extreme observations. . . . (3) The observations should be equally spaced on the frequency scale, that is, the difference between the plotting positions of the m+1'st and the m'th observation should be a function of n only, and independent of m. This condition need not be fulfilled if the probabilities of the mean, median, or modal m'th values are chosen as plotting positions. Furthermore the choice of the mean or median m'th values need not fulfill condition (2). (4) The plotting position ought to be analytically simple, and have an intuitive meaning. The writer [Gumbel (1945a)] has proposed to choose the probabilities of the most probable smallest and of the most probable largest value as plotting positions of the two extremes, and to determine the remaining n−2 plotting positions from linear interpolation. Kimball proposes to choose the mean frequency of the m'th observation. As the characteristic values of the extremes, the writer chooses their modes whereas Kimball chooses a new kind of average. Kimball's procedure is simpler since it does not require any analytic calculations. If the sample is sufficiently large, both methods are practically identical since the differences are only of the order $n^{-2}$. The test of fit on which Kimball is working is very interesting. The writer has calculated the expectation of the squared $\alpha$ difference of $G(x_m)$ from its expected value m/(n+1) which turns out to be 1/[6(n+1)]. If Kimball succeeds in finding the distribution of the squared difference, this test will henceforth rightly occupy the role which, up to now, is occupied, wrongly, by the Chi square test. Consequently the writer wants to encourage Kimball to continue this research." Kimball's rejoinder is given on pp. 952-953.

**References:** Gumbel (1943b), Wilks (1943), Gumbel (1945a), Kimball (1946a).

**Citation:** Gumbel (1948).

* Kontorova, T. A. (1946). The statistical theory of the scale factor. (Russian). *Sbornik Dokladov po Dinamicheskoi Prochnosti Detalei Mashin* [Collection of Papers on the Dynamic Strength of Machine Parts], pp. 178-184. Izd-vo Akademii Nauk SSSR, Moscow-Leningrad.

**Summary:** This paper deals primarily with the size effect on the transition temperature from ductile to brittle fracture. However, the author reviews the results given in her earlier papers [Kontorova (1940), Kontorova & Frenkel (1941)] on the statistical theory of brittle strength, based on the weakest-link or largest (most dangerous)-flaw theory. She writes [p. 181 (translation)]: "With the aid of very elementary considerations of the theory of probability it is possible to show that the probability W(F)dF that the brittle strength which corresponds to the most dangerous flaw, i.e., the strength of the specimen as a whole, will lie in an interval between F and F + dF will be determined by the function $W(F)dF = \overline{N}VCe^{-\alpha(F_0-F)^2}[1 - e^{-\alpha(F_0-F)^2}/2\sqrt{\pi\alpha}(F_0-F)]^{\overline{N}V}\,dF$, (6), where $F_0$ = the value of the brittle strength which corresponds to the most frequently encountered flaw, i.e., to the flaw with 'mean hazard level', $\overline{N}$ = the average number of flaws per unit volume of the material, V = the volume of the specimen, and $\alpha$ and C are constants. Equation (6) shows that the probability W(F)dF of encountering one value or another of the strength of the specimen, determined by consideration of all the numerous material defects, depends not only on the value of the strength F itself, but also on the volume V of the specimen. It is easy to show, further, that the most probable strength F* of a specimen of volume V, i.e., the strength which is determined by the condition $\partial W/\partial F = 0$, is also a function of V. With an increase in the volume V of the specimen, it diminishes acording to the law $F^* = F_0 - \sqrt{C_1 \lg V + C_2}$. (7)." She also shows that an increase in the size of the specimen is accompanied by a decrease in the scatter of values of the brittle strength F about its most probable value F*.

**References:** Kontorova (1940), Kontorova & Frenkel (1941).

* Mosteller, Frederick (1946). On some useful "inefficient" statistics. *Annals of Mathematical Statistics* **17,** 377-408. (MR 8, 447).

**Summary:** The author writes (Summary, p. 377): "Several statistical techniques are proposed for economically analyzing large masses of data by means of punched-card equipment; most of these techniques require only a counting sorter. The methods proposed are designed especially for situations where data are inexpensive compared to the cost of analysis by means of statistically 'efficient' or 'most powerful' procedures. The principal technique is the use of functions of order statistics, which we call *systematic statistics*. It is demonstrated that certain order statistics are asymptotically jointly distributed according to the normal multivariate law. For large samples drawn from normally distributed variables we describe and give the efficiencies of rapid methods: (i) for estimating the mean by using 1, 2, . . ., 10 suitably chosen order statistics; . . . (ii) for estimating the standard deviation by using 2, 4, or 8 suitably chosen order statistics; . . . (iii) for estimating the correlation coefficient whether other parameters of the normal bivariate distribution are known or not (three sorting and three counting operations are involved) . . . . The efficiencies of procedures ii) and iii) are compared with the efficiencies of other estimates which do not involve sums of squares or products."

**References:** Pearson (1920), Jackson (1921), Tippett (1925), Fisher & Tippett (1928), Pearson (1931), Gumbel (1935a), Smirnoff (1935, 1937b), Fisher & Yates (1938) [1943], Gumbel (1939f), Nair & Shrivastava (1942), Pearson & Hartley (1942), Kendall (1943), Wilks (1943), Brown & Tukey (1946).

**Citations:** Dodge (1933) [1951], Hastings, Mosteller, Tukey & Winsor (1947), Wilks (1948), Cole (1949), Godwin (1949b).

* Oding, I. A. (1946). Influence of size and shape of specimens on their fatigue strength. (Russian). *Sbornik Dokladov po Dinamicheskoi Prochnosti Detalei Mashin* [Collection of Papers on the Dynamic Strength of Machine Parts], pp. 141-156. Izd-vo Akademii Nauk SSSR, Moscow-Leningrad.

**Summary:** The author reviews the work of various writers on the effect of size and shape of specimens on the fatigue limit. He begins with the work of Peterson (1930), who showed that, with an increase in diameter of the specimen, the fatigue limit decreases, especially when the form is such as to produce stress concentration. Faulhaber, Buchholtz & Schulz (1933) performed experiments which confirmed Peterson's results. The author enumerates five causes of the scale effect, including the inhomogeneity of the strength and mechanical properties of the microvolumes making up the specimen (statistical theory of strength). He asserts that this theory (the weakest-link theory based on the theory of extreme values) can account for much but by no means all of the size effect observed in practice. He notes that Weibull (1939a), Kontorova (1940), and Kontorova & Frenkel (1941) applied this theory to the static strength of brittle materials, while Afanassiev (1940) used statistical theory for the direct solution of the problem of fatigue strength. The results of these authors show that the fatigue strength must decrease not only with an increase in the cross section of the specimen but also in its length. The author expresses the opinion that the technological factor (surface effects, etc.) is the most important of all factors in determining the size effect, playing an even larger role than statistical theory. He also reports the results of an experimental investigation of the scale factor which revealed the presence of a size effect for untempered but not for tempered specimens of structural steel.

**References:** Peterson (1930), Faulhaber, Buchholtz & Schulz (1933), Weibull (1939a), Afanassiev (1940), Kontorova (1940), Kontorova & Frenkel (1941).

* Roy, S. N. (1946a). Multivariate analysis of variance: the sampling distribution of the numerically largest of the p-statistics on the non-null hypothesis. *Sankhyā* **8**, 15-32. (MR 8, 475).

**Summary:** The author writes (Introduction, p. 15): "In earlier papers by the author [Roy (1939, 1942a,b)] it was shown that (i) the problem of testing the identity of population dispersion matrices for two multivariate normal populations, and (ii) the general problem of multivariate analysis of variance could each be tackled by a corresponding appropriate set of p-statistics having the same joint (and hence individual) sampling distribution on the null-hypotheses for (i) and (ii), but having entirely different joint (and hence individual) sampling distribution on the non-null hypotheses that are associated with (i) and (ii) respectively. The common joint distribution of the p-statistics on the null hypotheses for (i) and (ii), the joint distribution of the p-statistics on the non-null hypothesis for (i), the joint distribution on the non-null hypothesis for (ii) and the individual distributions of the p-statistics on the non-null hypotheses for (i) and (ii)—the same for the two cases—were all found by the author in earlier papers [Roy (1939, 1942a,b)] in the order mentioned here. It is the object of the present paper to obtain the sampling distribution of the numerically largest of the p-statistics on the non-null hypothesis for (ii), that is, for multivariate analysis of variance. It will be seen both from this as well as from the previous paper [Roy (1945)] that the distribution of the numerically smallest or any intermediate of the p-statistics on the non-null hypothesis for (ii) could be derived in general by the same method as that actually used for the largest statistic. But the case of the largest—and not of any other—is worked out in detail, for a special reason. It will be shown in the next paper [it is not clear what paper is meant, but it is not Roy (1946b)] that in a certain sense (which will be explained in that paper) the numerically largest of the p-statistics supplies the most powerful test and should thus be used in preference to any other of the p-statistics on the non-null hypothesis."

**Note:** The reviewer (G. W. Brown) points out that Anderson (1946) and Bartlett (1947a) have challenged the general result of Roy (1942b), except for the case of only one non-zero root. He infers [and Bartlett (1946b) explicitly states (p. 187)] that the results in the present paper are valid only for that special case.

**References:** Roy (1939, 1942a,b, 1945).

**Citations:** Roy (1946b), Bartlett (1947b).

* Roy, S. N. (1946b). A note on multi-variate analysis of variance when the number of variates is greater than the number of linear hypotheses per character. *Sankhyā* **8**, 53-66. (MR 8, 475).

**Summary:** The author writes (pp. 53-55): "The present note is mostly a sequel to two previous papers [Roy (1942a,b)] by the author and should be read together with those papers in particular and some other papers [Roy (1939, 1945, 1946a)] in general by the author. It is well known for univariate normal populations that the problem of analysis of variance associated with $(\ell-1)$ linear hypotheses can be made to depend upon the problem of testing the hypotheses of the equality of mean values for $\ell$ univariate normal populations. It can be also easily proved along similar lines that for multivariate normal populations the problem of analysis of variance associated with $(\ell-1)$ linear hypotheses per character can be easily linked up with the problem of testing the hypothesis of equality of mean values of each character for $\ell$ p-variate normal populations. For the case $\ell-1 \geq p$ a set of p-statistics appropriate to this latter situation were introduced by the author and their joint sampling distribution obtained both on the null hypothesis of equality and on the non-null hypothesis of inequality, in earlier papers [Roy (1939, 1942b)]. The present note proposes to discuss and solve a similar problem for the case $\ell-1<p$. We shall in the main follow the notation of two previous papers [Roy (1942a,b)] by the author, of which the present one is mostly a sequel. Given $\ell$ samples with sizes $n_r$, sample means $\bar{x}_{r,i}$ and sample dispersion

matrices $a_{r,ij}$—r=1,2, . . . $\ell$, referring to samples or populations and i,j=1,2 . . . p referring to characters—drawn from $\ell$ p-variate normal populations with population means $\zeta_{ri}$ and a common dispersion matrix $\alpha''_{ij}$ (i,j=1,2 . . . p) the problem was to test the hypothesis $\zeta_{1i}=\zeta_{2i}=\ldots=\zeta_{\ell i}$ with i=1,2, . . . p. A set of p-statistics $t_1, t_2, \ldots t_p$ which in a certain sense could be regarded as an appropriate generalisation of the usual statistic 'between variance/within variance' (for the univariate case) was constructed and the joint distribution of these both on the null as on the non-null hypotheses was worked out [Roy (1939, 1942b)]. This was for the case $\ell-1\geq p$ and the set of $t_i$'s came out as the p roots (all real) of the following determinantal equation in t: $|a'_{ij}-t^2a''_{ij}|=0$ (1.1) where $a'_{ij}=\Sigma_{r=1}^{\ell}n_r(\bar{x}_{ri}-\bar{x}_i)(\bar{x}_{rj}-\bar{x}_j)/(\ell-1)$, $a''_{ij}=\Sigma_{r=1}^{\ell}(n_r-1)a_{r,ij}/\Sigma_{r=1}^{\ell}(n_r-1)$, $\bar{x}_i=\Sigma_{r=1}n_r\bar{x}_{ri}/\Sigma_{r-1}^{\ell}n_r$. . . . The individual distribution of these $t_i$'s on the null hypothesis and in particular the distribution of the largest root $t_p$ and the smallest $t_1$ were worked out by the author in an earlier paper [Roy (1945)]. As observed earlier [Roy (1945, 1946a)] . . ., the largest root $t_p$ plays a specially important part and the sampling distribution of $t_p$ (or rather a function of it . . .) on the non-null hypothesis was also obtained earlier [Roy (1946a)] by the author. All this was, of course, for the case $\ell-1\geq p$. We shall now consider the case of $p>\ell-1$ and point out the consequent changes that should be introduced at the appropriate stages in the earlier papers [Roy (1942a,b)] . . . Put $\ell'\equiv\ell-1$. Then it is well known that if $p<\ell'$ the determinantal equation (1.1) will be a p-th degree equation in $t^2$, while if $p>\ell'$ it will be a $\ell'$th degree equation in $t^2$. Let the roots in the latter case be denoted by $t_1^2, t_2^2, \ldots t_{\ell'}^2$. We shall be concerned to find the joint distribution of these $t_i$'s (i=1,2, . . . $\ell'$)."

**Note:** The comments of Anderson (1945, 1946) and Bartlett (1947a) concerning the validity of the results of Roy (1942b) on the non-null hypothesis are equally applicable to the present paper [see Bartlett (1947b), p. 187]. This fact is also pointed out in the review by G. W. Brown (MR 8, 475).

**References:** Roy (1939, 1942a,b, 1945, 1946a).

**Citation:** Bartlett (1947b).

* Tintner, Gerhard (1946). Some applications of multivariate analysis to economic data. *Journal of the American Statistical Association* **41,** 472-500.

**Summary:** The author writes (pp. 472-473): "This essay proposes to introduce the economic statistician to some of the newer methods of multivariate analysis. The emphasis will be on methods of estimation and not on tests of hypotheses. Tests of significance will be indicated where they have been established. . . . We will deal only with the following methods: (1) Discriminant analysis: Here we propose to determine linear functions of 'indexes' computed from various measurable characteristics of certain data. The data have been classified into two groups. Discriminant analysis tries to establish linear functions of the characteristics which are such that they distinguish most successfully between these groups. This method was invented by R. A. Fisher. A test of significance utilizes earlier work of Harold Hotelling. (2) Principal components: We try to answer the following question: Is it possible to analyze a set of variables into a more fundamental set of components ('factors') possibly fewer in number? Which portion of the total variance can be accounted for by each component? The best method in this field is due to Harold Hotelling. (3) Canonical correlation: Assume we have two sets of variables. How can we determine linear combinations ('indexes') of the variables in each set in such a fashion that the correlation between the indexes becomes a maximum? This method is due to Harold Hotelling. (4) Weighted regression: Assume that we have a set of variables *all* of which are subject to disturbances ('errors'). How can we find a weighted linear regression function which will give us the 'best' estimates of the weighted regression coefficients? This method, evidently closely related to classical multiple regression analysis, is in its present form due to Tjalling Koopmans. It can also be used to answer a question previously raised by Ragnar Frisch: How many linear relationships exist probably between the variables (multicollinearity)?" The author points out that Methods (2)-(4) are all related to ordered roots of determinantal equations. The first principal component which accounts for most of the variance is the

*largest* root of a certain determinantal equation, as is the square of the maximum canonical correlation coefficient. The minimum of the weighted sum of squares Q in weighted regression is (N-1) times the *smallest* root of a certain determinantal equation, and the smallest, next smallest . . . etc. roots can be used in a test of collinearity.

**Note:** Fisher (1938) has shown that Method (1) also depends (in a different way) upon order statistics.

**References:** Hotelling (1933), Girshick (1936), Hotelling (1936a,b), Fisher (1938, 1939), Girschick (1939), Hsu (1939), Wald (1940), Davis (1941), Hsu (1941b), Wilks (1943), Tintner (1945).

**Citation:** Bartlett (1947b).

* Tukey, John W. (1946). Sampling from contaminated distributions (abstract of preliminary report). *Annals of Mathematical Statistics* **17,** 501; *Bulletin of the American Mathematical Society* **52,** 828.

**Summary:** The author writes (Abstract, p. 501): "A contaminated distribution is a nearly normal distribution in which extreme observations are more frequent than in a normal distribution. By studying the bias and variability of several measures of dispersion when applied to samples from particular one-parameter families of contaminated distributions it is shown that: (i) for nearly normal distributions, the mean deviation is often better than the standard deviation; (ii) small changes in the underlying distribution may increase the sampling variance of the standard deviation by a factor of three. This suggests that, in the broad class of cases, the mean deviation is safer than the standard deviation when a single dispersion is estimated from a set of data. This conclusion need not apply in an analysis of variance situation."

**Note:** The author does not state in the above abstract whether his mean deviation is taken from the arithmetic mean or from the median.

**Citations:** Tukey (1949c,e,f).

* Walsh, John E. (1946a). Some significance tests based on order statistics. *Annals of Mathematical Statistics* **17,** 44-52. (MR 7, 464).

**Summary:** The author writes (Summary, pp. 44-45): "In this paper significance tests are developed whose application requires only the determination of one order statistic and the computation of sums of sample values. The simplest case considered is that of testing a new sample value x on the basis of m previous sample values $y_1, \ldots, y_m$, all sample values being assumed from normal populations with the same variance. Two separate tests of whether the mean of the new population from which x was taken exceeds the mean of the population from which $y_1, \ldots, y_m$ were drawn consist in accepting the alternative that the new population mean exceeds the old population mean if (1) $x > [(\sqrt{m+1}+1)/m]\Sigma_1^m y_i - \sqrt{m+1}\, y_{(u)}$ [or] (2) $x > [(\sqrt{m+1}-1)/m]\Sigma_1^m y_i + \sqrt{m+1}\, y_{(m+1-u)}$, where $y_{(u)}$ is the $u^{th}$ largest of $y_1, \ldots, y_m$. It can be shown that both of these tests have the same power so that either one might be equally well selected for use. In practical application, however, there may exist reasons for preferring one test to the other. Similarly, the alternative that the new population mean is less than the old population mean will be accepted if (3) $x < [(\sqrt{m+1}+1)/m]\Sigma_1^m y_i - \sqrt{m+1}\, y_{(m+1-u)}$ [or] (4) $x < [(\sqrt{m+1}-1)/m]\Sigma_1^m y_i + \sqrt{m+1}\, y_{(u)}$. All four of these significance tests have the same power, also the same significance level $\alpha(u, m)$. By appropriate choice of u and m the significance level can be made to assume values suitable for significance tests. For example, $\alpha(1, 6) = .0156$, $\alpha(2, 10) = .0107$, $\alpha(3, 13) = .0110$, $\alpha(4, 16) = .0107$. The above tests are still valid if each of $x, y_1, \ldots, y_m$ equals a sum of r sample values. These order statistic tests are generalized to the case where x is a sum of r new sample values; $y_1, \ldots, y_m$ each equals a sum of s past sample values and another sum of relatively weighted past sample values is used but not as an order statistic. The introduction of this relatively weighted sum allows less reliable past information to be

lumped together and weighted according to its relative importance. In comparing the order statistic tests with the most powerful tests which could be used for these alternatives it is found that the size of the samples used must be increased in order to bring the efficiency of the order statistic test up to that of the corresponding most powerful test. Thus the advisability of using the order statistic test will depend upon whether it is more desirable to take larger samples but have less computation."

**Reference:** Wilks (1943).

**Citation:** Wilks (1948).

* Walsh, John E. (1946b). Some order statistic distributions for samples of size four. *Annals of Mathematical Statistics* **17,** 246-248. (MR 8, 43).

**Summary:** The author writes (Summary, p. 246): "Let $x_1, x_2, x_3, x_4$ represent the values of a sample of size four drawn from a normal population. There is no loss of generality in assuming that the distribution function of this population has zero mean and unit variance. Denote it by N(0,1). Let $x_{(i)}$ be the $i^{th}$ largest of $x_1, x_2, x_3, x_4$. The purpose of this note is to determine the joint distribution of $x_{(4)} + x_{(3)} - x_{(2)} - x_{(1)}, x_{(4)} - x_{(3)} + x_{(2)} - x_{(1)}$, and $x_{(4)} - x_{(3)} - x_{(2)} + x_{(1)}$, and derive from this joint distribution the joint distribution of these statistics taken in pairs, also the distribution of each statistic itself."

**Reference:** Wilks (1943).

**Citation:** Nair (1947).

* Winsten, C. B. (1946). Inequalities in terms of mean range. *Biometrika* **33,** 283-295. (MR 8, 43).

**Summary:** The author writes (Summary, p. 295): "Two inequalities are found in terms of mean range of samples of n(n = 2, 3, 4, . . .). The first is true, as is Tchebycheff's, for any frequency distribution whatsoever. The second holds for any unimodal symmetrical distribution. Both are shown to be the best possible of their type. Diagrams are given to facilitate the use of the inequalities for the cases n = 2, 3, . . ., 9, 10. A derivation is also given of the Gauss-Winkler inequalities analogous to that used for inequalities in terms of the mean range." He also writes (p. 283): "The new inequality is not an exact analogue of Tchebycheff's. Instead of considering the fixed interval $(\mu - t\sigma, \mu + t\sigma)$, we consider a variable interval $(x, x + tw_n)$ of fixed length $L = tw_n$ [$w_n$ is the mean range of samples of n]. For a given distribution d, and fixed t, the fraction of d falling outside this interval is a function of x, $p_d(x, t)$ say. We know that $p_d(x, t) \leq 1$, and it therefore follows that $p_d(x, t)$ has, for all x, an upper bound, $p_d(t)$ say. It is this upper bound, $p_d(t)$, that will be considered."

**References:** Kendall (1943), Barnard (1944).

* Bartlett, M. S. (1947a). The general canonical correlation distribution. *Annals of Mathematical Statistics* **18,** 1-17. (MR 8, 474).

**Summary:** The author writes (p. 1): "In the statistical theory of the relation of one vector variate with another (see Hotelling [(1936b)], the simultaneous distribution of the canonical correlations $r_i$, which are the roots of a certain determinantal equation, was first obtained in 1939 (Fisher [(1939)], Hsu [(1939)], Roy [(1939)] in the special but important case when the true roots or correlations $\rho_i$ are zero. Roy [(1942b)] has since investigated the case where the true roots are not zero when these non-zero values arise from non-central means. The present investigation is primarily intended to cover the alternative case where non-zero roots arise from the existence of true correlations $\rho_i$. The method developed is, however, also applicable to the case of non-central means; and it is shown that the general distribution, which for more than one non-zero root becomes very complicated, does not in the case of non-central means agree with the distribution given by Roy [(1942b)] except in the case of only one non-

zero root. . . . This conclusion has also been reached by T. W. Anderson [(1946)], who has given a solution of the non-central means problem in the cases of either one or two non-zero roots."

**References:** Hotelling (1936b), Fisher (1939), Hsu (1939), Roy (1939, 1942b), Anderson (1946).

**Citation:** Bartlett (1947b).

* Bartlett, M. S. (1947b). Multivariate analysis. *Journal of the Royal Statistical Society, Supplement* **9**, 176-190; discussion, 190-197. (MR 9, 453).

**Summary:** The author writes (pp. 176-179): "In matrix notation the analysis of variance and covariance of a number of correlated variates is very succinctly expressed . . . . For example, if the sample S is split up into two components which are mutually orthogonal, $S = X + Y$, (1), say, we have the corresponding analysis of sums of squares and products (analysis of variance and covariance): $SS' = (X + Y)(X + Y)' = XX' + YY'$, (2), where $X'$ is the transpose of X and the terms $YX'$ and $XY'$ vanish owing to the orthogonality of the components X and Y. . . . In cases where it is useful to make an over-all test of the effect of treatments (or other relevant classification) a test based on the general criterion $\Lambda = |YY'| / |SS'|$, (3), where X is the component to be tested, was proposed (this being equivalent to the appropriate likelihood criterion in the special case of testing the differences in means among k multivariate samples . . .). . . .It is well known that the splitting up of a sample in analysis of variance can be regarded as the regression analysis of the sample in terms of other variates or pseudo-variates, and . . . it is convenient to discuss the analysis in terms of regression. We therefore change the notation of equations (1) and (2), and write $S_2$ for S, $S_1$ for the set of 'independent variates' that we are taking the regression on, and $S_{2.1}$ for the 'residual component' of $S_2$ after the regression on $S_1$ has been removed. For equation (1) we now have . . . $S_2 = B_{21} S_1 + S_{2.1}$, (4), where the matrix of regression coefficients is $B_{21} = C_{21} C_{11}^{-1}$, (5) where $C_{21} = S_2 S_1'$, $C_{11} = S_1 S_1'$, $C_{22} = S_2 S_2'$ and $C_{22.1} = S_{2.1} S_{2.1}'$; and for equation (2) we have $C_{22} = C_{21} C_{11}^{-1} C_{12} + C_{22.1}$ (6). The analysis of a new dependent variate obtained by a linear transformation of the original set, $a'S_2$, say, is given by $a'S_2 = a'B_{21} S_1 + a'S_{2.1}$, (7), whence $a'C_{22} a = a'C_{21} C_{11}^{-1} C_{12} a + a'C_{22.1} a$ (8). Equation (8) represents an analysis of variance of the variate $a'S_2$; and the ratio of the first to the second sum of squares on the right-hand side of (8) measures the significance of the dependence of $a'S_2$ on $S_1$. To find the particular variate $a'S_2$ which maximizes this ratio we choose the coefficients a as a solution of the equation $(C_{21} C_{11}^{-1} C_{12} - R^2 C_{22}) a = 0$, (9) where $|C_{21} C_{11}^{-1} C_{12} - R^2 C_{22}| = 0$ (10). Multiplying (9) on the left by $a'$, we have $R^2 = a'C_{21} C_{11}^{-1} C_{12} a / a'C_{22} a$ (11) so that $R^2$ is the fraction of the total sum of squares that we have been maximizing, and must be the *largest* root of (10). It is also the square of the multiple correlation between $a'S_2$ and $S_1$. The relation of $a'S_2$ to $S_1$, while maximized, does not in general exhaust the real dependence of $S_2$ on $S_1$, and if the variate $a'S_2$ is removed, we may still find a significant relation from the remaining variates (chosen to be uncorrelated with $a'S_2$). In fact, a repeated maximization for such a reduced sample would be found to lead to another root, the second largest, of the original equation (10); and so on. Hotelling [(1936b)], to whom the above type of analysis is due, has shown that the original set $S_2$ can be transformed into an equivalent mutually uncorrelated set $A S_2$, say, each variate of which is associated with one of the roots $R^2$ in (10); such a system of variates with their corresponding partners from transformations of $S_1$ are called *canonical variates*, and the corresponding roots $R^2$, or rather their square roots R, *canonical correlations*. I shall for convenience refer to such an analysis as a *canonical analysis*. To relate these roots with the over-all test $\Lambda$ defined in equation (3), we may note that equation (10) may be written $|1 - C_{21} C_{11}^{-1} C_{12} C_{22}^{-1} - (1 - R^2)| = 0$, (12), whence $\Pi_{i=1}^{p} (1 - R_i^2) = |1 - C_{21} C_{11}^{-1} C_{12} C_{22}^{-1}| = |C_{22} - C_{21} C_{11}^{-1} C_{12}| / |C_{22}| = \Lambda$ (13). This relation is sometimes useful if we wish to test the significance of the remaining roots $R_{r+1}^2 \ldots R_p^2$, when the first r roots $R_1^2 \ldots R_r^2$, say, which are established to be associated with real relationships between $S_2$ and $S_1$, have been removed. We have $\Lambda = \Lambda' \Lambda'' = \Pi_{i=1}^{r} (1 - R_i^2) \Pi_{j=r+1}^{p} (1 - R_j^2)$, where the $\chi^2$ test approximation for $\Lambda$ may be extended to $\Lambda'' \ldots$, giving $\chi^2 = -\{n - (1/2)(p + q + 1)\} \log_e \Lambda''$ (14) with $(p - r)(q - r)$ degrees of freedom." The author also discusses (Section 6, pp. 186-188) the general sampling distribution of the canonical roots [Fisher (1939), Hsu (1939), Roy (1939, 1942a,b, 1945, 1946a,b), Anderson (1946), Bartlett (1947a)]. Relevant portions of the

discussion include (1) the comments of R. C. Geary concerning the interrelationships among the theory of the roots of determinantal equations (discovered by Fisher and developed by Hsu, Bartlett and others), Hotelling's principal component theory, and Tintner's theory of linear relationships between economic series and (2) the statement by Sir Cyril Burt that the method of principal components, usually associated with the name of Hotelling, was first suggested by Karl Pearson (no reference given) as early as 1901, and was in fact the earliest form of factor-analysis ever proposed.

**References:** Hotelling (1933, 1936b), Bartlett (1938), Fisher (1938, 1939), Girshick (1939), Hsu (1939), Roy (1939), Hsu (1941b), Roy (1942a,b), Wilks (1943), Roy (1945), Anderson (1946), Kendall (1946), Roy (1946a,b), Tintner (1946), Bartlett (1947a).

**Citations:** Bartlett (1948), Rao (1948).

* Bickerstaff, Thomas Alton (1947). Certain Order Probabilities in Non-Parametric Sampling. Unpublished Ph.D. Dissertation, University of Michigan, Ann Arbor.

**Summary:** The author writes (Introduction, pp. 2-4): "An ordered sample of n unequal variates $(x_i)$, $i = 1, 2, 3, \ldots, n$, defines n sample points and $n(n-1)/2$ sample intervals $(x_i, x_j)$ with $1 \leq i < j \leq n$. Let the universe percentages measured from zero be designated by Greek letters $\alpha < \beta < \gamma \ldots$ with range (0, 1) and let $p_i, p_j, p_k$, etc. be the unknown universe percentages below $x_i, x_j, x_k$, etc. respectively of the sample. Then in terms of these $n(n+1)/2$ sample statistics of order, and these comparable universe parameters of order, questions concerning probabilities such as the following may be phrased: (1) $P(p_i < \alpha)$; (2) $P(\alpha < p_i < \beta)$; (3) $P[(p_i, p_j) > \alpha]$, $(p_i, p_j)$ being an interval; (4) $P[\alpha < (p_i, p_j) < \beta]$; (5) $P(p_i < \alpha < p_j)$; (6) $P(p_i < \alpha < \beta < p_j)$; (7) $P(\alpha < p_i < p_j < \beta)$; (8) $P(\alpha < p_i < \beta < p_j)$; (9) $P(p_i < \alpha < p_j < \beta)$; (10) $P(p_i < \beta, p_j > \alpha)$; and (11) $P[(p_h, p_k) > \beta, (p_i, p_j) > \alpha]$, for $h < i < j < k$. A study of these probabilities and their values involves a number of considerations such as sample size and form. We shall deal here with the infinite universe of one variable assuming a continuous distribution function but specifying nothing about parameters or distribution form. . . . Chapter I provides such fundamentals and auxiliary matter as are needed. A notation system is introduced, the probability integral transformation is applied, distribution functions of order statistics are provided, and general moment characteristics are derived. Chapter II deals with questions involving a single sample point. Special cases such as the median, an extreme, or a quartile are included. Probabilities (1) and (2) above are evaluated. Chapter III deals with the single sample interval including the sample range. The presence of bias is observed. Probabilities (3) and (4) above are evaluated. The study of the individual behavior of two sample points distributed jointly provides the subject for Chapter IV. Confidence limits find their logical place here and limits for universe *intervals* as well as for universe points are established. The probabilities (5) to (10) above are found expressible in terms of certain incomplete double integrals. Chapter V extends Chapter IV to include joint distributions of any two order statistics without common elements. Such probabilities as (11) above are considered. Joint distributions of two statistics with common elements are considered in Chapter VI. Included is the study of the median and a central interval considered jointly with the question of independence involved. The joint probability surface is developed and a regression system in such a case is provided. Chapter VII is concerned with the study of the joint probability surface and the incomplete double integral involved in Chapters IV and V. Certain tables of these integrals are exhibited and problems illustrating the theory are solved. Chapter VIII consists of a concluding summary."

**Note:** For an application of the results of this dissertation and further results on order statistics obtained by the use of probability integral transformations, see the dissertation by Bennett (1952).

**References:** Neyman & Pearson (1928), Pearson & Adyanthāya (1928), Rider (1929), Hojo (1931), Pearson & Pearson (1931), Craig (1932a), Thompson (1936), Neyman (1937a), Savur (1937b), Nair (1940b), Wilks (1941, 1942, 1943).

* Cohan, G. M. (1947). New uses for the average range. *Industrial Quality Control* **3** (6), 17-19.

**Summary:** The author writes (pp. 17-18): "It is a well-known fact that no matter how well centered a process is, if the variability exhibited in sampling is excessive, the process will yield scrap or rework or both. To determine excessive variability the use of the standard deviation or the average range is employed. The standard deviation may be computed by the root-mean-square method from the first fifty pieces taken from the process, or by taking much smaller samples at intervals and computing the standard deviation from the average range. If six standard deviations exceed the total tolerance, scrap or rework, or both, will result. This procedure may be shortened and used to advantage in many instances by making use of the average range in terms of percentage of the total tolerance rather than following the procedure of computing sigma. The use of average range in terms of per cent of the total tolerance is more direct and can be more readily understood by plant personnel. This device may be used to advantage in the following instances: 1) To determine when the range of a sample or a series of samples is large with relation to the specification limits. 2) To determine the amount of scrap or rework being produced under (1). 3) To aid in setting limits beyond which $\overline{\overline{X}}$ should not deviate when $\overline{R}$ is small with relation to the specification limits. 4) To determine when and how much to reset a process that has deviated from nominal. . . . With these advantages in mind a table is presented (for samples of 5) which may be used as a ready reference for estimating in any one of the above instances. The range is presented in terms of per cent of the total tolerance so that it will fit any particular specification . . . ." He also writes (p. 19): "Tables for samples of any other size may be compiled in the same manner by substituting the proper value of $d_2$ in computing the standard deviation. A note of warning should accompany the use of these tables. They are based on the assumption that a normal distribution of individual observations and their averages exists which very often is not the case especially in small runs. However, if these limitations are realized and appreciated the tables may be used to advantage in many instances."

* Davidenkov, N.; Shevandin, E.; Wittmann, F. (1947). The influence of size on the brittle strength of steel. *Journal of Applied Mechanics* **14,** A-63-A-67. (AMR 1 #87).

**Summary:** The authors write (p. A-63): "Experiments on static tension and bending of cylindrical specimens of brittle phosphorous steel in liquid air reveal the statistical nature of the size effect and give a good qualitative verification of Weibull's theory." In the section on statistical theory of size effect, they write (pp. A-64-A-65): "Considering now the statistical theory of the size effect, it may be recalled that this approach was the (qualitative) explanation, proposed long ago by Alexandrov and Jurkov [(1933)] in their investigation of the great strength of thin glass threads. . . . The statistical theory of strength, whose mathematical interpretation was given independently by W. Weibull [(1939a)] and T. Kontorova and J. Frenkel [(1941)] is based upon the assumption that brittle failure is determined not by the value of the average, but of the local stress in the piece, at the locus where the most dangerous structural defect is located. The specimen can be represented as consisting of a set of volume elements of various strengths connected in series, the strength being distributed according to the probability law. The larger the piece, the lower the strength of its weakest element and therefore the lower the strength of the piece itself. According to Weibull the distribution of local strength of the material is arbitrarily assumed to be described by a power law. In the expression for the 'risk of rupture' $B = \int n(\sigma)\, dv$, the function $n(\sigma)$ is written as $(\sigma/\sigma_0)^m$ where m and $\sigma_0$ are constants of the material, m being a characteristic of the degree of homogeneity of the material and is larger the higher the homogeneity. Both constants must be determined experimentally, and it is only necessary to know a set of strength values for a number of specimens. . . . According to the theory of T. Kontorova and J. Frenkel, the distribution of the faults is assumed to follow the normal Gaussian law, while the number of constants characterizing the material increases to three. In spite of its advantage over Weibull's theory in the sense of stricter physical foundation, this theory is, however, less convenient for practical purposes." In their conclusion, the authors state (p. A-67): "The experimental data . . . lead to the conclusion that the statistical theory of strength explains satisfactorily and without inner contradictions the influence of size on the brittle strength of steel. . . . The brittle strength varies with increase of size more and more slowly, so it can be assumed that this tends to some definite limit. The theoretical meaning of this limit consists in the fact that starting with a sufficiently large specimen, a complete set of all possible nonhomogeneities will be present. The larger the specimen, the closer this limit will be approached."

**References:** Alexandrov & Žurkov (1933), Weibull (1939a), Kontorova & Frenkel (1941).

**Citations:** Epstein (1948a,b), McAdam, Geil, Woodard & Jenkins (1948), Peterson (1949a).

* Domb, C. (1947). The problem of random intervals on a line. *Proceedings of the Cambridge Philosophical Society* **43,** 329-341. (MR 8, 591).

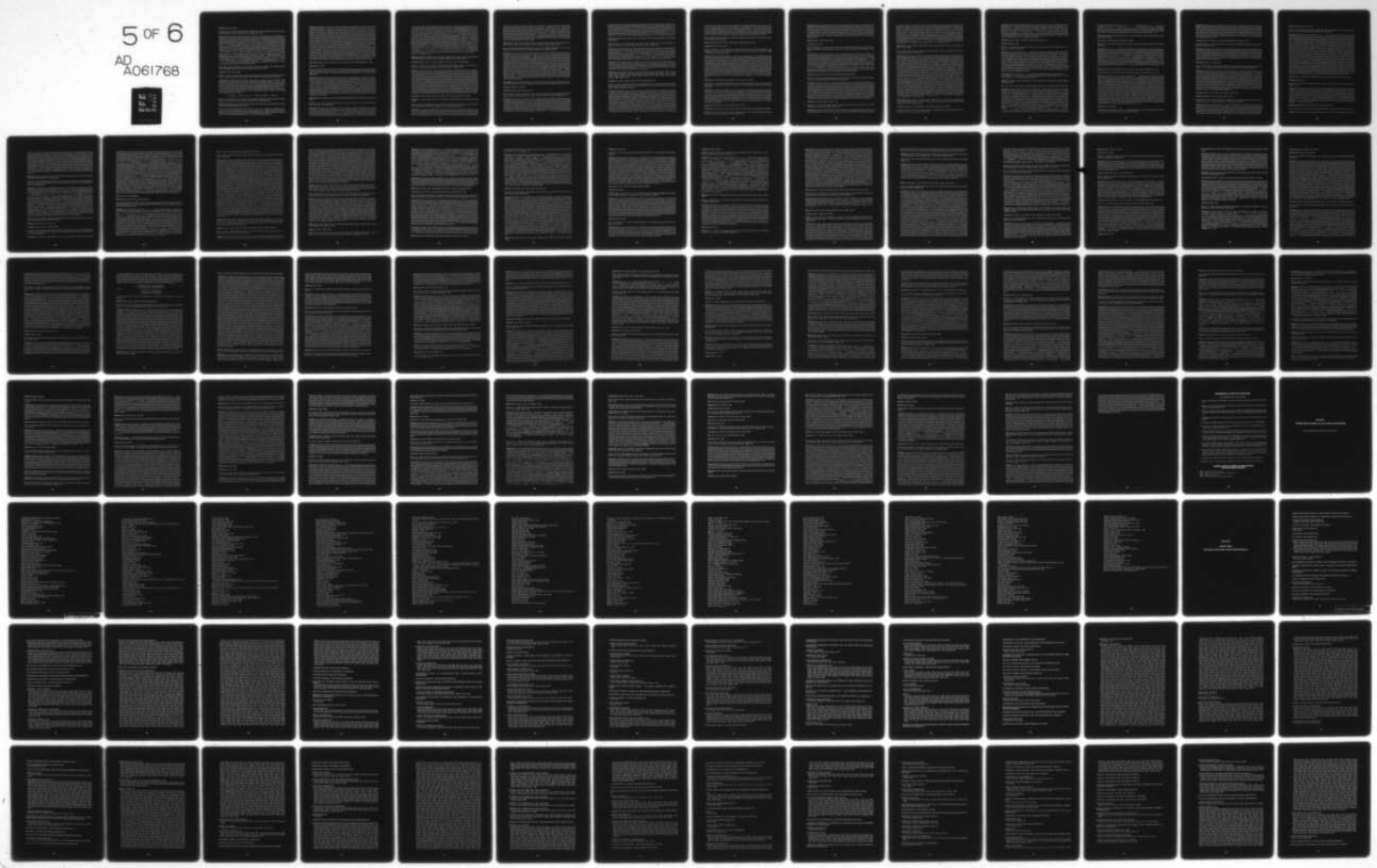
**Summary:** The author states the problem as follows (p. 329): "Suppose that events occur at random points on a line from $t = -\infty$ to $+\infty$, the probability of an event occurring between t and $t + dt$ being $\lambda\, dt$. If we select any interval of the line, say the interval $[0, y]$, there will be a finite probability that it contains 0, 1, 2, . . ., r, . . . events; in fact, it is not difficult to show that these probabilities form a Poisson distribution, the probability that the interval contains r events being $\lambda^r y^r e^{-\lambda y}/r!$. . . . Consider the case when each event consists of an interval of length $\alpha$ (an event being characterized by its first point). What is the probability that the covered portion of the $[0, y]$ lies between x and $x + dx$? A related problem, that of n given intervals at random points on the circumference of a circle, has been discussed by W. L. Stevens [(1939)]. The methods used in the present paper, however, are basically different from those of Stevens, since the concept of random events on an infinite line leads naturally to the use of continuous processes. Some of the results given by Stevens will be deduced in the course of the work." In Section 7 (p. 334), the author considers the case of n given events and derives Stevens' solution. In Sections 8 and 9 (pp. 334-337), he discusses the distribution of gaps and the moments of that distribution.

**Comments:** For the special case where n is fixed and $\alpha = 0$, the gaps are those between the order statistics of a sample of size n from a uniform distribution. This case is related to work on the random division of an interval by various authors, including Clifford (1866), von Bortkiewicz (1915), Morant (1921), Fisher (1940), Garwood (1940), and Moran (1947).

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**Reference:** Stevens (1939).

* Duncan, David B. (1947). Significance Tests for Differences between Ranked Variates Drawn from Normal Populations. Ph.D. thesis, Iowa State College, Ames, Iowa.

**Summary:** The author writes (Summary, p. 115): "Consider n variates drawn from n normal populations with the same variance for which an independent estimate based on p degrees of freedom is available. A multiple choice significance test is required for testing the $n(n-1)/2$ differences between the variates taken in all possible pairs. The main requirement of the test is that it has, as closely as possible, the same properties with respect to each difference as an $\alpha$-level t test has to the single difference in the simple case, n = 2. This type of problem is commonly met for example, in varietal yield trials and similar agricultural experiments. A test is proposed which places upper bounds on errors comparable to type 1 errors in the case n = 2 and at the same time compares favorably with other alternatives in minimizing a weighted average risk function relative to the remaining errors. The test is given in detail for the cases n = 3, $\alpha$ = .05, p = $\infty$; n = 3, $\alpha$ = .05, p = 10; and n = 4, $\alpha$ = .05, p = $\infty$." On p. 52 he states: "The most convenient form in which the proposed test can be tabled is in terms of significant values for the differences $k_{uv}[=g_{v+u} - g_v]$ between the ranked variates $g_i$. It is helpful . . . to refer to these differences as 'ranked' differences and to the subscripts u and v as the lag and the position of each respectively. Thus $k_{uv}$ may be referred to as the '$v^{th}$', 'lag u', ranked difference. The significant value $k^s_{uv} = k(\alpha, p, n, u, v, k_{1\beta})$ for a given $k_{uv}$ depends on $\alpha$, p, n, u, v and $k_{1\beta}$ where $k_{1\beta}$ are the first u − 1 lag one differences $k_{1\beta}(\beta = v, v+1, \ldots, v+u-1)$ into which $k_{uv}$ can be subdivided." These values are tabulated in Tables 12.1 - 12.3 (pp. 54-55) for the cases mentioned in the author's summary.

**References:** Whittaker & Robinson (1924), Fisher (1935) [1942], Snedecor (1937) [1946], Fisher & Yates (1938)[1943], Wald (1939), Pearson & Hartley (1943), Wilks (1943).

**Citations:** Tukey (1948c, 1949a).

* Egudin, G. I. (1947). Certain relations between the moments of the distribution of extreme values in random samples. (Russian). *Doklady Akademii Nauk SSSR* (N.S.) **58,** 1581-1584. (MR 9, 295).

**Summary:** The author points out that many recent authors [e.g. Tippett (1925), E. S. Pearson (1926, 1932), McKay & Pearson (1933), and Newman (1939)] have studied the distribution of extreme values and range in samples of a given size, mainly from a normal population, and that the calculations of certain constants which their publications contain are extremely significant for statistical quality control, especially by the range method as recommended by Dudding & Jennett (1942). Let $X_{1,n}, X_{2,n}, \ldots, X_{n,n}$ be the order statistics of a sample of size n from a population with cdf F(x). Then the author shows that the mean of the moments of order r of the order statistics about the origin is equal to the corresponding population moment, i.e.

$$(1/n)\Sigma_{k=1}^{n}\overline{X_{k,n}^{r}} = (1/n)\Sigma_{k=1}^{n}\int_{-\infty}^{\infty}x^{r}F_{k,n}(x) = \int_{-\infty}^{\infty}x^{r}dF(x), \text{ where } \overline{X_{k,n}^{r}} = \int_{-\infty}^{\infty}x^{r}dF_{k,n}(x)$$

and $F_{k,n}(x)$ is the cdf of the $k^{th}$ order statistic. The author also establishes certain recurrence relations between the moments of extreme values in samples from a normal population.

**References:** Tippett (1925), E. S. Pearson (1926, 1932), McKay & Pearson (1933), Romanovsky (1933), Newman (1939), Dudding & Jennett (1942).

* Eisenhart, Churchill; Hastay, Millard M.; Wallis, W. Allen (1947). Acceptance sampling when lot quality is measured by the range (based on a memorandum by M. A. Girschick). *Selected Techniques of Statistical Analysis* (edited by Churchill Eisenhart, Millard M. Hastay and W. Allen Wallis), Chapter 5 (pp. 225-234). McGraw-Hill Book Company, Inc., New York-London. (MR 9, 365).

**Summary:** The authors write (p. 227): "The range, defined as the difference between the largest and smallest observations in a sample, is familiar in industrial experimentation and quality control as a measure of dispersion in samples. Many applications of the range are to normal or nearly normal populations, for which the variability is better characterized by the standard deviation. The range is used to measure dispersion in samples because of the ease with which it can be determined, and because in small samples it is almost as efficient a measure of variability as the standard deviation. The present chapter is concerned with an interesting reversal of these roles, where the variability of the lot or process is characterized in terms of the range itself but where the standard deviation is used as the sampling statistic to determine whether the range meets specified requirements. One type of problem in which the range is the parameter of ordinary interest is that of ensuring the virtual simultaneity of a set of events." They also write (p. 229): "For a normal distribution the probability that the range of k independent observations is less than some preassigned number U depends only on the standard deviation $\sigma$. . . . Let $\sigma_1$ be the value of the lot standard deviation that defines good quality. Then $\sigma_1$ depends on the group size k, on the acceptable value U of the range in groups of k, and on the proportion of ranges that can be allowed to exceed U. These three quantities are related to $\sigma_1$ through the cumulative distribution of the standardized range, which gives the probability that the range $R_k$ of k items from a normal distribution will exceed any specified multiple $W_k$ of the population standard deviation $\sigma$. Tables of the cumulative distribution of the range have been published by Pearson [& Hartley (1942)]." Test procedures, in the cases of single sampling and of sequential sampling, are given for testing the hypothesis that the range of a sample of size k does not exceed a specified value, based on the sample standard deviation.

**References:** Tippett (1925), Pearson (1931), Fisher & Yates (1938), Pearson [& Hartley] (1942).

**Citation:** Grant (1946).

* Eisenhart, Churchill; Solomon, Herbert (1947). Significance of the largest of a set of sample estimates of variance. *Selected Techniques of Statistical Analysis* (edited by Churchill Eisenhart, Millard M. Hastay and W. Allen Wallis), Chapter 15 ( pp. 383-394). McGraw-Hill Book Company, Inc., New York-London. (MR 9, 365).

**Summary:** The authors write (pp. 386-387): "Bartlett's test, though applicable, is somewhat inappropriate when the alternative to homogeneity that is most likely to represent the true situation, or that is of special importance when it does obtain, is that the expected value of some one of the mean squares (variance estimates) substantially exceeds the expected values of the others, which may be homogeneous or not. By 'some one' it is meant that prior to the analysis of the data — or, better, prior to the collection of the data — there is no basis for identifying the anomalous one of the set. . . . In situations of this type what is needed is a test of the statistical significance of the amount by which the largest of the variance estimates exceeds the others. Cochran [(1941)] has developed such a test and has provided a small table of 5 per cent significance levels to facilitate its use." A more extensive table of 5% significance levels and a table of 1% levels, which were not tabulated by Cochran, are given in this chapter, and the methods of computation are discussed.

**References:** Davies & Pearson (1934), Pearson & Chandra Sekar (1936), Cochran (1941), Thompson (1941).

* Elfving, G. (1947). The asymptotical distribution of range in samples from a normal distribution. *Biometrika* **34,** 111-119. (MR 8, 395).

**Summary:** The author writes (Introductory, p. 111): "Consider a sample of n observations, taken from an infinite normal population with the mean 0 and the standard deviation 1. Let a be the smallest and b the greatest of the observed values. Then $w = b - a$ is the *range* of the sample. For certain statistical purposes knowledge of the sampling distribution of range is needed. The distribution function, however, involves a rather complicated integral, whose exact calculation is, for $n > 2$, impossible. Tippett (1925),

E. S. Pearson (1926, 1932) and McKay & Pearson (1933) have studied and calculated the mean, the standard deviation and the Pearson constants $\beta_1$, $\beta_2$ of the range. Fitting appropriate Pearson curves to the distribution by means of these parameters, Pearson (1932) has computed approximate percentage points for it. Later on, Hartley (1942) and Hartley & Pearson [Pearson & Hartley] (1942) have, by numerical integration, tabulated the distribution function for $n = 2, \ldots, 20$. As pointed out by Pearson, the distribution of range is very sensitive to departures from normality in the tails of the parental distribution. The effect of such departures becoming more perceptible for increasing n, the practical importance of the range distribution is, perhaps, small for large samples. Nevertheless, it seems to be at least of theoretical interest to investigate the *asymptotical* distribution of the range for $n \to \infty$. This is the purpose of the present paper. The results are summarized in a theorem at the end of the inquiry." In the above-mentioned theorem, which is stated on p. 118, the author expresses in terms of a Hankel function the common limit $F(x)$ as $n \to \infty$ of the cumulative distribution functions (c.d.f.'s) $F_n(x)$ and $F_n^*(x)$ of $x = 2n\sqrt{\Phi(a)\Phi(-b)}$ and $x^* = 2n\Phi(-w/2)$, where w is the range of a sample of size n from a normal population with c.d.f. $\Phi(x) = (2\pi)^{-1/2} \int_{-\infty}^{x} \exp(-u^2/2)\, du$. He also states that, for $n \geq 12$, $F_n(x)$ satisfies the inequalities $|F_n(x) - F(x)| < (8/n)(1 + 3x^2/n)$, $|F_n(x) - F(x)| < (4x^2/n)[\log(\sqrt{n}/x) + 1/2]$. The proof does not actually require normality; the theorem is valid if the frequency function is of the form $g(x) = C \exp[-|x|^p/p]$, where $1 < p \leq 2$.

**References:** Tippett (1925), Pearson (1926), Fisher & Tippett (1928), Pearson (1932), McKay & Pearson (1933), Gumbel (1935a), Hartley (1942), Pearson & Hartley (1942), Cramér (1946).

**Citations:** Gumbel (1947), Cox (1948), Gartšteĭn (1948), Wilks (1948), Gumbel (1949).

* Fisher, J. C.; Hollomon, J. H. (1947), A statistical theory of fracture. *Transactions of the American Institute of Mining and Metallurgical Engineers* **171,** 546-561. (AMR 2 #1142).

**Summary:** The authors endeavor to construct a consistent theory to explain the size effect in solids, the scatter of fatigue-stress values, and the dependence of fracture stress upon strain, and to suggest a quantitative relation between the structure and the fracture stress. They assume that the material has an idealized structure, that of an elastic solid containing many cracks which are oriented at random. They assume an exponential distribution of crack sizes (widths) and work out the probability of failure, when subjected to a specified stress, of a specimen containing a single crack and of one containing many cracks. One of their conclusions is that there is a marked decrease in strength with increasing number of cracks, with no marked size effect to be expected in the case of materials containing many cracks. They point out that the theory of Kontorova & Frenkel (1941), which is based on the assumption of a normal (Gaussian) distribution of flaw strengths, breaks down (for different reasons) when the number of flaws is either too large or too small. They also point out that any quantitative theory of fracture must take account of the fact that the defect producing the largest stress concentration in the specimen must be responsible for initiating fracture. Their own theory differs from that of Kontorova & Frenkel mainly in assuming an exponential distribution of flaw widths instead of a normal distribution of flaw strengths.

**References:** Griffith (1920), Kontorova & Frenkel (1941) [Frenkel & Kontorova (1943)].

**Citations:** Gensamer, Saibel & Lowrie (1947), Epstein (1948b), McAdam, Geil, Woodard & Jenkins (1948).

* Gensamer, Maxwell; Saibel, Edward; Lowrie, Robert E. (1947). Report on the fracture of metals, Part II. *Welding Journal (Welding Research Supplement)* **26,** 472s-484s. (AMR 1 #276).

**Summary:** In a section on nonuniform stress and strain and the size effect, the authors write (pp. 476s-477s): "The effects of stress and strain gradients have been studied considerably in connection with the so-called 'size effect'. However, as was stated in the initial survey [Gensamer, Saibel & Ransom (1947)], 'size effect' probably consists of three factors which must be separated from one another, namely,

the statistically greater possibility of finding a given defect in a larger section, the smaller stress gradients in large sizes, and induced stresses in large sizes due to restriction of strain. . . . In general, little experimental evidence has been found of size effects in tension tests of metals. However, the preparation of homogeneous, metallurgically identical large and small specimens is very difficult. Thus, the gathering of really significant data requires very exacting care. . . . A number of attempts have been made to account for size effect in fracture strength. . . ." In particular, the authors summarize the results of Kontorova (1940), Kontorova & Frenkel (1941), and Fisher & Hollomon (1947). In their final section, they outline a recommended research program involving further work on the effect of volume of notched and unnotched specimens on fracture strength.

**References:** Griffith (1920), Kontorova (1940), Kontorova & Frenkel (1941) [Frenkel & Kontorova (1943)], Gurney (1945), Fisher & Hollomon (1947), Gensamer, Saibel & Ransom (1947).

* Gensamer, Maxwell; Saibel, Edward; Ransom, John T. (1947). Report on the fracture of metals, Part I. *Welding Journal (Welding Research Supplement)* **26,** 443s-472s. (AMR 1 #276).

**Summary:** In a section on size effects (pp. 453s-454s), the authors write: "The effect of stress and strain gradients has not received much direct study. Most of the work has involved gradients produced by notches, and is therefore open to question. . . . A phenomenon intimately connected with gradients is the 'size effect'. This question is of considerable practical importance since small scale tests often predict safe structures which subsequently fail when made full scale. The size effect can apparently be resolved into three separate effects (1) the smaller stress gradients in large sizes, (2) the greater statistical possibility of finding a defect in large sections and (3) the induced stresses due to reduction of strain by large sizes. Most of the work on size effect has been done by bending notched bars with little attempt to separate these factors. The general rule of size has been that as long as geometrical similarity is observed, tensile properties will be the same. However, in the presence of gradients, scaling the external dimensions usually changes the gradient in some irregular way so that properties naturally change. . . . The true effects of size and gradients have received practically no fundamental treatment. The experimental program in this survey outlines an attack to study these variables with reference to both the flow and fracture strength. . . ." In a section on microcrack theory and size effect, the authors describe briefly a statistical analysis of crack distribution and density that has been developed in an effort to explain the various aspects of size effect.

***Comments:*** Of the three separate aspects of the size effect mentioned by the authors, only (2) is primarily statistical in nature. It is amenable to treatment by the statistical theory of extreme values.

**References:** Griffith (1920, 1924).

**Citation:** Gensamer, Saibel & Lowrie (1947).

* Grubbs, Frank E.; Weaver, Chalmers L. (1947). The best unbiased estimate of population standard deviation based on group ranges. *Journal of the American Statistical Association* **42,** 224-241.

**Summary:** The authors write (Abstract, p. 224): "In order to better the efficiency of the statistic range for sample sizes greater than eleven, a method, which possesses practical efficiency, is given in this paper for dividing a sample into groups so that an estimate of the standard deviation of the normal population sampled can be formed from a linear function of the group ranges. The estimate so obtained is 'best' in the sense that it is unbiased and has a smaller variance than any other estimate which is based on a linear compound of group ranges. In dividing a sample into groups for using the range of groups, it is shown in the Appendix so that the most efficient group size is eight [cf. Pearson (1932)]. Table I of this paper gives the approximate percentage points [upper and lower 0.5%, 1%, and 5% points] for the best

unbiased estimate based on group ranges [for total sample size 2(1)100] and Table II gives approximate percentage points [same levels] for the estimate based on mean range determined from r groups [r = 1(1)15] or samples of equal size n [=6(1)10]." Table A of the Appendix gives (to four decimal places) the standard deviations of the unbiased estimates $s/c_n$ and $R_n/d_n$ of the population standard deviation $\sigma$, where s and $R_n$ are the standard deviation and the range, respectively, of a sample of size n = 2(1)25. Table B gives the following moment constants of the range for n = 2(1)12: Mean ($d_n$), Std. dev. ($k_n$), $\alpha_3$, $\alpha_4$, $d_n/k_n^2$, $(d_n/k_n)^2$, $(d_n/k_n)^3\,\alpha_3$ [all to five significant figures] and $(d_n/k_n)^4\{\alpha_4 - 3\}$ [to four significant figures].

**References:** Tippett (1925), Pearson (1932), Davies & Pearson (1934), Pearson & Hartley (1942), Wilks (1943) [1946].

* Gumbel, E. J. (1947). The distribution of the range. *Annals of Mathematical Statistics* **18,** 384-412. Abstract, *Bulletin of the American Mathematical Society* **53,** 68. (MR 9, 195).

**Summary:** By convolution of the asymptotic distribution of the two extremes, the author deduces, for unlimited symmetrical distributions of exponential type, the asymptotic distribution of the "reduced" sample range $R = \alpha_n(w - 2u_n)$, where w is the ordinary range, and $\alpha_n$, $u_n$ are parameters depending on the initial distribution and the sample size n. He determines, in a similar manner, the asymptotic distribution of the reduced midrange $v = \alpha(x_1 + x_n)$, and obtains the asymptotic distributions of the reduced extremes by introducing the transformations $y_1 = \alpha(x_1 + u)$ and $y_n = \alpha(x_n - u)$. He gives several tables and graphs, and generalizes his results to the mth range and to the asymmetrical case.

**Comments:** The author's result for the range in the symmetric case, like that of Elfving (1947), is expressed in terms of Hankel functions; Elfving's $2n\Phi(-w/2)$ corresponds asymptotically to Gumbel's $2e^{-R/2}$. A comparison with the tables of Pearson & Hartley (1942) shows that the accuracy of the two methods is about the same for samples of size n ≈ 20 from a normal population, but that Elfving's method is somewhat more accurate than Gumbel's for small ranges.

**References:** von Bortkiewicz (1922a,b), Tippett (1925), E. S. Pearson (1926), Student (1927), Fisher & Tippett (1928), K. Pearson (1931), McKay & Pearson (1933), Gumbel (1935a), McKay (1935), Hartley (1942), Pearson & Hartley (1942), Gumbel (1943a), Kendall (1943), Gumbel (1944), Carlton (1946), Gumbel (1946a), Elfving (1947).

**Citations:** Cox (1948), Gartštein (1948), Wilks (1948), Gumbel (1949).

* Gurney, C. (1947). The Statistical Estimation of the Effect of Size on the Breaking Strength of Rods. Reports and Memoranda No. 2157, Aeronautical Research Council (London), March 1945 (issued in 1947). (AMR 2 #628) [See Note below].

**Summary:** The author writes (Conclusion, p. 1963): "A practically important size effect is to be expected in materials with large variation in strength of nominally identical pieces, the variation probably being largest in material whose average strength approaches the maximum possible in flawless material. A brittle material may be expected to fail at the breaking stress of the weakest of the n elements of which the material may be considered to be composed. In considering the strength of rods of ductile material, the effect of length may be expected to follow the 'least of n' theory, while the mean strength of rods of given cross-sectional area may be expected to exceed that of the mean of the n unit pieces of which the cross section may be considered to be composed. In both 'least of n' and 'mean of n' theories, n may be greater or less than unity. The fact that n can take values less than unity is of interest, as n less than unity has no meaning in usual sampling theory. From the discussion on the practical application of the theory of the least of n, it is concluded that fitting a normal curve to experimental strength data is not likely to lead to accurate results [since, under the 'least of n' theory, if the strength is normal for one length, it is non-normal for every other length]. The first three moments of the experimental results should be

computed and the length to give a normal distribution of strengths estimated from the value of $\sqrt{\beta_1}$ [$= \alpha_3$], $\sigma$ or $\mu$ according as one, two or three lengths have been tested. The use of $\sqrt{\beta_1}$ is limited to cases where the length tested is between 1/10 and 100 times the normal length. . . . From the discussion on the source of strength variation in materials, it is concluded that many materials are not statistically homogeneous and to such materials the theory of the least of n does not strictly apply. Application of the theory to such cases will usually give a strength/size gradient greater than that in practice. . . ."

**Note:** It appears that the reviews of this report and the following one [Gurney & Pearson (1947)] have been inadvertently interchanged.

**References:** Tippett (1925), Peirce (1926), Weibull (1939a,b), Gurney (1945).

**Citation:** Gurney & Pearson (1947).

* Gurney, C.; Pearson, S. (1947). The Effect of Length on the Strength of a Steel Wire. Reports and Memoranda No. 2158, Aeronautical Research Council (London), November 1945 (issued in 1947). (AMR 2 #481) [See Note on Gurney (1947)].

**Summary:** The authors write (Summary, p. 1965): "Sets of strength tests on two lengths cut from the same coil of steel have been made, each set consisting of about one hundred tests on nominally identical test pieces. The lengths tested were 1 in. and 240 in. Three pairs of sets of tests were made, there being some difference in the handling of the wire and the testing apparatus between each pair of sets. The results have been compared with those predicted by statistical theory [see Gurney (1947)] assuming that a normal distribution of strength would be obtained for a unique, but initially unknown length of wire, and that the strength of the wire is statistically homogeneous. The experiments show small, but significantly different [sic; small, but significant differences in the] distributions of strength of the long and short wires, the general trends of the results being in agreement with the theory. Mean strength and scatter of test results decrease with increasing length and the skewness tends to become more negative. The experiments are too few to enable an accurate comparison with theory to be made, but it is fairly certain that influences other than those which the theory is intended to cover are having significant effects. . . ."

**References:** Peirce (1926), Fisher & Tippett (1928), Gurney (1945, 1947).

* Hastings, Cecil, Jr.; Mosteller, Frederick; Tukey, John W.; Winsor, Charles P. (1947). Low moments for small samples: A comparative study of order statistics. *Annals of Mathematical Statistics* **18,** 413-426. (MR 9, 195).

**Summary:** The authors write (Summary, p. 413): "The means, variances, and covariances [of the order statistics] for samples of size $\leq 10$ from the normal distribution, a selected long-tailed distribution [$r(u) = (1-u)^{-1/10} - u^{-1/10}$, where u has the uniform distribution on the interval [0, 1] and $x = r(u)$ is the variable considered], and the uniform distribution are tabled and compared with the usual asymptotic approximations. The methods of computation used and the accuracy expected are discussed. Use is made of the representation of an arbitrarily distributed variate as a monotone function of a uniformly (rectangularly) distributed variate. It is hoped that these tables will encourage experimentation with new statistical procedures."

**References:** Fisher & Yates (1938) [1943], Mosteller (1946).

**Citations:** Jones (1948), Wilks (1948), Godwin (1949a,b), Nair (1949a), Purcell (1949), Walsh (1949d).

* Jacobson, Paul H. (1947). The relative power of three statistics for small sample destructive tests. *Journal of the American Statistical Association* **42,** 575-584.

**Summary:** The author writes (Abstract, p. 575): "The probability distributions are given for the mean of the three smallest ordered observations (m) and for the median of all observations ($m_d$) in samples of size five drawn from a normal distribution with unit variance and zero mean. The Neyman-Pearson theory is used to demonstrate the extent by which $\overline{X}$ affords a more powerful test method than m or $m_d$ for testing the hypothesis that $\mu = 0$ under the alternative hypothesis that $\mu > 0$. Also, m is shown to be superior to $m_d$, and both are evaluated in terms of the power of $\overline{X}$ in samples of size three and four. These results are applied to destructive tests, and a special kind of problem is cited for which there are economic grounds for the use of m."

**References:** Hojo (1931), Wilks (1943).

**Citation:** Purcell (1949).

* Kimball, Bradford F. (1947). Some basic theorems for developing tests of fit for the case of the non-parametric probability distribution function, I. *Annals of Mathematical Statistics* **18,** 540-548. (MR 9, 295).

**Summary:** The author writes (Summary, p. 540): "In developing tests of fit based upon a sample $O_n(x_i)$ in the case that the cumulative distribution function F(X) of the universe of X's is not necessarily a function of a finite number of specific parameters — sometimes known as the non-parametric case — it has been pointed out by several writers that the 'probability integral transformation' is a useful device . . . . The author finds that a modification of this approach is more effective. This modification is to use a transformation of ordered sample values from a random sample $O_n(x_i)$ based on successive *differences* of the cdf values $F(x_i)$. A theorem is proved giving a simple formula for the expected values of the products of powers of these differences, where all differences from 1 to n + 1 [$u_1 = F_1, u_r = F_r - F_{r-1} (r = 2, 3, \ldots, n)$, $u_{n+1} = 1 - F_n$, where $F_i \leq F_{i+1}$ and $0 \leq F_i \leq 1$] are involved in a symmetrical manner. The moment generating function of the test function defined as the sum of m squares of these successive differences is developed and the application of such a test function is briefly discussed."

**References:** Neyman (1937a), Gumbel (1942a), Scheffé & Tukey (1945).

* Kincaid, W. M. (1947). Numerical methods for finding characteristic roots and vectors of matrices. *Quarterly of Applied Mathematics* **5,** 320-345. (AMR 1 #205).

**Summary:** The author writes (p. 320): "The present paper treats the problem of finding the characteristic roots and vectors of a matrix (having linear elementary divisors). The emphasis throughout is on methods of getting numerical results in practical cases rather than on theoretical questions. Symmetric matrices are taken up first, and methods are discussed for finding (a) the largest characteristic root and corresponding vector and (b) the other roots and vectors. All these methods are variants of the iteration process. They are then extended to general matrices, with particular reference to the case of complex roots. The paper closes with a brief discussion of the solution of algebraic equations by means of matrices. Among earlier work on this subject, we may mention that of Hotelling [(1936a)] and Aitken [(1937)]. Indeed, much of the material in the present paper is taken from Aitken's, though some modifications have been introduced, particularly with regard to the determination of roots other than the largest. . . ."

**References:** Hotelling (1936a), Aitken (1937).

* Lehmann, E. L. (1947). On optimum tests of composite hypotheses with one constraint. *Annals of Mathematical Statistics* **18,** 473-494. (MR 9, 454).

**Summary:** The author writes (Summary, p. 473): "This paper is concerned with optimum tests of certain composite hypotheses. In section 2 various aspects of a theorem of Scheffé concerning type $B_1$ tests are

discussed. It is pointed out that the theorem can be extended to cover uniformly most powerful tests against a one-sided set of alternatives. It is also shown that the method for determining explicitly the optimum test region may in certain cases be reduced to a simple formal procedure. . . . In section 4 the totality of similar regions is obtained for a large class of probability laws which admit a sufficient statistic. [Let $p(x) = f(x)/g(\theta)$ be a pdf and let (a) $k(\theta) \leq x \leq c$ or (b) $\theta \leq x \leq b(\theta)$. It is shown that in case (a) the smallest value in the sample is a sufficient statistic for $\theta$, while in case (b) the smallest and largest sample values are jointly sufficient.] Some composite hypotheses concerning exponential and rectangular distributions are treated in section 5. It is proved that the likelihood ratio tests of these hypotheses have various optimum properties. [For the exponential, use is made of the transformation (introduced by Sukhatme (1937)) $Z_1 = nY_1, Z_i = (n - i + 1)(Y_i - Y_{i-1}), (i = 2, \ldots, n)$ where $Y_i$ is the ith of the x's in order of magnitude; for the rectangular, use is made of $Y_k/Y_{k+1}$, as well as $Y_1$ and $Y_n$.]"

**References:** Neyman & Pearson (1928), Pitman (1936), Neyman (1937b), Sukhatme (1937), Paulson (1941), Cramér (1946).

* Lord, E. (1947). The use of range in place of standard deviation in the t-test. *Biometrika* **34,** 41-67; corrigenda, **39** (1952), 442. (MR 8, 394).

**Summary:** In the Introduction (pp. 41-42), the author summarizes work on the range of samples from a normal population by Tippett (1925), E. S. Pearson (1926, 1932), McKay & Pearson (1933), Davies & Pearson (1934), Pearson & Haines (1935), Hartley (1942), and Pearson & Hartley (1942). He then writes (p. 42): "As a result of the work outlined above, the range is now of considerable importance in many fields, especially in industrial quality control, where its simplicity has enabled it to be extensively and easily applied to the measurement of fluctuations in the variability of quality of a manufactured article or material. In the present paper an investigation is made of the use of range estimates of standard deviation in the consideration of the statistical significance of deviations of sample means in normal random sampling theory. This use of range estimates of standard deviation is analogous to the use of root-mean-square estimates in the well-known t-test. [The s of Student's ratio is replaced by an estimate of $\sigma$ based on the range, $\bar{w}(m,n)/d_n$, where $\bar{w}(m,n)$ is the mean value of m ranges w, obtained from m independent samples or subgroups, each containing n observations, and $d_n$ is the expected value of the range in samples of n from a normal population with unit standard deviation.] Tables of $u(m,n) = x\, d_n/\bar{w}(m,n)$, where x is N(0,1), are given, at several probability levels, and these may be employed in determining the statistical significance of either the deviation of a sample mean from some fixed or hypothetical population value, or the difference between the means of two samples. These tables may also be used for obtaining rapid estimates of the accuracy of a sample mean from the variation within the sample as measured by the range. The use of range, in place of root-mean-square estimates of standard deviation, in this modified form of the t-test necessarily entails some loss of precision. It will, however, be shown in a future paper [Lord (1950)] that this reduction in accuracy is negligible for all practical purposes. Furthermore, this slight disadvantage of the new test is compensated by its greater simplicity, involving a reduced amount of computing compared with the usual t-test." At the end of the paper examples are given of the use of the range in biological and medical as well as industrial experimentation. Tables are given of the upper $100\alpha\%$ points of $u(m,n)$ for $\alpha = .10, .05, .02, .01, .002, .001$; $m = 1(1)6(2)10(5)20, 30$; $n = 2(1)10, 12, 20$.

**References:** Tippett (1925), E. S. Pearson (1926), K. Pearson (1931), E. S. Pearson (1932), McKay & Pearson (1933), Davies & Pearson (1934), Pearson (1935), Pearson & Haines (1935), Hartley (1942), Pearson & Hartley (1942), Daly (1946).

**Citations:** Snedecor (1937) [1956], Walsh (1947), Tukey (1949b), Walsh (1949b).

* von Mises, R. (1947). On the asymptotic distribution of differentiable statistical functions. *Annals of Mathematical Statistics* **18,** 309-348. (MR 9, 194).

**Summary:** This paper does not deal primarily with order statistics. However, the following portion (p. 310) is relevant: ". . . Statistical functions are known whose distributions assume, asymptotically, a form different from the Gaussian. One example is Pearson's Chi-square, another the test function $\omega^2$, introduced by H. Cramér [(1928)] and the author [von Mises (1931)]: $f = \omega^2 = \int g'(x)\,[S_n(x) - \overline{V}_n(x)]^2\,dx$ where $g'(x) > 0$ and $\overline{V}_n(x) = (1/n)\,[V_1(x) + V_2(x) + \ldots + V_n(x)]$. N. V. Smirnoff [(1936, 1937a)] computed the asymptotic distribution of $\omega^2$ for the case that all $V_n(x)$ and, therefore, $\overline{V}_n(x)$ equal one and the same distribution function $V(x)$. The result differs widely from the Gaussian distribution." The above-stated result for the distribution of $\omega^2$ is derived on pp. 346-348.

**References:** Cramér (1928), von Mises (1931), Smirnoff (1936, 1937a).

**Citation:** Hoeffding (1948).

* Moran, P. A. P. (1947). The random division of an interval. *Journal of the Royal Statistical Society, Supplement* **9,** 92-98. (MR 9, 291).

**Summary:** The author writes (p. 92): ". . . Greenwood [(1946)] has proposed the following problem, which is of interest in providing a test of significance in the statistical study of infectious diseases. Let n points be distributed at random in the unit interval (0,1), thus forming n+1 intervals $I_1, \ldots, I_{n+1}$. What is the statistical distribution of $S = \Sigma_{i=1}^{n+1}|I_i|^2$? In the present paper we discuss this problem. The theory of various distributions connected with the random division of an interval has been discussed by many writers (Clifford [(1866)], v[on] Bortkiewicz [(1915)], Morant [(1921)], Fisher [(1940)], Garwood [(1940)], and others) and has many applications to statistical problems. Both Clifford and Fisher remark that this problem is equivalent to considering n+1 quantities $x_i = |I_i|$ which are non-negative and satisfy $\Sigma_1^{n+1} x_i = 1$, and whose probability distribution is such that equal areas on the positive part of the n-dimensional plane defined by this equation correspond to equal probabilities for the corresponding division of the interval." The author computes the first four moments of the random variable S and shows that the distribution of S is asymptotically normal.

**Comments:** The n+1 intervals $I_1, \ldots, I_{n+1}$ may be thought of as the gaps between order statistics $x_1, \ldots, x_n$ of a sample of size n from a uniform distribution on the interval (0,1), the first gap being $x_1 - 0 = x_1$ and the last being $1 - x_n$—see comments on Domb (1947).

**References:** Clifford (1866), von Bortkiewicz (1915), Morant (1921), Fisher (1940), Garwood (1940), Greenwood (1946).

* Nair, K. R. (1947). A note on the mean deviation from the median. *Biometrika* **34,** 360-362. (MR 9, 363).

**Summary:** The author points out that, for samples from a normal population, the exact sampling distribution is known for m, the mean deviation from the sample mean, but not for m′, the mean deviation from the sample median; however, it is known that $E(m') \leq E(m)$ [the equality holds only for sample size $n = 2$ and asymptotically as $n \to \infty$]. He shows that, for samples of size 3 or 4, the standard error of m′ is less than that of m; however, the coefficient of variation of m′ is the same (to at least five decimal places) as that of m for samples of size 3 and slightly greater than that of m for samples of size 4. He writes (p. 362): "In conclusion, it seems worth making the following point: (a) if expressions for the expectation and variance of m′ were available and tables of its probability integral worked out, (b) if the efficiency of the m′ estimate compared to the m estimate for $n > 4$ was not appreciably worse than for the case $n = 4$, there would be strong practical grounds for using m′ rather than m in view of greater simplicity in calculation. In both cases we must first arrange the observations in order of magnitude. Then if $x_1 \leq x_2 \leq \ldots \leq x_n$, m′ may be calculated from the formula $m' = (1/n)\{x_{n-t+1} + x_{n-t+2} + \ldots + x_n) - (x_1 + x_2 + \ldots + x_t)\}$, where $t = n/2$ or $(n-1)/2$ according as n is even or odd. For m, however, we must also calculate the arithmetic mean $\bar{x}$ and look for $x_k$ and $x_{k+1}$ between which $\bar{x}$ lies. Then m can be obtained

from one of the three formulae $nm/2k = \bar{x} - (x_1 + x_2 + \ldots + x_k)/k$, $nm/2(n-k) = (x_{k+1} + \ldots + x_n)/(n-k) - \bar{x}$, $n^2m/2k(n-k) = (x_{k+1} + \ldots + x_n)/(n-k) - (x_1 + \ldots + x_k)/k$. This certainly involves a rather long process. It is interesting to note that m′ becomes a special case of the measure of dispersion based on difference between the sums of the first and the last r observations (in order of magnitude) suggested by Jones (1946), the range becoming another special case of the same measure, when $r = 1$."

**References:** Helmert (1876a), Fisher (1920), E. S. Pearson (1926), Pearson & Hartley (1942), Jones (1946), Walsh (1946b).

**Citations:** Nair (1948b), Godwin (1949b), Nair (1949a).

* Plackett, R. L. (1947). Limits of the ratio of mean range to standard deviation. *Biometrika* **34,** 120-122. (MR 8, 395).

**Summary:** Values of $d_n = \bar{w}_n/\sigma$, where $\bar{w}_n$ is the mean range in samples of size n and $\sigma$ is the population standard deviation, have been tabulated for samples from a normal population and used extensively in control chart work to estimate $\sigma$ from the mean range of a number of small samples. The author shows that populations exist for which $d_n$ is arbitrarily near to zero, while for no population will it exceed the value $d_n(\max.) = n\sqrt{(2\{(2n-2)! - [(n-1)!]^2\}/(2n-1)!)}$, which tends to $\sqrt{(n+1/2)}$ for large n. He exhibits populations for which $d_n$ attains the value $d_n(\max.)$; these populations are rectangular for $n = 2, 3$. He tabulates $\sqrt{(n+1/2)}$, $d_n(\max.)$, $d_n(\text{normal})$, and $d_n(\text{rectangular})$ for $n = 2(1)12$; all are given to five decimal places except $d_n(\text{normal})$, which is given to three decimal places. He points out that for $n \leq 12$, $d_n(\text{normal})$ and $d_n(\text{rectangular})$ agree to within 10%; however, the difference increases with n. The value of $d_n(\max.)$ approaches the asymptotic value $\sqrt{(n+1/2)}$ quite rapidly, differing from it by less than 0.1% for $n = 12$.

**References:** Pearson & Adyanthāya (1928), K. Pearson (1931), Pearson [& Hartley] (1942).

* Purcell, Warren B. (1947). Saving time in testing life. *Industrial Quality Control* **3** (5), 15-18.

**Summary:** The author writes (pp. 15-16): "In testing life, . . . use of the statistics of average and range requires that each test be continued until the last unit has reached the end of its life. The consequent delay in obtaining the data is fatal to the effective use of this Modern Quality Control Technique. In many cases, the difficulty has been partially overcome by increasing the severity of the life test. A classic example is the shortening of the life of incandescent lamps from 1000 hours to about 5 hours by burning at 150% of rated voltage. Even at this acceleration, however, it often requires 7 or 8 hours to burn out the last lamp of a sample; yet the final result might yield an average so low or a range so great that need of corrective action is indicated. Attempts to shorten the life further cause abnormal arcs and do not give a true picture of normal life. Since the test cannot be further accelerated, it is necessary to accelerate the statistics. In conventional statistics, average is used to indicate a shift in the whole population; range is used to indicate a change in population dispersion. A shift in the whole population can also be indicated by the median, or central failure in the sample; a population shift or change in dispersion can be indicated by the minimum, or first failure in the sample. The use of median and minimum as statistics substantially accelerates the results. A tendency toward short life which is indicated by the sample is indicated quickly. It is not necessary to continue the test beyond the median, and corrective action can be started promptly." On page 17, the author gives formulas for control limits for medians and minima, and tabulates factors for computing them for samples of size $n = 3(1)9$.

**Reference:** Kendall (1943).

* Ruggles, Richard; Brodie, Henry (1947). An empirical approach to economic intelligence in World War II. *Journal of the American Statistical Association* **42,** 72-91.

**Summary:** The authors report the story of the development of a technique, used by the Economic Warfare Division of the American Embassy in London, starting in early 1943, for analyzing markings and serial numbers obtained from captured German equipment in order to obtain estimates of German war production and strength. Given a sample of serial numbers, the total number in the series was estimated to be the largest serial number in the sample plus the average gap between successive ordered serial numbers, as we can see from the following statement (p. 82): "The highest mold number [for tank tires] of one maker was 77 and the average gap in the series was about 3 cases, so it was estimated that this plant had about 80 molds."

**Comments:** For an analysis of the statistical methods used, see Goodman (1952).

* Tanenhaus, Seaman J. (1947). The median as a typical value for Walker yarn abrader data. *Textile Research Journal* **17,** 281-286.

**Summary:** The author writes (Abstract, p. 281): "The data obtained on the Walker yarn abrader in the determination of abrasion-resistance of yarns form frequency distributions which are decidely positively skewed. The mean of such distributions, being affected by the extremes, tends to be atypical. The modal class is too broad for precise differentiation and the first breaks are too varible to be of use. The median, which is independent of the extreme values, is more representative. Furthermore, by computing the average median of several sets of yarns rather than determining the lot median, a saving of time of approximately 50 per cent is effected."

**Reference:** Dodge et al. (1933)[1943].

Tukey, John W. (1947). Non-parametric estimation II. Statistically equivalent blocks and tolerance regions — the continuous case. *Annals of Mathematical Statistics* **18,** 529-539. (MR 9, 295).

**Summary:** The author writes (Summary, p. 529): "Wald [(1943)] extended the usefulness of tolerance limits to the simplest multi-dimensional cases. His principle [successive elimination] is here used to provide many new ways of using a sample of n to divide the range of the population into n + 1 blocks of known behavior. The exact tolerance distribution for the proportions of the population covered by these blocks is extended from the case of a continuous probability density function to the case of a continuous cumulative distribution function. Such an extension is needed in dealing completely with multivariate cases *even* when the underlying distribution is as smooth as a multivariate normal distribution. The devices used in Paper I [Scheffé & Tukey (1945)] to extend the usefulness of tolerance limits to the case of a discontinuous underlying distribution will be applied in the next paper of this series, with some extension, to extend the usefulness of these general tolerance regions to the case of a discontinuous distribution. Some of these results specialize into new results for the univariate case, although they do not seem to have any immediate practical application. . . ."

**References:** Wilks (1941), Wald (1943), Scheffé & Tukey (1945).

**Citations:** Murphy (1948), Tukey (1948a), Wilks (1948).

* Walsh, John E. (1947). An extension to two populations of an analogue of Student's t-test using the sample range. *Annals of Mathematical Statistics* **18,** 280-285. (MR 9, 48).

**Summary:** The author writes (Summary, p. 280): "The modified t-test [based on the sample range] considered by Daly [(1946) and also by Lord (1947)] is used to develop one-sided significance tests to decide whether the mean of a new normal population exceeds the mean of an old normal population having the same variance. Significance tests are also developed to decide whether the mean of the new population is less than the mean of the old population. These tests require very little computation for their application and are approximately as powerful as the most powerful tests for these hypotheses."

**References:** Daly (1946), Lord (1947).

Anderson, T. W. (1948). The asymptotic distributions of the roots of certain determinantal equations. *Journal of the Royal Statistical Society B* **10,** 132-139. (MR 10, 553).

**Summary:** The author writes (Introduction, p. 132): ". . . Geary [(1948)] has used the roots of a certain determinantal equation to construct a criterion for testing hypotheses about the number of linear relations holding between the coefficients of linear regression functions of multivariate normal variates. One of the matrices involved in the determinantal equation is the covariance matrix of the variates; the other is composed of weighted sums of the sample regression coefficients. The criterion suggested by Geary is similar to that suggested by P. L. Hsu [(1941b)] for the case in which the covariance matrix is replaced by its estimate from the sample. If r is the number of independent linear relations among the regression coefficients, the asymptotic distribution of the r smallest roots is the same as in Hsu's case [Hsu (1941a)]; the asymptotic distribution of the other is modified by scale factors which make the distribution more concentrated (Theorem 1). The asymptotic distribution of the criterion, which is the sum of the r smallest roots, is the $\chi^2$–distribution (Theorem 2). Geary also studies an approach to the problem of linear relations suggested by Gerhard Tintner [(1945)]. In this case the expected values are not assumed to be linear functions of known fixed variates, but are arbitrary constants satisfying a number of linear equations. For testing hypotheses about the number of equations Tintner has suggested the use of roots of a determinantal equation involving the covariance matrix of the variates and the sample covariance matrix. The asymptotic distribution of all the roots of this equation are given in Theorem 3. The r smallest roots are distributed asymptotically like the characteristic roots of a matrix with a certain normal distribution. The asymptotic distribution of the criterion (properly normalized) is normal (Theorem 4). This note has been prompted by the use that Geary has made of the roots of these two determinantal equations. Although the proofs of the theorems in this note are not new, being merely adaptations of Hsu's proof, and although parts of the theorems have been anticipated by Geary and others, it seems useful to give these results explicitly for the benefit of other investigators, and to verify that they can be derived rigorously. The note concludes with some remarks about the models of Geary and Tintner, and an indication of more general conditions under which the theorems hold."

**References:** Fisher (1939), Hsu (1939, 1941a,b), Anderson (1945), Tintner (1945), Geary (1948).

**Citation:** Geary (1948).

* Bartlett, M. S. (1948). Internal and external factor analysis. *British Journal of Psychology, Statistical Section* **1,** 73-81.

**Summary:** The author notes that he has discussed and illustrated at some length elsewhere [Bartlett (1947b)] the mutual relation between two sets of variables [Hotelling (1936b)]. He stresses that the canonical transformation of one group of variables into "factors" is made with reference to an external criterion, the other group of variables; for that reason he refers to it as an external factor analysis, in contrast with the internal canonical or factor analysis of a single set of variables. He discusses some tests of significance to be applied to the roots of a determinantal equation: (1) a $\chi^2$ approximation for the significance of the k smallest roots and (2) the significance of the ratio of the two smallest roots. He tabulates the 5% and 1% critical values of this ratio for external and internal analysis with degrees of freedom s = 1(1)10(2)20(5)50(10)100.

**References:** Hotelling (1933, 1936b), Wilks (1943), Bartlett (1947b).

* Cox, D. R. (1948). A note on the asymptotic distribution of range. *Biometrika* **35,** 310-315. (MR 10, 466).

**Summary:** The author writes (Introduction and summary, p. 310): "In a recent paper Elfving (1947) has

given an asymptotic form for the distribution of range in large samples. In the present note, two other methods of obtaining an asymptotic form for the range distribution are discussed which have the advantage of being expressed directly in terms of the range, while Elfving's form involved a non-linear transformation of range. Numerical results are given for a normal population comparing the exact distribution of range with the two approximations discussed here, and with Elfving's approximation." The two methods discussed by the author are (1) derivation from the results of Fisher & Tippett (1928) on the distribution of extremes, which was previously discussed in detail by Gumbel (1947), as the author acknowledges in a footnote and (2) an asymptotic form obtained by the method of steepest ascents. The author tabulates the exact values of the mean, standard deviation, $\beta_1$ and $\beta_2$ for the range of normal samples of size n = 10, 20, 50, 100, 500, 1000, together with the results of method (1) for the same values of n and those of method (2) and Elfving's approximation for n = 20, 50. The results for method (2) are more accurate than those for method (1) [Gumbel's method] but less accurate than those for Elfving's method.

**References:** Tippett (1925), E. S. Pearson (1926), Fisher & Tippett (1928), K. Pearson (1931), Pearson & Hartley (1942), Elfving (1947), Gumbel (1947).

* Eisenhart, Churchill; Deming, Lola S.; Martin, Celia S. (1948a). The probability points of the distribution of the median in randon samples from any continuous population (abstract). *Annals of Mathematical Statistics* **19,** 598-599.

**Summary:** The authors write (pp. 598-599): "The abscissa of the (one-tail) $\varepsilon$-probability point of the distribution of the median in random samples of size $n = 2m + 1$ ($m \geq 0$) from any continuous population is identical with the abscissa of the corresponding $P_{\varepsilon,n}$-probability point of the parent distribution, where $P_{\varepsilon,n}$ is determined by (1) $\Sigma_{k=(n+1)/2}^{n} C_k^n \, p_{\varepsilon,n}^k \, (1 - P_{\varepsilon,n})^{n-k} = \varepsilon$, $(0 \leq \varepsilon \leq 1)$. From (1) it follows that (2) $P_{1-\varepsilon,n} = 1 - P_{\varepsilon,n}$ and that (3) $P_{\varepsilon,n} = x_\varepsilon(n + 1, n + 1) = [1 + F_\varepsilon(n + 1, n + 1)]^{-1} = \{1 + \exp[2Z_\varepsilon(n + 1, n + 1)]\}^{-1}$, where $x_\varepsilon(\nu_1, \nu_2)$, $F_\varepsilon(\nu_1, \nu_2)$ and $Z_\varepsilon(\nu_1, \nu_2)$ denote the $\varepsilon$-probability points of the incompete-beta-function distribution, Snedecor's F-distribution and Fishers's z-distribution, for $\nu_1$ (=2q) and $\nu_2$ (=2p) 'degrees of freedom', respectively. The foregoing results are certainly not 'new': Harry S. Pollard )[(1934)] implicitly used the first equality on the extreme left of (3) . . ., and John H. Curtiss [1943)] has given the generalization of (1) appropriate to the case of the 'r th position' in random samples from any continuous population . . . and utilized (3) explicitly to obtain the 5% point of the distribution of the median in random samples of size n = 23. The aim of the present paper is to give these results somewhat greater publicity—they are hardly 'well-known'. To this end a table . . . is given of the values of $P_{\varepsilon,n}$ to 5 significant figures for $\varepsilon$ = 0.001, 0.005, 0.01, 0.025, 0.05, 0.10, 0.20, 0.25 and n = 3(2)15(10)95, together with expressions from which $P_{\varepsilon,n}$ can be evaluated accurately and conveniently for values of n (and $\varepsilon$) not included in the table. Numerical examples illustrate the use of the table and formulas. Concise derivations of the fundamental relations and formulas are given in an appendix."

**Note:** The full text and tables (slightly edited) of this paper and the two succeeding ones were published much later [Eisenhart, Deming & Martin (1963)].

**References:** Pollard (1934), Curtiss (1943).

**Citation:** Eisenhart, Deming & Martin (1948b).

* Eisenhart, Churchill; Deming, Lola S.; Martin, Celia S. (1948b). On the arithmetic mean and the median in small samples from the normal and certain non-normal populations (abstract). *Annals of Mathematical Statistics* **19,** 599-600.

**Summary:** The authors write (pp. 599-600): "Let $\bar{x}_{\varepsilon,n}$ and $\tilde{x}_{\varepsilon,n}$ denote the abscissae of the one-tail

$\varepsilon$-probability points of the arithmetic mean and the median, more specifically, the abscissae exceeded with probability $\varepsilon$ by the mean and the median, respectively, in random samples of size $n (= 2m + 1)$ from any specified population, and let $\sigma_{\bar{x}_n}$ and $\sigma_{\tilde{x}_n}$ denote the standard deviations of the mean and the median in such samples, respectively. The following symmetrical populations with zero location parameters and unit scale parameters are considered in this paper: normal (Gaussian) $\exp(-x^2/2)/\sqrt{2\pi}$, $-\infty < x < \infty$; double-exponential (Laplace) $\exp(-|x|)/2$, $-\infty \leq x \leq \infty$; rectangular (uniform) 1, $-1/2 \leq x \leq 1/2$; Cauchy $[\pi(1 + x^2)]^{-1}$, $-\infty < x < \infty$; sech (sech x)/$\pi$, $-\infty \leq x \leq \infty$; sech$^2$ (derivative of 'logistic') $\text{sech}^2 x/2$, $-\infty \leq x \leq \infty$. Using the basic table, relating probability points of the distribution of the median to probability points of the parent distribution, given . . . [by Eisenhart, Deming & Martin (1948a)], values of $\tilde{x}_{\varepsilon,n}$ for random samples from each of the above distributions have been evaluated, and are tabulated to 5 decimal places in the present paper, for n = 3(2)15(10)95 and $\varepsilon$ = 0.001, 0.005, 0.01, 0.025, 0.05, 0.10, 0.20, 0.25. In the case of the normal distribution, values of $\bar{x}_{\varepsilon,n}$ to 5 decimal places are given also for the aforementioned combinations of $\varepsilon$ and n. Comparison of the values of $\tilde{x}_{\varepsilon,n}$ and $\bar{x}_{\varepsilon,n}$ gives precise numerical meaning to the well-known lesser accuracy of the median as an estimator of the center of a normal population, for samples of any odd size (n = 2m + 1). . . . In the case of the double-exponential distribution, values of $\bar{x}_{\varepsilon,n}$ are given to 4 decimal places for n = 3(2)11, and $\varepsilon$ = 0.005, 0.01, 0.025, 0.05, 0.10, 0.25, for comparison with the corresponding values of $\tilde{x}_{\varepsilon,n}$. It is found that when n = 3, $\bar{x}_{\varepsilon,3} < \tilde{x}_{\varepsilon,3}$ for = 0.005, 0.01, and 0.025, indicating that in random samples of 3 from a double-exponential distribution the arithmetic mean furnishes narrower confidence limits for the center of the distribution at 0.95, 0.98 and 0.99 levels of confidence. When n = 5, the mean is 'better' at the .98 and .99 levels of confidence; and, when n = 7, at the 0.99 level. For all other combinations of $\varepsilon$ and n ($\geq$ 3), the median is 'better'. In the case of the rectangular distribution, values of $\bar{x}_{\varepsilon,n}$ are tabulated to 4 decimals for n = 3(2)9 and values of $\breve{x}_{\varepsilon,n}$, the $\varepsilon$-probability point of the mid-range in samples of n, for n = 3(2)15(10)95, in each instance for $\varepsilon$ = 0.005, 0.01, 0.025, 0.05, 0.10, 0.25, and in the case of $\bar{x}_{\varepsilon,n}$ for $\varepsilon$ = 0.001 also. The superiority of the midrange over the mean and the median, well-known but here exhibited numerically for the first time, is truly amazing. . . ."

**Note:** See note on preceding paper.

**Reference:** Eisenhart, Deming & Martin (1948a).

* Eisenhart, Churchill; Martin, Celia S. (1948). The relative frequencies with which certain estimators of the standard deviation of a normal population tend to underestimate its value (abstract). *Annals of Mathematical Statistics* **16,** 600.

**Summary:** The authors write (p. 600): "Let $x_1, x_2, \ldots, x_n$ denote a random sample of n independent observations from a normal population with mean $\mu$ and standard deviation $\sigma$. Common estimators of $\sigma$ are $s_1 = \sqrt{\Sigma_{i=1}^n (x_i - \bar{x})^2}/n$, $s_2 = s_1\sqrt{n}/(n-1)$, $s_3 = s_1/c_2$, $m_1 = \sqrt{\pi/2}\,\Sigma_{i=1}^n |x_i - \bar{x}|/n$, $m_2 = m_1\sqrt{n}/(n-1)$, and $R_1 = (x_L - x_S)/d_2$, where $\bar{x} = \Sigma_{i=1}^n x_i/n$, $x_L$ is the largest and $x_S$ the smallest of the x's, $c_2 = E(s_1)$, and $d_2 = E(x_L - x_S)$ . . . . A table is given that shows to 3 decimals the relative frequencies (probabilities) with which these estimators tend to underestimate $\sigma$ when n = 2(1)10, 12, 15, 20, 24, 30, 40, 60. The results show among other things that, for very small samples (n < 10) such as chemists and physicists commonly use, Bessel's formula for the probable error, which is based on $s_2$, has a marked downward bias in the probability sense (in addition to its known slight downward bias in the mean value sense), whereas Peters' formula, which is based on $m_2$, has only a slight downward bias in the probability sense and no bias in the mean value sense. A table of divisors is given by means of which 'median estimators' of $\sigma$ can be computed readily from the basic quantities $\Sigma_{i=1}^n (x_i - \bar{x})^2$, $\Sigma_{i=1}^n |x_i - \bar{x}|$, and $(x_L - x_S)$, that is, estimators that will over-and under-estimate $\sigma$ equally often in repeated use. An application to control charts is noted. Median estimators, like maximum likelihood estimators ('modal estimators') have the useful property that if $T_{1,2}$ is a median estimator of $\theta$, then $f(T_{1,2})$ is a median estimator of $f(\theta)$, a property unfortunately not possessed by the customary 'unbiased' ('mean') estimators."

**Note:** See note on paper by Eisenhart, Deming & Martin (1948a).

* Epstein, Benjamin (1948a). Statistical aspects of fracture problems. *Journal of Applied Physics* **19,** 140-147. (MR 9, 360).

**Summary:** The author writes (Abstract, p. 140): "In recent years there has been an increasing interest in the development of statistical theories of strength. A main aim of these theories is to explain in a reasonable way such things as the dependence of the strength of specimens on their volume or length. In this paper it is pointed out that the problems posed by these models are equivalent to an important problem in mathematical statistics, namely, the distribution of the smallest value in samples of size n drawn from a population having some probability density function f(x). The calculations made by mathematical statisticians give a far more complete description of the results to be expected than do the estimates to be found up to now in the technical literature." After reviewing the work of Griffith (1920, 1924), Peirce (1926), Weibull (1939a, b), Kontorova (1940), Kontorova & Frenkel (1941) [Frenkel & Kontorova (1943)] and Davidenkov, Shevandin & Wittmann (1947) on the "weakest link" concept, that of Tippett (1925), Fisher & Tippett (1928) and Gumbel (1935a, 1937a, 1941a) on extreme-value theory and that of Daniels (1945) on the statistical theory of bundles of threads, the author gives his own exposition of the statistical theory of extreme values. In his concluding remarks, he writes (pp. 146-147): "When one looks at the problem of fracture in its most general setting it is clear that the 'weakest link' concept, when applicable, may be the key to the occurrence of certain observed relationships between the strength of specimens and their size. . . . From a statistical point of view, certain phenomena, which may on the surface appear to be different, are really equivalent since they lead in each instance to the same problem, namely, the distribution of the smallest value in large samples. Now, it is not unreasonable to suppose that the distribution of strengths due to flaws is Gaussian or perhaps more generally of the form $A \exp(-B|x-\mu|^p)$ for large values of $|x-\mu|$ (where A, B, $\mu$, and p are positive constants). Under such hypotheses statistical theory leads to the prediction that the most probable value of the smallest value in samples of size n must decrease as $(\log n)^{1/p}$ and that the distribution of smallest values in samples of size n must be negatively skewed. . . . One other important point that we wish to emphasize is that one should not expect the 'weakest link' hypothesis, attractive as it may be, to explain all phenomena relating to the strength of materials. In the first place it is implicitly assumed in this paper that the flaws are all independent and do not in any way influence each other. There are physical situations [for example, notched specimens] where such an assumption is not justified. . . . In the second place there have been attempts (see, for example, A. M. Freudenthal [(1946)]) to apply the weakest link concept to the study of the fatigue of materials. We believe that such an attack is basically incorrect even if it leads to the prediction of relationships which agree reasonably well with what is found in nature. For example, the relationship between the length of time required to break a specimen and the applied load is in its very essence a time-dependent problem. The specimen is changing and, therefore, any essentially static approach which uses the 'weakest link' concept without modification leaves out certain basic features of the process. . . ."

**References:** Griffith (1920, 1924), Tippett (1925), Peirce (1926), Fisher & Tippett (1928), K. Pearson (1931), Gumbel (1935a, 1937a), Weibull (1939a, b), Kontorova (1940), Gumbel (1941a), Kontorova & Frenkel (1941) [Frenkel & Kontorova (1943)], Daniels (1945), Cramér (1946), Freudenthal (1946), Davidenkov, Shevandin & Wittmann (1947).

**Citations:** Epstein (1948b), Epstein & Brooks (1948), Epstein (1949a), Peterson (1949a,b).

* Epstein, Benjamin (1948b). Application of the theory of extreme values in fracture problems. *Journal of the American Statistical Association* **43,** 403-412.

**Summary:** This paper covers much of the same material as the preceding one [Epstein (1948a)]. The author shows that the theory of extreme values is pertinent to the treatment of certain aspects of

fracture or breakdown of materials used in modern technology and makes an attempt to integrate some of the results scattered throughout the statistical literature. He writes (pp. 403-404): "In essence, all the statistical models proposed in the study of fracture take as a starting point Griffith's theory [Griffith (1920, 1924)] which states that the difference between the calculated strengths of materials and those actually observed resides in the fact that there exist flaws in the body which weaken it. Accepting this point of view is equivalent to assuming that there will be a distribution of strengths in a given specimen in the sense that a different amount of force will be needed to fracture a specimen at one or another point. There are many physical situations where failure at one point means failure of the entire specimen. But this simply means statistically that it is the worst flaw among N flaws (where N is the number of flaws in the specimen) which determines the strength of a specimen and therefore the theory of extreme values is immediately applicable. Clearly N increases as the specimen size increases and, therefore, the dependence of strength on specimen size is equivalent statistically to the problem of how the distribution of smallest (or largest) values depends on N. Generally speaking, asymptotic theory is applicable since N, the number of flaws, is large in most practical problems." The author proceeds to summarize work on the strength of materials by Peirce (1926), Weibull, (1939a,b), Kontorova (1940), Kontorova & Frenkel (1941) [Frenkel & Kontorova (1943)], Davidenkov, Shevandin & Wittmann (1947) and other authors, as well as the related work on the theory of extreme values by Tippett (1925), Fisher & Tippett (1928), Gumbel (1935a) and others. He gives the asymptotic distributions of smallest and largest values in samples of size n (n large) for the distributions of Laplace, Gauss, and Weibull and the mode, the mean, and the variance for each of these asymptotic distributions.

**References:** Griffith (1920, 1924), Peirce (1926), Fisher & Tippett (1928), Gumbel (1935a, 1937a), Weibull (1939a,b), Kontorova (1940), Gumbel (1941a), Kontorova & Frenkel (1941) [Frenkel & Kontorova (1943)], Tucker (1941), Daniels (1945), Cramér (1946), Davidenkov, Shevandin & Wittmann (1947), Fisher & Hollomon (1947), Epstein (1948a), Epstein & Brooks (1948).

* Epstein, Benjamin; Brooks, Hamilton (1948). The theory of extreme values and its implications in the study of the dielectric strength of paper capacitors. *Journal of Applied Physics* **19,** 544-550.

**Summary:** The authors apply the theory of extreme values to the study of the dielectric strength of paper capacitors and its dependence on capacitor size. The statistical theory presented is essentially the same as in the two preceding papers [Epstein (1948a,b)], except that interest now centers in the distribution of largest (rather than smallest) values. The reason for this is given in the following statement (p. 545): "Since the dielectric strength of a capacitor in the immediate neighborhood of a conducting particle is determined by the size of the particle, it is immaterial from a statistical point of view whether one considers, on the one hand, the distribution of dielectric strengths in the neighborhood of flaws, or on the other, the distribution of the sizes of the conducting particles. In this paper it will prove convenient to study the distribution of dielectric strengths of capacitors by considering the distribution of the largest values in samples of size n drawn from a population of conducting particles following some size distribution law described by a continuous probability density function f(x)." The authors write down the basic formulas for such a distribution. In applying the results to the dielectric strength of capacitors, they assume, on the basis of empirical evidence, that the size distribution of conducting particles, f(x), is of exponential type $f(x) = \lambda e^{-\lambda x}$, $x > 0$.

**References:** Griffith (1920, 1924), Peirce (1926), Fisher & Tippett (1928), Gumbel (1935a, 1937a, 1941a), Cramér (1946), Epstein (1948a).

**Citations:** Epstein (1948b, 1949a).

* Feller, W. (1948). On the Kolmogorov-Smirnov limit theorems for empirical distributions. *Annals of Mathematical Statistics* **19,** 177-189; corrections, **21** (1950), 301-302. (MR 9, 599; 11, 674).

**Summary:** The author writes (Summary, p. 177): "Unified and simplified derivations are given for the limiting forms of the difference (1) between the empirical distribution of a large sample and the corresponding theoretical distribution and (2) between the distributions of two large samples." He defines the empirical distribution of the ordered sample $X_1^* \leq X_2^* \leq \ldots \leq X_N^*$ from a distribution with c.d.f. $F(x)$ to be the step function $S_N(x) = 0$ for $x < X_1^*$, $S_N(x) = k/N$ for $X_k^* < x < X_{k+1}^*$, $S_N(x) = 1$ for $x \geq X_N^*$. He gives simplified proofs of the following theorems: Theorem 1 (Kolmogoroff [(1933)]). Suppose that $F(x)$ is continuous and define the random variable $D_N$ by $D_N = \text{l.u.b.} |S_N(x) - F(x)|$. Then for every fixed $z \geq 0$ as $N \to \infty$, $\Pr\{D_N \leq z N^{-1/2}\} \to L(z)$ where $L(z)$ is the cumulative distribution function which for $z > 0$ is given by either of the following equivalent relations, $L(z) = 1 - 2\sum_{\nu=1}^{\infty}(-1)^{\nu-1} e^{-\nu^2 z^2} = (2\pi)^{1/2} z^{-1} \sum_{\nu=1}^{\infty} e^{-(2\nu-1)^2\pi^2/8z^2}$. For $z \leq 0$ we have, of course, $L(z) = 0$. Theorem 2 (Smirnov [(1939a,b)]). Let $(X_1, \ldots, X_m)$ and $(Y_1, \ldots, Y_n)$ be two samples of mutually independent random variables having a common continuous distribution $F(x)$. Let $S_m(x)$ and $T_n(x)$ be the corresponding empirical distribution functions and define a new random variable $D_{m,n}$ by $D_{m,n} = \text{l.u.b.} |S_m(x) - T_n(x)|$. Put $N = mn/(m+n)$ and suppose that $m \to \infty$, $n \to \infty$ so that $m/n \to a$, where $a$ is a constant. Then for every fixed $z \geq 0$, $\Pr\{D_{m,n} \leq z N^{-1/2}\} \to L(z)$, where $L(z)$ is the same as in Theorem 1. He also proves the following theorem. With the notations and assumptions of Theorem 1 let $D_N^+ = \text{l.u.b.} \{S_n(x) - F(x)\}$. Then $\Pr\{D_N^+ \leq z N^{-1/2}\} \to 1 - e^{-2z^2}$.

**Comments:** The compiler has already pointed out, in his comments on the paper by Smirnov (1939b), that $D_{m,n}$ does not depend upon the order statistics of either of the two samples, but only on the ranks of the observations from the two samples in the combined sample. However, because the statistic $D_{m,n}$ has the same limiting distribution as the statistic $D_N$, which does depend upon the sample order statistics, papers dealing with $D_{m,n}$, and not just those dealing with $D_N$, are included in this bibliography.

**References:** von Mises (1931), Kolmogoroff (1933), Smirnov (1939a,b), Kolmogoroff (1941).

**Citations:** Smirnov (1948), Chung (1949), Dobb (1949), Kac (1949), Maniya (1949).

* Frankel, Jacob Porter (1948). Relative strengths of Portland cement mortar in bending under various loading conditions. *Proceedings of the American Concrete Institute* **45,** 21-32; discussion, 32-1-32-5.

**Summary:** The author compares the results of applying the theory of Tucker (1927, 1941), based [the author says] on the normal distribution, and that of Weibull (1939a,b), based on the Weibull distribution, to the relative strengths of mortar in bending. According to Weibull's theory, the probability P of rupture is given by $P = 1 - \exp\{-kV[(S-S_\ell)/S_0]^m\}$, where k is a constant depending on the method of loading, V is the volume of the specimen, S is the applied stress, $S_\ell$ is the minimum stress (location or threshold parameter) below which no rupture will occur, $S_0$ is a particular stress constant (scale parameter), and m is a (shape) parameter to be determined. The author states that Tucker's theory is essentially a special case of Weibull's with $m = 2$. On the basis of tests performed on 99 small mortar beams under sixth-, third-, and center-point loading, he finds that $m = 2.9$ and that there is excellent agreement between Weibull's theory and the experimental findings. Further light is shed on the relative strengths and weaknesses of the two theories (Weibull's and Tucker's) by the discussion, which consists of comments by Tucker and a closure by the author. Of particular interest is Tucker's assertion that the author's statement that he (Tucker) assumes a normal strength distribution is incorrect; he states that the only essential in his philosophy is the author's Eq. (2), $1 - P_V = (1 - P_0)^V$, where $P_0$ is the probability of failure of a unit volume and $P_V$ that of a specimen of volume V.

**Comments:** The author's statements that Tucker's theory is based on the normal distribution and that it is a special case of Weibull's for $m = 2$ are inconsistent, since the Weibull distribution is not normal for shape parameter $m = 2$ (or for any other value of m); it has skewness $\alpha_3 = 0$ for $m \doteq 3.6$, while $\alpha_3 \doteq .631$ for $m = 2$. Tucker's denial that his theory is based on the normal distribution resolves the discrepancy.

**References:** Tippett (1925), Tucker (1927), Weibull (1939a,b), Tucker (1941, 1945a,b).

* Gartštejn, B. N. (1948). On certain limit laws for the range. (Russian). *Doklady Akademii Nauk SSSR* (N. S.) **60** (7), 1119-1121. (MR 10, 51).

**Summary:** Let the random values of a finite sequence $\xi_1, \xi_2, \ldots, \xi_n$ be independent and have the same continuous c.d.f. (cumulative distribution function) $F(x)$. The random value $\rho_n = \zeta_n - \eta_n$ is called the range, where $\zeta_n = \max(\xi_1, \xi_2, \ldots, \xi_n)$, $\eta_n = \min(\xi_1, \xi_2, \ldots, \xi_n)$. The author generalizes the work of several recent authors, including Gumbel (1947) and Elfving (1947), on the limiting distribution of the range. Let $\Phi_n(r)$ be the c.d.f. of the range $\rho_n$ and let $\Psi_n(r) = F^n(r) * [1 - F(-r)]^n$ be the convolution of the c.d.f.'s of $\zeta_n$ and $-\eta_n$. The author states (without proof) Theorem 1: In order for the function $\Phi_n(a_n u + b_n)$, for the proper selection of the constants $a_n > 0$ and $b_n$, to converge as $n \to \infty$ toward some limiting c.d.f. $\Phi(u)$ which is different from the improper c.d.f. which is 0 for $u \leq a$ and 1 for $u > a$, where $a$ = constant, it is necessary and sufficient for $\Psi_n(a_n u + b_n)$ to converge similarly, for the same $a_n$ and $b_n$, and in that case the two limiting c.d.f.'s coincide. He then imposes an additional limitation on the initial c.d.f. $F(x)$, namely that $F(x)$ is such that limiting distributions of the maximum $\zeta_n$ and the minimum $\eta_n$ exist. He points out that Gnedenko (1943) has shown that these limiting distributions must be of one of the following types: (1) $\Phi_\alpha(u) = 0$ for $u \leq 0$, $= e^{-u^{-\alpha}}$ for $u > 0$; (2) $\Psi_\alpha(u) = e^{-(-u)^\alpha}$ for $u \leq 0$, $= 1$ for $u > 0$; (3) $\lambda(u) = e^{-e^{-u}}$, where $\alpha$ is a positive constant. The author then proves (with the aid of a lemma) Theorem 2: The class of possible limiting laws for convolutions of c.d.f.'s of the maximum and minimum terms consists of laws of the following six types: $\Phi_\alpha(u)$, $\Psi_\alpha(u)$, $\lambda(u)$, $\Phi_\alpha(u) * \Phi_\alpha(au)$, $\Psi_\alpha(u) * \Psi_\alpha(au)$, $\lambda(u) * \lambda(au)$, where $a$ = constant. It follows, as a direct consequence of Theorems 1 and 2, that the limiting distribution of range can belong only to one of the indicated six types. The author's results, together with those of Gnedenko (1943), make it possible to obtain those of Gumbel (1947).

**References:** Gnedenko (1943), Elfving (1947), Gumbel (1947).

* Geary, R. C. (1948). Studies in relations between economic time series. *Journal of the Royal Statistical Society B* **10,** 140-158. (MR 10, 553).

**Summary:** The author writes (Introduction, p. 140): "The present paper owes its origin to a suggestion by Richard Stone, who . . . was struck by the fact that Harold Hotelling's principal component theory [Hotelling (1933)] depends on the ascertainment of the *largest* significant latent roots of a certain determinant, whereas in Gerhard Tintner's theory [Tintner (1945)] . . . the smallest latent roots of the same determinant are utilized. He suggested that I should examine the genesis of the relationship between the two theories, and indeed largely anticipated the findings at the end of Section 1 of this paper." In Section 1, the author establishes the complementarity of the Hotelling principal component theory and the Tintner theory. In Section 2, he makes some observations on the Tintner theory; in Section 3, he develops a new theory; and in Section 5, he investigates the consistency and efficiency of the estimates of the principal components. Concerning these results, he writes (Summary, pp. 157-158): "In the Tintner theory of relations between p economic series the incidental estimates of the systematic parts between which r linear relations subsist are linear functions of the first $(p-r)$ principal components of the original series as observed. While the incidental estimates of the systematic parts are consistent, they are not—and they are not claimed to be by Tintner—valid estimates because in utilizing so many parameters in the process of deriving the maximum likelihood estimates they take on to themselves a proportion of the error variance. . . . To overcome the difficulty that in the Tintner theory the incidental estimates of the systematic parts are not satisfactory . . . a new theory is developed (in section 3) appropriate to the case in which the systematic parts can be estimated by an expansion in a finite number of orthogonal functions. It is shown (in section 5) that in this case the estimates of the coefficients are more efficient than are those found using the Tintner method. A $\chi^2$ test is developed for ascertaining the number of linear relations. The large sample variance of the ratio of a pair of coefficients as estimated by the method in the many variable case assumes . . . [a] remarkably simple form . . . ."

**References:** Hotelling (1933), Fisher & Yates (1938)[1943], Hsu (1941b), Tintner (1945), Anderson (1948).

**Citation:** Anderson (1948).

* Gumbel, E. J. (1948). The Statistical Forecast of Floods. Bulletin No. 15, Ohio Water Resources Board, Columbus.

**Summary:** The author writes (Abstract, p. 1): "In Bulletin No. 7 of the Ohio Water Resources Board, a probability paper based on the statistical theory of extreme values was used for the floods. If this theory holds, the observed floods for a given stream plotted on this paper in a prescribed way are scattered around a straight line. It is our purpose now to trace this theoretical line to be used for a statistical forecast, and to construct the control curves valid for all floods. First we show these methods for an arbitrary continuous unlimited distribution involving two constants. Then we apply it to the floods, and give a numerical example. Finally we show a procedure deciding whether historical, scattered records of peaks may be combined with continuous present-day records of floods." The author discusses both the construction of probability paper for any specified continuous distribution and the plotting of points on such a paper. Concerning the latter, he writes (p. 2): "The plotting positions $F_1$ and $F_n$ of the smallest and the largest observations $x_1$ and $x_n$ are chosen as being the probabilities $\Phi(\tilde{x}_1)$ and $\Phi(\tilde{x}_n)$ of the most probable smallest and largest values. The plotting positions of the $n-2$ intermediate observations are obtained by linear interpolation between $F_n$ and $F_1$. A simplification of this procedure has recently been proposed by B. F. Kimball [(1946b)]. To the m'th observation he attributes its mean frequency $F_m = m/(n+1)$. If n is sufficiently large, say at least 50, the difference between the two procedures is small. Consequently Kimball's procedure may be used as an alternative for large n."

**References:** Gumbel (1935a, 1942c, 1945b), Kimball (1946b).

**Citation:** Press (1949).

* Higuchi, Takeru; Leeper, H. M.; Davis, D. S. (1948). Determination of tensile strength of natural rubber and GR-S. Effect of specimen size. *Analytical Chemistry* **20** (11), 1029-1033.

**Summary:** The authors write (Summary, p. 1030): "Peirce's equation [based on the weakest-link theory], which relates observed tensile strength of textile fibers with their length, was found to be applicable to rubberlike material if specimen volume is used in place of specimen length. Experiments in which the tensile strength of GR-S and natural rubber compositions were determined for a range of specimen volumes yielded results in close accord with theory. A tenfold increase in the volume of the material results in a decrease of 308 and 339 and of 204 pounds per square inch in the observed tensile strength of GR-S and comparable natural rubber stocks, respectively. The numerical magnitudes of the slopes of the straight lines obtained when tensile strengths were plotted against the logarithms of the relative specimen volumes are shown to bear direct relationships to the homogeneity of the stocks under test. The use of a dumbbell sample with a constricted center should result in a relatively simple means of measuring quantitatively the degree of homogeneity of rubber compositions."

**Reference:** Peirce (1926).

* Hill, L. R.; Schmidt, P. L. (1948). Insulation breakdown as a function of area (abstract). *Electrical Engineering* **67,** 76.

**Summary:** The authors write (Abstract, p. 76): "It is found in practice that large areas of insulation have a lower average breakdown voltage and less variability than small areas of the same insulation. This paper applies probability theory [extreme value theory] to give a theoretical explanation of this phenomenon on the basis of insulation variability, excluding electrode edge and contact effects. General expressions are derived from which tables and curves are calculated for rapid and easy translation from a distribution of an elemental area to that of n areas of insulation in parallel."

**Citations:** Peterson (1949a,b).

* Hoeffding, Wassily (1948). A class of statistics with asymptotically normal distribution. *Annals of Mathematical Statistics* **19**, 293-325. (MR 10, 134).

**Summary:** The author writes (Summary, p. 293): "Let $X_1, \ldots, X_n$ be n independent random vectors, $X_\nu = (X_\nu^{(1)}, \ldots, X_\nu^{(r)})$, and $\Phi(x_1, \ldots, x_m)$ a function of m ($\leq$ n) vectors $x_\nu = (x_\nu^{(1)}, \ldots, x_\nu^{(r)})$. A statistic of the form $U = \Sigma'' \Phi(X_{\alpha_1}, \ldots, X_{\alpha_m}) / n(n-1) \ldots (n-m+1)$, where the sum $\Sigma''$ is extended over all permutations $(\alpha_1, \ldots, \alpha_m)$ of m different integers, $1 \leq \alpha_i \leq n$, is called a U-statistic. . . . It is shown that if $X_1, \ldots, X_n$ have the same distribution and $\Phi(x_1, \ldots, x_m)$ is independent of n, the d.f. of $\sqrt{n}(U - \theta)$ tends to a normal d.f. as $n \to \infty$ under the sole condition of the existence of $E\Phi^2(X_1, \ldots, X_m)$. Similar results hold for the joint distribution of several U-statistics (Theorems 7.1 and 7.2), for statistics U′ which, in a certain sense, are asymptotically equivalent to U (Theorems 7.3 and 7.4), [and] for certain functions of statistics U or U′ (Theorem 7.5). . . . Examples of statistics of the form U or U′ are the moments, Fisher's k-statistics, [and] Gini's mean difference . . . ." Concerning the mean difference and the coefficient of concentration, the author writes (pp. 313-314): "If $Y_1, \ldots, Y_n$ are n independent real-valued random variables, Gini's mean difference (without repetition) is defined by $d = \Sigma_{\alpha \neq \beta} |Y_\alpha - Y_\beta| / n(n-1)$. If the $Y_\alpha$'s have the same distribution F, the mean of d is $\delta = \int\int |y_1 - y_2| \, dF(y_1) \, dF(y_2)$, and the variance . . . is $\sigma^2(d) = 2\{2\zeta_1(\delta)(n-2) + \zeta_2(\delta)\}/n(n-1)$, where $\zeta_1(\delta) = \int \{\int |y_1 - y_2| \, dF(y_2)\}^2 dF(y_1) - \delta^2$, $\zeta_2(\delta) = \int\int (y_1 - y_2)^2 \, dF(y_1) \, dF(y_2) - \delta^2 = 2\sigma^2(Y) - \delta^2$. . . . U. S. Nair [(1936)] has evaluated $\sigma^2(d)$ for several particular distributions. By Theorem 7.1, $\sqrt{n}(d - \delta)$ is asymptotically normal if $\zeta_2(\delta)$ exists. If $Y_1, \ldots, Y_n$ do not assume negative values, the coefficient of concentration (cf. Gini [(1914a)]) is defined by $G = d/2\overline{Y}$, where $\overline{Y} = \Sigma Y_\alpha / n$. G is a function of two U-statistics. If the $Y_\alpha$'s are identically distributed, if $E\{Y^2\}$ exists, and if $\mu = E\{Y\} > 0$, then, by Theorem 7.5, $\sqrt{n}(G - \delta/2\mu)$ tends to be normally distributed with mean 0 and variance $\delta^2 \zeta_1(\mu)/4\mu^4 - \delta\zeta_1(\mu,\delta)/\mu^3 + \zeta_1(\delta)/\mu^2$, where $\zeta_1(\mu) = \int y^2 \, dF(y) - \mu^2 = \sigma^2(Y)$, $\zeta_1(\mu,\delta) = \int\int y_1 |y_1 - y_2| \, dF(y_1) \, dF(y_2) - \mu\delta$, and $\zeta_1(\delta)$ is given [above]."

**References:** Yule (1911) [Yule & Kendall (1937)], Gini (1914a), Nair (1936), Cramér (1946), von Mises (1947).

* Housner, G. W.; Brennan, J. F. (1948). The estimation of linear trends. *Annals of Mathematical Statistics* **19,** 380-388. (MR 10, 201).

**Summary:** The authors write (Summary p. 380): "This paper deals with the problem of bivariate regression where both variates are random variables having a finite number of means distributed along a straight line. A regression statistic is derived which is independent of change in scale so that a prior knowledge of the frequency distribution parameters is not required in order to obtain a unique estimate. The statistic is shown to be consistent. The efficiency of the estimate is discussed and its asymptotic distribution is derived for the case when the random variables are normally distributed. A numerical example is presented which compares the performance of the statistic in this paper with that of other commonly used statistics. In the example it is found that the method of estimation proposed in this paper is more efficient." For the case of ungrouped data (one sample point for each x), the proposed estimate of the regression coefficient (slope) b of the line $Y = a + bX$ reduces to the form $\hat{b} = \Sigma_{i=1}^n y_i(i - \bar{i}) / \Sigma_{i=1}^n x_i(i - \bar{i})$, where the x's are ordered by magnitude. In the special case when the x's are equally spaced, the proposed estimate is identical with the least squares estimate.

**Reference:** Wald (1940).

* Huzurbazar, V. S. (1948). The likelihood equation, consistency and the maxima of the likelihood function. *Annals of Eugenics* **14,** 185-200. (MR 10, 388).

**Summary:** The author writes (Summary, pp. 199-200): "It is shown under fairly general conditions that the probability that the likelihood function of a sample of n independent observations from a distribution has a maximum approaches certainty as $n \to \infty$, thus justifying in general terms the word 'maximum' appearing in 'Maximum Likelihood'. . . . Interest lies in a stationary maximum when the range is independent of the parameter and in a terminal maximum (i.e. a maximum occurring at an end-point of an interval) when the range depends upon the parameter (except in the case discussed in §6.2 of this paper [the case $I(\theta) \equiv 0$, where $I(\theta) = E\{(\partial/\partial\theta) \log f(x, \theta)\} = a'f(a, \theta) - b'f(b, \theta)$ and $f(x, \theta)$ is the p.d.f. of the distribution under consideration, with range (a,b), a and b being functions of $\theta$]) because it is with them that the optimum properties of consistency and efficiency are associated in the respective cases in general. It is proved that when the range is independent of the parameter, a consistent solution of the likelihood equation is unique. When the range depends upon the parameter, the likelihood equation has no consistent solution under certain conditions. But then the two extreme observations in a sample play the almost analogous role of the likelihood equation in providing a fairly good consistent estimator (except in the case of §6.2 where again the likelihood equation triumphs). It is pointed out that under general conditions the actual proof of 'maximum' follows that of 'consistency'. The statement 'Maximum likelihood estimators are consistent', familiar in the literature, would, with due regard to the logical order in which things can be actually proved under general conditions, run as 'The consistent solution of the likelihood equation when the range is independent of the parameter, as well as the consistent estimator provided by the two extreme observations in a sample when the range depends on the parameter, is a maximum of the likelihood function with a probability approaching certainty as $n \to \infty$.' In Section 5.2, the author quotes a simple theorem on order statistics first proved by Dodd (1923), and in Section 5.3, he obtains the limiting forms of the greatest and the smallest members in a sample and formulae for the order of their variances in large samples.

**References:** Fisher (1922), Dodd (1923), Pitman (1936), Kendall (1943) [1945], Cramér (1946).

* Jones, Howard L. (1948). Exact lower moments of order statistics in small samples from a normal distribution. *Annals of Mathematical Statistics* **19,** 270-273. (MR 9, 601).

**Summary:** The author writes (Summary, p. 270): "Exact means in samples of size $\leq 3$, and exact second moments and product-moments in samples of size $\leq 4$, are given . . . in terms of $\pi$ for order statistics selected from the normal distribution N(0, 1). The derivation employs multiple integration and some general properties of the moments." The following general properties are used: $E(x_i) = -E(x_{n-i+1}), E(x_i^2) = E(x_{n-i+1}^2), E(x_i x_j) = E(x_{n-i+1} x_{n-j+1}), \Sigma_{i=1}^{n-1} \Sigma_{j=i+1}^{n} E(x_i x_j) = 0, \Sigma_{i=1}^{n} E(x_i) = 0, \Sigma_{i=1}^{n} E(x_i^2) = n, \Sigma_{j=1}^{n} E(x_i x_j) = 1 (i = 1, 2, \ldots, n)$.

**References:** Cramér (1946), Hastings, Mosteller, Tukey & Winsor (1947).

**Citations:** Godwin (1949a), Nair (1949a).

* McAdam, D. J., Jr.; Geil, G. W.; Woodard, D. H.; Jenkins, W. D. (1948). Influence of size and the stress system on the flow stress and fracture stress of metals. *Metals Technology* **15,** Technical Paper No. 2373. (AMR 2 #738).

**Summary:** The authors study the applicability of the statistical theory of fracture and the size effect (based on the assumption that fracture is determined by the largest defect) to ductile metals. After reviewing the work of Griffith (1920), Weibull (1939a), Kontorova & Frenkel (1941), Davidenkov, Shevandin & Wittmann (1947), and Fisher & Hollomon (1947), they present the results of their own experimental investigation on notched and unnotched specimens of oxygen-free copper and of carbon steel. They state the following conclusions (p. 11): "The statistical theory of fracture is not applicable to the fracture of metals after even slight plastic deformation. Since the use of notched specimens for the investigation of the influence of the stress system on fracture requires enough ductility to relieve most of the stress concentration, the statistical theory is not applicable to such fractures. The results of this investigation, in which notched and unnotched specimens varying greatly in size were tested to fracture,

show conclusively that the increase in the fracture stress with increase in the sharpness of the notch is due to the increase in the ratio of transverse to longitudinal tensile stress, not to a size effect."

**References:** Griffith (1920), Weibull (1939a), Kontorova & Frenkel (1941) [Frenkel & Kontorova (1943)], Davidenkov, Shevandin & Wittmann (1947), Fisher & Hollomon (1947).

* Murphy, R. B. (1948). Non-parametric tolerance limits. *Annals of Mathematical Statistics* **19,** 581-589. (MR 10, 414).

**Summary:** The author writes (Summary, p. 581): "In this note are presented graphs of minimum probable population coverage by sample blocks determined by the order statistics of a sample from a population with a continuous but unknown cumulative distribution function (c.d.f.). The graphs are constructed for the three tolerance levels .90, .95, and .99. The number, m, of blocks excluded from the tolerance region runs as follows: m = 1(1)6(2)10(5)30(10)60(20)100, and the sample size, n, runs from m to 500. Thus the curves show the solution, $\beta$, of the equation $1 - \alpha = I_\beta(n - m + 1, m)$ for $\alpha$ = .90, .95, .99 over the range of n and m given above, where $I_x(p,q)$ is Pearson's notation for the incomplete beta function. Examples are cited . . . for the one- and two-variate cases. Finally, the exact and approximate formulae used in computations for these graphs are given."

**References:** Thompson (1941), Wilks (1942), Wald (1943), Tukey (1947, 1948a).

* Nair, K. R. (1948a). The studentized form of the extreme mean square test in the analysis of variance. *Biometrika* **35,** 16-31. (MR 9, 601).

**Summary:** Part I (pp. 16-21) of this paper is devoted to an extension of Hartley's results on expansion of the studentized integral, and is not directly relevant to our subject, though reference is made (on p. 21) to the studentized range and the studentized extreme deviate. Part II (pp. 21-30) deals with application to tests regarding the largest and smallest of several variances. The author writes (pp. 21-22): "Let $s_1^2, \ldots, s_k^2$ be independent estimates of an unknown variance $\sigma^2$ of a normal population each calculated from sums of squares having n degrees of freedom and arranged in ascending order of magnitude. Let $s_0^2$ be another independent estimate based on $\nu$ degrees of freedom against which each of the first k estimates is to be tested for significant differences from $s_0^2$. In routine analysis of variance where such tests frequently occur, it is customary to test each of the k variances against $s_0^2$ and separately declare whether there is a significant difference or not. Forming now for each of the independent $s_i^2$ the ratio $F_i = s_i^2/s_0^2$, it is obvious that the largest ratio, $F_k = s_k^2/s_0^2$, is more likely to be declared significant than any other variance ratio in the set. Indeed, for k = 20 it would be expected to be significant at the 5% level of F for m and $\nu$ degrees of freedom. This source of bias can be eliminated if the probability integral of the largest variance ratio is numerically evaluated. Although the theory discussed in this paper applies to any value of m, owing to practical difficulties, tables [to 2 decimal places] of the upper 5 and 1% points of the largest variance ratio have been prepared only when m = 1 [for k = 1(1)10, $\nu$ = 10, 12, 15, 20, 30, 60, $\infty$], which is the most important case in practice. Thus, as Wishart (1938) has pointed out, it has useful application in the analysis of variance for $2 \times 2 \times 2 \times \ldots$ factorial design. It could in fact be applied to the analysis of any designed experiment where the total variance due to treatment effects is split up into components each having a single degree of freedom. Another possible application is in the fitting of curves with orthogonal terms, e.g. orthogonal polynomials and harmonic analysis. It may well happen that while, taken as a whole there is no significant curvilinearity, the coefficient of a single high order term turns out to be very large compared to the rest. The new tables will be useful in testing whether such isolated coefficients are significantly large. The theory developed for the largest variance ratio could easily be extended to any other *ranked* ratio. Of these the smallest, namely, $F_1 = s_1^2/s_0^2$ has been considered. Occasions to test the significance of the smallest of k variance ratios are much less frequent than that of the largest. In the former case we are concerned with the *lower* percentage points of the

probability integral, in order that a test could be made whether an observed smallest ratio is significantly small. In field experiments, the smallest of k variances tested may become significantly small compared to the 'error' variance ($s_0^2$) if the m+1 groups of plot values from which the m degrees of freedom of that variance were obtained show a high negative intra-class correlation. The extreme value that this correlation can have, if there are $\ell$ plots in each group, is $-1/(\ell-1)$, so that it can be $-1$ only if $\ell = 2$. Wishart (1938) again gives an example where a test for significance of smallest variance ratio when $m = 1$ is appropriate. . . ." The author shows that the probability that the smallest of k variance ratios, $F_1$, is $\leq q^2$ is given by $1 - {}_{\nu}p_k(q) = \{1 - a_0 - (a_1/\nu)(1-1/8\nu)\}$, where $a_0$ and $a_1$ are quantities which he tabulates (to 6 and 5 decimal places, respectively) for k = 2(1)10 and q = 0.01(0.01)0.10.

**References:** Neyman & Pearson (1928), Sukhatme (1937), Wishart (1938), Cochran (1941), Finney (1941), Thompson (1941), Pearson & Hartley (1943), Hartley (1944), Nair (1948b).

* Nair, K. R. (1948b). The distribution of the extreme deviate from the sample mean and its studentized form. *Biometrika* **35,** 118-144. (MR 9, 602).

**Summary:** The author reviews various tests, based on order statistics, for the presence of outliers in samples from a normal population with mean $\mu$ and standard deviation $\sigma$, where either $\mu$ or $\sigma$ (or both) may be unknown. He writes (p. 119): "In this paper attention is mainly concentrated on the McKay [(1935)] statistic, $u = (x_n - \bar{x})/\sigma$ (or $(\bar{x} - x_1)/\sigma$) and its studentized form [$Q =$] $(x_n - \bar{x})/s_\nu$ (or $(\bar{x} - \bar{x}_1)/s_\nu$) [where $s_\nu$ is an estimate of $\sigma$, with $\nu$ degrees of freedom, which is independent of the sample $x_1, \ldots, x_n$]. It is shown that the distribution of u can be obtained by a more direct method than was employed by McKay and that it can be reduced to certain integrals which have recently been termed G-functions by Godwin . . . in his representation of the distribution of the mean deviation. Tables of the probability integral of u and of its studentized form have been prepared. Apart from serving as a criterion for rejection of an outlying observation in a 'normal' sample when $\mu$ and $\sigma$ are unknown, the studentized form of u has useful applications in judging the significance of a single outstanding treatment (best or worst) in a group of treatments tried out in a designed experiment. Some illustrations are given in the paper." Tables include a 6-decimal-place table of the extreme deviate u for n = 3(1)9 and u = 0.01(0.01)4.70; a 2-decimal place table of the lower and upper 0.1%, 0.5%, 1.0%, 2.5%, 5.0%, and 10.0% points of the extreme deviate; a table (varying numbers of decimal places) of three auxiliary quantities required for calculating the probability integral of the studentized extreme deviate Q for n = 3(1)9 and Q = 0.00(0.20)4.00; a 2-decimal-place table of the lower 5% and 1% points of Q for n = 3(1)9 and $\nu$ = 10,15,30,$\infty$; and a similar table of the upper 5% and 1% points of Q for $\nu$ = 10(1)20,24,30,40,60,120,$\infty$.

**References:** Irwin (1925b), Tippett (1925), Student (1927), McKay & Pearson (1933), McKay (1935), Thompson (1935), Pearson & Chandra Sekar (1936), Pearson & Hartley (1943), Hartley (1944), Jones (1946), Nair (1947).

**Citations:** Nair (1948a), Tukey (1948c), Godwin (1949b), Nair (1949a), Tukey (1949a).

* Nanda, D. N. (1948a). Distribution of a root of a determinantal equation. *Annals of Mathematical Statistics* **19,** 47-57. (MR 9, 453).

**Summary:** The author writes (Summary, p. 47): "S. N. Roy [(1945)] obtained . . . the distribution of the maximum, minimum and any intermediate one of the roots of certain determinantal equations based on covariance matrices of two samples on the null hypothesis of equal covariance matrices in the two populations. The present paper gives a different method [based on a result of Hsu (1939)] of working out the distribution of any of these roots under the same hypothesis. The distribution of the largest, smallest and any intermediate root when the roots are specified by their position in a monotonic arrangement has been derived for p = 2, 3, 4, and 5 by the new method. The method is applicable for obtaining the distribution of roots of an equation of any order, when the distributions of the roots of lower order equations have been worked out."

**References:** Hsu (1939), Roy (1945).

**Citation:** Nanda (1948b).

* Nanda, D. N. (1948b). Limiting distribution of a root of a determinantal equation. *Annals of Mathematical Statistics* **19,** 340-350. (MR 10, 135).

**Summary:** The author writes (Summary, p. 340): "The exact distribution of a root of a determinantal equation when the roots are arranged in a monotonic order was obtained by S. N. Roy [(1945)] . . . . A different method for deriving the distribution of any one of these roots has been described by the author [Nanda (1948a)]. In the present paper the limiting forms of these distributions are obtained. This paper gives a method by which the limiting distributions can be obtained without undergoing an inordinate amount of mathematical labor."

**References:** Hsu (1939), Roy (1945), Nanda (1948a).

* Noether, Gottfried E. (1948). On confidence limits for quantiles. *Annals of Mathematical Statistics* **19,** 416-419. (MR 10, 135).

**Summary:** The author writes (pp. 416-417): "In finding confidence limits for quantiles it is usual to determine two order statistics $Z_i$ and $Z_j$ which with a given probability contain the unknown quantile between them. The values of i and j corresponding to a given confidence coefficient can be determined with the help of the distribution laws of order statistics as is shown, e.g., in Wilks [(1948)]. The purpose of this note is to determine i and j with the help of a confidence band for the unknown cumulative distribution function." For any arbitrary value x, the author considers the sample c.d.f. $F_n(x)$ as the sample estimate of the unknown parameter $p = F(x)$ of a binomial distribution. He proceeds to use the tables of percentage points of the incomplete beta function [C. M. Thompson (1941)], as suggested by Scheffé, to find the confidence limits. He shows that the resulting confidence limits for the median are the same as those found by W. R. Thompson (1936), who used a different method.

**References:** W. R. Thompson (1936), C. M. Thompson (1941), Scheffé & Tukey (1945), Wilks (1948).

* Pillai, K. C. S. (1948). A note on ordered samples. *Sankhyā* **8,** 375-380. (MR 10, 723).

**Summary:** The author studies the effect of ordering a sample with special reference to problems relating to Fisher's theory of information and the distribution of the semirange. The amount of information concerning an unknown parameter contained in a random sample of size n is n times that in a single observation. When the sample is ordered, however, the information contained in an observation depends upon its rank in the ordered sample. The author obtains a general expression for the information in the i-th order statistic of a sample of size n concerning an unknown parameter $\theta$ of any population whose p.d.f. is continuous and twice differentiable. He also obtains expressions for the information concerning the mean a and the standard deviation $\sigma$ contained in the i-th order statistic of a sample of size n from a normal population. He tabulates the results (to 2 decimal places) for $n = 2(1)12$ for a and $n = 2(1)10$ for $\sigma$, with $i = 1(1)[n+1)/2]$ in each case. Because of symmetry, the $(n-i+1)$-th observation contains the same amount of information as the i-th observation concerning either a or $\sigma$. The central observations give the most information concerning a and the extreme observations give the most information concerning $\sigma$. The author gives an unbiased estimator of $\sigma$ which is a function of n multiplied by the semirange $(x_n - x_1)/2$. He also gives expressions for the p.d.f. and the c.d.f. of the semirange in terms of the normal probability integral.

**Reference:** Hojo (1931).

* Rao, C. Radhakrishna (1948). Tests of significance in multivariate analysis. *Biometrika* **35,** 58-79. (MR 9, 602).

**Summary:** The author shows how to solve a number of problems in multivariate analysis by the use of discriminant functions. The only mention of methods based on order statistics is the following (p. 59): "The introduction of the discriminant function led to a new method of deriving test criteria suitable for multiple variates. The problem is reduced to the case of a single variate by using a linear compound of the several variables, where the compounding coefficients are chosen so as to maximize the value of a statistic suitable for a single variate. The application of this method to test the difference in mean values for several groups gave rise to the theory of [ordered] canonical roots of determinantal equations (Roy, 1939; Fisher, 1939; Hsu, 1939). The distribution of the individual roots and the exact nature of the tests require further study. Wilks' statistic, which is a symmetric function of the canonical roots, may be considered as providing an overall test of the hypotheses concerned."

**References:** Neyman & Pearson (1928), Hotelling (1936b), Bartlett (1938), Fisher (1938, 1939), Hsu (1939), Roy (1939), Bartlett (1947b).

* Schützenberger, Marco P. (1948). An abac for the sample range. *Psychometrika* **13** (2), 95-97.

**Summary:** The author writes (p. 95): "An abac is computed which gives the probability that at least q% of the whole population will be included in the range of a random sample of given size. Applications are suggested for testing homogeneity of a sampling from a given population. Let a sample of N values be randomly drawn from an infinite continuous distribution. The present chart gives the probability p that at least q per cent of the whole population lie between the extreme values of the sample. . . . It is easy to prove that the probability p, the percentage q, and the sample size N are related by $p = 1 - [m-1)/m]^{N-1}$ $[(N+m-1)/m]$, (1) where $m = 1/(1-q)$. For large enough N and m, (1) may be closely approximated by $p = 1-(r+2)e^{-r-1}$, where $r = N/m - 1$. The important fact is that (1) holds for every *continuous infinite* distribution, even leptokurtic or platykurtic, skew or multimodal." Values of q, from 50% to 99.8%, are plotted on the ordinate, and values of N, from 5 to 1000, are plotted on the abscissa, with both scales logarithmic. Equi-probability curves are drawn for p = .999, .99, .9, .75, .5, .25, .1, .01, and .001. Applications in the field of psychology are discussed.

**Reference:** Wilks (1943).

* Smirnov, N. (1948). Table for estimating the goodness of fit of empirical distributions. *Annals of Mathematical Statistics* **19,** 279-281. (MR 9, 599).

**Summary:** An editorial note (p. 279) states: "The table presented on pp. 280-281 was originally published in [Smirnov (1939b)]. It gives values of $L(z) = 1 - 2\sum_{\nu=1}^{\infty}(-1)^{\nu-1}e^{\nu^2z^2} = (2\pi)^{1/2}z^{-1}\sum_{\nu=1}^{\infty}e^{-(2\nu-1)^2\pi^2/8z^2}$, which is also derived in [Feller (1948)]. Let $(X_1, \ldots, X_n)$ be a sample of independent variables with the same continuous cumulative distribution $F(x)$, and let $N(z)$ be the number of $X_i$ which are $\leq z$. By empirical distribution is meant the step-function $F_n^*(z) = N(z)/n$. The maximum $D_n$ of the difference $|F_n^*(z) - F(z)|$ is a random variable and $L(z)$ is the limiting cumulative distribution function of $n^{1/2}D_n$. If $D_{m,n}$ is the maximum of the difference $|F_m^*(z) - F_n^{**}(z)|$ between the empirical distributions of two independent samples of sizes m and n, respectively, then $L(z)$ is also the limiting cumulative distribution function of $(mn/(m+n))^{1/2}D_{m,n}$." Values of $L(z)$ are tabulated to 6 DP for z = .28(.01)2.49, to 7 DP for z = 2.50(0.05)2.80, and to 8 DP for z = 2.85(0.05)3.00. They vary from .000001 for z = 0.28 to .99999997 for z = 3.00.

**References:** Smirnov (1939b), Feller (1948).

**Citations:** Doob (1949), Noether (1949).

* Stevens, W. L. (1948). Control by gauging. *Journal of the Royal Statistical Society B* **10,** 54-98; discussion, 98-108.

**Summary:** The author writes (Summary, p. 55): "Although it has long been realized that the 'quality control' of a metrical character can be achieved by testing the product against a pair of gauges (instead of making exact measurements), progress has been delayed by the widely held but erroneous belief that the efficiency of such a technique is so low as to make it almost always uneconomical. The purpose of this paper is to provide a theoretical basis for control by gauging, thus permitting an accurate evaluation of the system in comparison with the more orthodox methods. The interesting result emerges that it is possible to achieve with *ten* gauged articles the same sensitivity of control of the mean as is given by *eight* articles exactly measured. When the possibility of this high efficiency has been demonstrated, it is clear that the case for control by gauging must be reopened. . . ." The author points out that using a single gauge at the mean is equivalent to using the median to estimate the mean of a population (efficiency $2/\pi = 63.7\%$ for a normal distribution). For the usual two gauges, he shows (p. 72) that, for estimating the standard deviation, the efficiency of gauging is somewhat lower (around 60%, he says) than for estimating the mean (around 80%). However, he writes (p. 74): "It should be noted that the efficiency of the estimate of $\sigma$ derived from a gauged sample has been based on a comparison with the efficient estimate derived from exact measurements, i.e. $\sqrt{\{S(x-\bar{x})^2/(n-1)\}}$. It is, however, usual in quality control to employ, not this estimate, but the sample range. The range leads to an estimate of standard deviation which is nearly 100% efficient in small samples, but which begins to fall appreciably below full efficiency when n exceeds 10. Hence in comparison with the use of a range chart, control of the standard deviation by gauging is somewhat more efficient than . . . [in comparison with use of the sample standard deviation]." In his contribution to the discussion (pp. 101-102), E. S. Pearson writes ". . . equivalance in sensitivity was not for n = 10 using a + c [where a is the number of articles passing the smaller gauge and c the number failing to pass the larger gauge] and n = 6 using S. D. or range [as indicated by the author's 60% efficiency value], but more nearly for n = 10 using a + c and n = 4 using S. D. or range."

**References:** Fisher (1922), Dudding & Jennett (1942), Pearson & Hartley (1942), Deming (1943).

* Thomas, Harold A., Jr. (1948). Frequency of minor floods. *Journal of the Boston Society of Civil Engineers* **35,** 425-442.

**Summary:** The author discusses the limitations of the probability method of flood prediction, the return period of floods, the distribution of the largest flood, plotting positions of floods, and a non-parametric method of flood estimation. If the probability is p that a design flood will not be exceeded in a given year, he reasons that the probability that it will not be exceeded for $t-1$ consecutive years and will then be exceeded in the $t^{th}$ year is $p^{t-1}(1-p)$, so that the average return period will be $\bar{t} = 1(1-p) + 2p(1-p) + 3p^2(1-p) + \ldots + tp^{t-1}(1-p) + \ldots = 1/(1-p)$. Concerning the distribution of the largest flood, he writes (pp. 434-435): "In a subgroup of n floods the probability $d\theta$ that $(n-m)$ floods will be smaller than a certain value Q, $(m-1)$ floods will be larger than Q, and one flood will fall in the range $Q \pm dQ/2$ is $d\theta = \binom{n}{m} m p^{n-m}(1-p)^{m-1}\,dp$, (4), where p is the probability that a flood of size Q will not be exceeded during any year. The probability $\theta$ that the actual p-value of the $m^{th}$ of n floods will be less than a certain fixed value, say $p_0$, may be obtained by integrating Equation (4). $\theta = \binom{n}{m} m \int_0^{p_0} p^{n-m}(1-p)^{m-1}dp$. (5). For the case of the largest floods ($m = 1$), $\theta = n \int_0^{p_0} p^{n-1}dp = p_0^n$. (6). . . ." Concerning the plotting positions of floods, he writes (p. 436): "According to Equation (4), the probability that the actual p-value of the largest flood ($m = 1$) lies within a range of $p \pm dp/2$ is as follows: $d\theta = np^{n-1}dp$. Taking moments of p about the origin zero, the *mean* p-value of the largest of n floods is given by the relation $\bar{p} = \int_0^1 p\,d\theta = n\int_0^1 pp^{n-1}dp = n/(n+1)$. (7). . . . The theory underlying Equation (7) may be generalized to apply to the determination of

plotting positions of floods other than the largest. For the $m^{th}$ of n floods, $\bar{p} = (n-m+1)/(n+1)$. (8)." In his discussion of the non-parametric method of flood estimation, he shows (p. 437) that the probability that in t future years the $m^{th}$ of n past floods will be exceeded *exactly* k times is $\Phi_k = m\binom{t}{k}\binom{n}{m}/(m+n)\binom{t+n}{m+k}$. The author summarizes his results as follows (p. 442): "The probability method of flood prediction, properly applied, will yield useful estimates of the frequency of minor floods. Of the two forms of applications of the method, parametric and non-parametric, the latter is to be preferred, since it gives exact, not approximate flood probabilities."

* Tukey, John W. (1948a). Nonparametric estimation, III. Statistically equivalent blocks and multivariate tolerance regions—the discontinuous case. *Annals of Mathematical Statistics* **19,** 30-39. (MR 9, 453).

**Summary:** The author writes (Summary, p. 30): "In Paper II of this series [Tukey (1947)] it was shown that if n functions and a sample of n were used to divide the population space into n + 1 blocks in a particular way, and *if the joint cumulative of the functions were continuous*, then the n + 1 fractions of the population, corresponding to the n + 1 blocks, were distributed symmetrically and simply. In Paper I of this series [Scheffé & Tukey (1945)] it was shown that the one-dimensional theory of tolerance regions could be extended to the discontinuous case, if equalities were replaced by inequalities. In this paper the results of Paper II will be extended to the discontinuous case with the same weakening of the conclusion. The devices involved are more complex, but the nature of the results is the same . . . . As a tool, it is shown that any n-variate distribution can be represented in terms of an n-variate distribution with a continuous joint cumulative (in fact, with uniform univariate marginals), where each variate of the given distribution is a different monotone function of the corresponding variate from the continuous distribution." He continues (Introduction, pp. 30-31): "The importance of extending the simple results of the continuous case to the more complex results of the discontinuous case may not be clear at first thought. Yet *all the data with which the statistician actually works comes from discontinuous distributions.* Often these distributions are very fine-grained . . ., but *all* their probability is concentrated at discrete points. . . . In order to deal with the discontinuous case, we must face two problems: . . . (1) What to do about 'ties'? (2) Finite probabilities associated with cuts." He solves the first problem by using the idea of lexicographical ordering. He points out that if we want a certain set of blocks to cover *at least* a certain amount with a certain probability we must add the adjoining cuts and if want a certain set of blocks to cover *at most* a certain amount with a certain probability we may add only those cuts which do not adjoin blocks not in our set. He then solves the second problem by introducing the sets explicitly.

**References:** Scheffé & Tukey (1945), Tukey (1947).

**Citation:** Murphy (1948).

* Tukey, John W. (1948b). Some elementary problems of importance to small sample practice. *Human Biology* **20,** 205-214.

**Summary:** The author discusses several problems. His proposed solution of one of them involves the use of order statistics. He writes (pp. 206-207): "It would be highly desirable to have a modified version of the t-test with a greater resistance to skewness when used as a one-sided test. . . . Casting about for a possible way in which to bring about this modification, our attention is struck by the work of Egon Pearson [& N. K. Adyanthāya (1929), pp. 280-286] and Walsh (unpublished) on the use of the expression (midrange-assumed mean)/range in very small samples. The efficiency is very high when the underlying distribution is normal, and the distribution is remarkably little affected by deviations from normality. It

thus seems possible that an improved modification of the t-test might (1) be based on: the sample mean, the sample variance, the sample size, the upper and lower sample semiranges (upper semirange = greatest observation minus sample mean, lower semirange = sample mean minus smallest observation); (2) approximately reduce to a constant times the 'midrange/range' expression in samples of 3 or 4 (samples of 2 are trivial); and (3) approximately reduce to t for large samples without an unusually discrepant observation. An example of an expression with roughly plausible properties is

$$t + K\frac{\text{upper semirange} - \text{lower semirange}}{\text{upper semirange} + \text{lower semirange}}t^2$$

where

$$t = \frac{\text{sample mean} - \text{assumed mean}}{\sqrt{\text{sample variance}/\text{sample size}}}$$

and K may depend on the sample size. Whether this is the right sort of expression is, of course, doubtful. . . ."

**Reference:** Pearson [& Adyanthāya] (1929), Walsh [(1949b)].

* Tukey, John W. (1948c). Comparing Individual Means in the Analysis of Variance. Memorandum Report No. 6, Statistical Research Group, Princeton University, Princeton, NJ.

**Summary:** The author writes (Summary, p. 1): "The practitioner of the analysis of variance often wants to draw as many conclusions as are reasonable about the relation of the true means for individual 'treatments', and a statement by the F-test (or the z-test) that they are not all alike leaves him thoroughly unsatisfied. The problem of breaking up the treatment means into distinguishable groups has not been discussed at much length, the solutions given in the textbooks differ and, what is more important, seem solely based on intuition. After discussing the problem on a basis of combining intuition with some hard, cold facts about the distributions of certain test quantities (or 'statistics') a simple and definite procedure is proposed for dividing treatments into distinguishable groups, and for determining that the treatments within some of these groups are different, although there is not enough evidence to say 'which is which'. The procedure is illustrated on examples." More specifically, he writes (pp. 4-6): "When the real differences between variety means are large, how do we realize this fact? Three vague criteria come naturally to mind: (1) There is an unduly wide gap between adjacent variety means when arranged in order of size, (2) One variety mean straggles too much from the grand mean, (3) The variety means taken together are too variable. It is these three criteria we are going to apply in order to break up an observed set of means. We need, then, quantitative tests for detecting (1) excessive gaps, (2) stragglers, (3) excessive variability. These must be used when the variance of an individual observed mean is not known exactly, but rather when it is estimated from some other line of an analysis of variance table. The tests which we use must therefore be Studentized tests. Exact tests for (2) [based on the Studentized extreme deviate—see Nair (1948b)] and (3) [based on the familiar F-test (or z-test) of the analysis of variance] are available, but for the present we shall confine ourselves to an approximate and conservative test for (1) [given in an appendix on the largest gap]. . . . We propose to use these tests successively, and in the following order and manner. First, apply the gap test to break up the means into one or more broad groups. Second, apply the straggler test *within* these groups to further break off stragglers within groups. Third, apply the F-test to these new subgroups to detect excess variability. . . ."

**References:** Fisher (1935)[1947], Snedecor (1937)[1946], Fisher & Yates (1938)[1948], Newman (1939), Duncan (1947), Nair (1948b).

* Wilks, S. S. (1948). Order statistics. *Bulletin of the American Mathematical Society* **54,** 6-50. (MR 9, 601).

**Summary:** The author writes (Introduction, pp. 7-8): "There are many problems of statistical inference in which one is unable to assume the functional form of the population distribution. Many of these problems are such that the strongest assumption which can be reasonably made is continuity of the cumulative distribution function of the population. An increasing amount of attention is being devoted to statistical tests which hold for all populations having continuous cumulative distribution functions. Problems of this type in which the distribution function is arbitrary within a broad class are referred to as *non-parametric* problems of statistical inference. An excellent expository account of the theory of non-parametric statistical inference has been given by Scheffé [(1943)]. In non-parametric problems it is being found that *order statistics,* that is, the ordered set of values in a random sample from least to greatest, are playing a fundamental rôle. . . . There are both theoretical and practical reasons for this increased attention to nonparametric problems and order statistics. From a theoretical point of view it is obviously desirable to develop methods of statistical inference which are valid with respect to broad classes of population distribution functions. It will be seen that a considerable amount of new statistical inference theory can be established from order statistics assuming nothing stronger than continuity of the cumulative distribution function of the population. Further important large sample results can be obtained by assuming continuity of the derivative of the cumulative distribution function. From a practical point of view it is desirable to make the statistical procedures themselves as simple and as broadly applicable as possible. This is indeed the case with statistical inference theory based on order statistics. Order statistics also permit very simple 'inefficient' solutions of some of the more important *parametric* problems of statistical estimation and testing of hypotheses. The solutions are inefficient in the sense that they do not utilize all of the information contained in the sample as it would be utilized by 'best possible' (and computationally more complicated) methods. But this inefficiency can be offset in many practical situations where the size of the sample can be increased by a trivial amount of effort and expense. It is the purpose of this paper to present some of the more important results in the sampling theory of order statistics and of functions of order statistics and their applications to statistical inference together with some reference to important unsolved problems at appropriate places in the paper." After making some remarks on the history of order statistics, starting with the work of Karl Pearson (1902) on the Galton difference problem, the author proceeds to the technical discussion, which covers the following topics and more [see Note below]: sampling distributions for coverages in the case of one dimension; examples of direct applications of coverage distributions for the case of one dimension (confidence limits of median, quartiles and other quantiles; population tolerance limits); distribution of single order statistics (exact and limiting distributions of $x_{(r)}$; limiting distribution of largest or smallest value in a sample); joint distributions of several one-dimensional order statistics and applications (distribution of sample range; limiting distribution of two or more order statistics in large samples; estimation of population parameters by order statistics; application of order statistics to breaking strength of thread bundles); confidence bands for the cdf $F(x)$; sampling distributions of coverages for the case of two or more dimensions (for regions determined by one-dimensional order statistics; for rectangular regions in the case of independent random variables; Wald's results for rectangular regions in case the random variables are not independent; Tukey's generalization of Wald's results; extensions of distribution theory of coverages and order statistics to the discrete case); application of multi-dimensional coverages and order statistics to estimation problems; one-dimensional parametric tests involving order statistics.

**Note:** The author's definition of order statistics is somewhat broader than the compiler's. The former includes, but the latter excludes, nonparametric methods based on ranks, runs, and randomization.

**References:** K. Pearson (1902, 1920), von Bortkiewicz (1922a,b), Dodd (1923), Irwin (1925a), Tippett (1925), Fréchet (1927), Fisher & Tippett (1928), Craig (1932a), Kolmogoroff (1933), McKay & Pearson (1933), Davies & Pearson (1934), Gumbel (1935a), McKay (1935), Smirnoff (1935), von Mises (1936), Thompson (1936), Gumbel (1937b), Savur (1937b), Smirnoff (1937b), Thompson (1938), Smirnov

(1939a,b), Wald & Wolfowitz (1939), Nair (1940a,b), Gumbel (1941a), Kolmogoroff (1941), Mood (1941), Wald & Wolfowitz (1941), Wilks (1941), Hartley (1942), Pearson & Hartley (1942), Wald (1942), Wilks (1942), Mathisen (1943), Pearson & Hartley (1943), Scheffé (1943), Wald (1943), Wilks (1943), Gumbel (1944), Robbins (1944a), Scheffé & Tukey (1944), Daniels (1945), Scheffé & Tukey (1945), Carlton (1946), Cramér (1946), Daly (1946), Gumbel (1946a), Mosteller (1946), Walsh (1946a), Elfving (1947), Gumbel (1947), Hastings, Mosteller, Tukey & Winsor (1947), Tukey (1947), Walsh (1949b).

**Citation:** Noether (1948).

* Bartlett, M. S. (1949). Fitting a straight line when both variables are subject to error. *Biometrics* **5,** 207-212.

**Summary:** The author presents and illustrates a modification of the method of Wald (1940) for fitting a straight line when both variables are subject to error. This modification involves the use of three groups (containing, as nearly as possible, equal numbers of points) instead of two, the result being the line through the centroid of the middle group with slope equal to that of the line joining the centroids of the two extreme groups, not the latter line itself, as proposed by Nair & Shrivastava (1942). The author shows that his method has, in general, greater efficiency than Wald's. He states that it is theoretically preferable to the method of Nair & Shrivastava.

**References:** Wald (1940), Nair & Shrivastava (1942), Nair & Banerjee (1943).

* Benson, F. (1949). A note on the estimation of mean and standard deviation from quantiles. *Journal of the Royal Statistical Society B* **11,** 91-100.

**Summary:** The author writes (Introduction, p. 91): "Kendall (1943, p. 36) defines quantiles as the n-1 variate values which divide the total frequency into n equal parts. We propose, however, to follow a more general definition given by Cramér (1946, p. 181). For a distribution with the distribution frunction $F(x)$, we define a quantity $\zeta_p$ by the equation $F(\zeta_p) = p$, where p is any number such that $0 < p < 1$. Then $\zeta_p$ is the quantile of order p of the distribution. The quantile of order p is, then, the variate value which exceeds in value a proportion p of all the variates in the population. . . . Consider now a large random sample of n from a normal distribution. Let us denote by $z_p$ and $z_{1-p}$ the quantiles of the sample of order p and 1 – p, where, following Cramér (1946, p. 367), we define the sample quantiles of order p in the following way: Suppose that np is not an integer and that the sample values are arranged in order of magnitude $x_1 \leq x_2 \leq \ldots \leq x_n$. Then there is a unique quantile $z_p$ equal to the sample value $x_{\mu+1}$, where $\mu = [np]$ denotes the greatest integer $\leq np$. If np is an integer, then $z_p$ can be any value in the interval $x_{np}$, $x_{np+1}$. . . ." He summarizes his results as follows (p. 99): "It is sometimes convenient to use quantiles for estimating the mean and standard deviation of a distribution which may not be normal. It is shown that, if the distribution is not markedly non-normal, bias in the estimates can be minimized if the estimate of the mean is half the sum of the quantiles of order 1/6 and 5/6 and the estimate of the standard deviation is the difference between the quantiles of order 0.93 and 0.07 divided by 2.95. Moreover, if the distribution is normal, the estimate of the mean will have an efficiency of 75 per cent, which is not much less than the maximum efficiency of 81 per cent for estimation from a single pair of quantiles. The estimate of the standard deviation will be the most efficient possible for a single pair of quantiles.

**References:** Yule (1911) [Yule & Kendall (1946)], Pearson (1920), Kendall (1943), Cramér (1946).

* Birnbaum, Z. W.; Zuckerman, H. S. (1949). A graphical determination of sample size for Wilks' tolerance limits. *Annals of Mathematical Statistics* **20**, 313-316. (MR 10, 724—listing only).

**Summary:** The authors write (Summary, pp. 313-314): "To determine the smallest sample size for which the minimum and the maximum of a sample are the 100 $\beta\%$ distribution-free tolerance limits at the probability level $\varepsilon$, one has to solve the equation (1) $N\beta^{N-1} - (N-1)\beta^N = 1-\epsilon$ given by S. S. Wilks [(1943)]. A direct numerical solution of (1) by trial requires rather laborious tabulations. An approximate formula for the solution has been indicated by H. Scheffé and J. W. Tukey [(1944)], however an analytic proof for this approximation does not seem to be available. The present note describes a graph which makes it possible to solve (1) with sufficient accuracy for all practically useful values of $\beta$ and $\epsilon$."

**References:** Wilks (1943), Scheffé & Tukey (1944).

* Chung, Kai-Lai (1949). An estimate concerning the Kolmogoroff limit distribution. *Transactions of the American Mathematical Society* **67**, 36-50. (MR 11, 606).

**Summary:** The author writes (pp. 36-37): "We consider a sequence of independent random variables having the common distribution function $F(x)$ which is assumed to be continuous. Let $nF_n(x)$ denote the number of random variables among the first n of the sequence whose values do not exceed x. Write (1.1) $d_n = \sup_{-\infty<x<\infty}|n(F_n(x)-F(x))|$. Kolmogoroff [(1933)] proved that the probability (1.2) $P(d_n \leq \lambda n^{1/2})$, where $\lambda$ is a positive constant, tends as $n\to\infty$ uniformly in $\lambda$ to the limiting distribution (1.3) $\Phi(\lambda) = \sum_{-\infty}^{\infty}(-1)^j e^{-2j^2\lambda^2}$. Smirnoff [(1939a)] extended this result and recently Feller [(1948)] has given new proofs of these theorems. In this paper we shall obtain an estimate of the difference between (1.2) and (1.3) as a function of n, valid not only for $\lambda$ equal to a constant but also for $\lambda$ equal to a function $\lambda(n)$ of n which does not grow too fast. Since this estimate of the 'remainder' will be of the order of magnitude of a negative power of n, it is futile to consider $\lambda(n)$ which is beyond the order of magnitude of lg n. In fact, a glance at (1.3) will show that already for $\lambda(n) = \lg n$ we have $\Phi(\lambda)$ differ from 1 by a term of an order of magnitude smaller than that of any negative power of n, thus smaller than our estimate of the remainder. For a similar reason it is also futile to consider $\lambda(n)$ whose order is less than $(\lg n)^{-1}$. Keeping these facts in mind we state our result as follows: Theorem 1. If $A_0 > 0$ is an arbitrary constant and (1.4) $(A_0 \lg n)^{-1} \leq \lambda(n) \leq A_0 \lg n$, we have (1.5) $|P(d_n \leq \lambda(n)n^{1/2}) - \Phi(\lambda(n))| \leq An^{-1/10}\{1+(\lg n)^{1/2}(\lambda^{-1}(n)+\lambda^{-2}(n))\}$ where A is a constant depending only on $A_0$. . . ." The author also proves Theorem 2*: We have $P(\limsup_{n\to\infty} d_n/(2^{-1}n \lg_2 n)^{1/2} = 1) = 1$.

**References:** Kolmogoroff (1933), Smirnov (1939a), Feller (1948), Doob [(1949], Kac [(1949)].

* Cole, Randal H. (1949). An R-ply Range Estimation of Mean and Standard Deviation. Memorandum Report No. 20, Statistical Research Group, Princeton University, Princeton, NJ.

**Summary:** The author writes (Summary, p.1): "An r-ply range is defined to be the difference between the average of the r largest and the average of the r smallest sample values from a sample of size n. The estimate of the population standard deviation is obtained by using a table of expected values of the semi r-ply range. The midpoint of the r-ply range is used as the estimate of the mean. Tables of the standard deviations of these estimates [for a normal population] are given. By consulting the tables it is possible to determine the most economical pair of values of n and r to use in a specific situation."

**Note:** The r-ply range, as defined by the author, is the average of the quasi-ranges of orders $0,1,\ldots,r-1$ and the r-ply midrange is the average of the quasi-midranges of orders $0,1,\ldots,r-1$, where the quasi-range of order 0 is the range and the quasi-midrange of order 0 is the midrange.

**References:** Jones (1946), Mosteller (1946).

* Cox, D. R. (1949). The use of the range in sequential analysis. *Journal of the Royal Statistical Society B* **11**, 101-114. (MR 11, 262).

**Summary:** The author writes (Summary, pp. 113-114): "Previous authors have given sequential tests [one-sample and two-sample] for hypotheses about the variances of normal populations. The present paper gives arithmetically more convenient tests involving the range. The sampling distribution of range is approximated by a $\Gamma$ distribution. This leads to convenient sequential and fixed sample size tests whose properties are discussed. The adequacy of the $\Gamma$ approximation in sequential analysis is considered."

**References:** E. S. Pearson (1926), Student (1927), K. Pearson (1931), Pearson & Hartley (1942).

* Doob, J. L. (1949). Heuristic approach to the Kolmogorov-Smirnov theorems. *Annals of Mathematical Statistics* **20,** 393-403. (MR 11, 43).

**Summary:** The author writes (Introduction and summary, p. 393): "Asymptotic theorems on the difference between the (empirical) distribution function calculated from a sample and the true distribution function governing the sampling process are well known. Simple proofs of an elementary nature have been obtained for the basic theorems of Kolmogorov [(1933)] and Smirnov [(1939a,b)] by Feller [(1948)], but even these proofs conceal to some extent, in their emphasis on elementary methodology, the naturalness of the results (qualitatively at least), and their mutual relations. Feller suggested that the author publish his own approach (which had also been used by Kac [(1949)]), which does not have these disadvantages, although rather deep analysis would be necessary for its rigorous justification. The approach is therefore presented (at one critical point) as heuristic reasoning which leads to results in investigations of this kind, even though the easiest proofs may use entirely different methods. No calculations are required to obtain the qualitative results, that is the existence of limiting distributions for large samples of various measures of the discrepancy between empirical and true distribution functions. The numerical evaluation of these limiting distributions requires certain results concerning the Brownian movement stochastic process and its relation to other Gaussian processes which will be derived in the Appendix."

**References:** Kolmogoroff (1933), Smirnov (1939a,b), Feller (1948), Smirnov (1948), Kac [(1949)].

**Citations:** Chung (1949), Kac (1949).

* Epstein, Benjamin (1949a). A modified extreme value problem. *Annals of Mathematical Statistics* **20,** 99-103. (MR 10, 550).

**Summary:** The author writes (Introduction and summary, p. 99): "Consider the following problem. Particles are distributed over unit areas in such a way that the number of particles to be found in such areas is a random variable following the law of Poisson, with $\nu$ equal to the expected number of particles per unit area. Furthermore, the particles themselves are assumed to vary in magnitude according to a size distribution specified (independently of the particular unit area chosen) by a d.f. $F(x)$ defined over some interval $a \leq x \leq b$, with $F(a) = 0$ and $F(b) = 1$. The problem is to find the distribution of the smallest, largest, or more generally the nth smallest or nth largest particle in randomly chosen unit areas. The problem as stated is not completely specified. . . . In the case of the distribution of the nth smallest or nth largest particle [$n = 1, 2, \ldots$], one must give a rule for dealing with those areas which contain $(n-1)$ or fewer particles. There are at least two possible alternatives. One alternative is to omit none of the areas from consideration by setting up the following rule: . . . if $(n-1)$ or fewer particles are found in a given unit area then this area will be considered as one for which the nth smallest size particle is $x = b$ and for which the nth largest size particle is $x = a$. A second alternative is to restrict attention to those areas which contain . . . at least n particles . . . . In other words, this means finding the relevant conditional distribution. From the point of view of the application of the theory of extreme values to fracture problems, there are some situations where the first model and other situations where the second model is the more appropriate in describing the phenomenon under investigation. In this paper section 2 will be devoted to a derivation of the distributions associated with the first alternative; in section 3 the conditional distributions will be described briefly."

**References:** Cramér (1946), Epstein (1948a), Epstein & Brooks (1948).

Epstein, Benjamin (1949b). The distribution of extreme values in samples whose members are subject to a Markoff chain condition. *Annals of Mathematical Statistics* **20,** 590-594; correction: **22** (1951), 133-134. (MR 11, 375).

**Summary:** Let $x_1, x_2, \ldots, x_n, \ldots$ be observations taken in order at time $t = 1, t = 2, \ldots, t = n, \ldots$ from a stationary stochastic process subject to the condition Prob $(x_n \leq x | x_1 \leq x, x_2 \leq x, \ldots, x_{n-1} \leq x)$ = Prob $(x_n \leq x | x_{n-1} \leq x)$, which is not [as W. Feller pointed out to the author, and he in turn to the reviewer (M. Loève)], a Markoff chain condition, as indicated in the title. The author finds the distributions of smallest, largest, second largest, and second smallest values in samples of size n.

* Godwin, H. J. (1949a). Some low moments of order statistics. *Annals of Mathematical Statistics* **20,** 279-285. (MR 10, 722).

**Summary:** The author writes (Introduction, p. 279): "In a paper on order statistics from several populations [Hastings, Mosteller, Tukey & Winsor (1947)], there were given, among other results, the means, variances, covariances, and correlations of order statistics in samples of ten or less from a normal population. These were obtained by numerical integration, and on account of the difficulties arising therefrom, some results were given to only two decimal places. More recently, Jones [(1948)] has shown that some of the integrals, for sample sizes not greater than four, can be evaluated explicitly. In this note these results are supplemented in two ways. For a paper which the author has recently submitted to *Biometrika* [Godwin (1949b)] integrals were evaluated which can be used to give some of the results in [Hastings, Mosteller, Tukey & Winsor (1947)] to more places of decimals. It is also shown that the table of explicit values can be extended [to sample sizes up through six]." The integrals available from Godwin (1949b) are $\psi(i) = \int_{-\infty}^{\infty} F^i(x)\,(1-F(x))^i dx (1 \leq i \leq 5)$ and $\psi(i,j) = \int_{-\infty}^{\infty} F^i(x) \int_{x}^{\infty} (1-F(y))^j\, dxdy$ $(1 \leq i,j;\ i+j \leq 10)$, where $F(x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{x} (1/\sqrt{2\pi}) \exp(-t^2/2)\, dt$. These integrals were evaluated to ten decimal places, the last place possibly being in error by one or two units. The author also defines $\alpha(i,j) = \int_{-\infty}^{\infty} x F^i(x)\,(1-F(x))^j dx = -\alpha(j,i)$, and $\beta(i,j) = \int_{-\infty}^{\infty} x^2 f(x) F^i(x)\,(1-F(x))^j\, dx = \beta(j,i)$ and uses recurrence relations to construct tables of the $\alpha$'s and $\beta$'s. For the order statistics of samples up through size ten from N(0,1), he tabulates the means and standard deviations (7DP), their variances and covariances (5DP), and their correlations (4DP).

**References:** Irwin (1925a), Hastings, Mosteller, Tukey & Winsor (1947), Jones (1948).

**Citations:** Godwin (1949b), Howell (1949).

* Godwin, H. J. (1949b). On the estimation of dispersion by linear systematic statistics. *Biometrika* **36,** 92-100. (MR 11, 673).

**Summary:** The author writes (Introduction, p. 92): "The purpose of this paper is to discuss the efficiency of estimates of the standard deviation of a population which are obtained by ranking the observations of a sample and taking a linear combination of them. Such statistics are termed systematic by Mosteller (1946). . . . The best known statistic satisfying the requirements is the range whose probability integral in samples from a normal population was tabulated by Pearson . . . and Hartley (1942). Earlier, Tippett (1925) had tabulated its mean value for sample sizes 2-1000, and given formulae for higher moments, while E. S. Pearson (1926) had calculated second, third and fourth moments for several sample sizes. Other measures which are the difference of two observations have been proposed, such as the interquartile range, discussed by Hojo (1931), and the difference of quindeciles, suggested by K. Pearson (1920). Mosteller (1946) discusses these and other differences of symmetrically placed ranks which he calls quasi-ranges. Recently, Nair (1947) has considered the mean deviation from the median. He propounds several questions as to the usefulness of this statistic and the present paper has grown from an attempt to answer these. I had obtained the frequency function of this statistic in normal samples (as Nair

(1948[b]) observed, after he had obtained it as a special case of a more general result), but finding it intractable for further development did not attempt to publish it. The method used here is to consider the first and second moments of difference of consecutive ranks. They were studied by Irwin (1925[a]) and Pearson & Pearson (1931), the last authors deriving results about the differences from a knowledge of the moments and correlations of the ranks themselves. Recently, Hastings, Mosteller, Tukey & Winsor (1947), have also tabulated the means, variances and covariances of ranks, and either of their results or my results on rank differences given below is deducible from the other. However, my results are computed to more places than theirs, and I have thus been able to give a more accurate version of their table (Godwin (1949[a])). Some of the results which they computed have been given exactly by Jones (1948), who has evaluated certain integrals connected with the normal distribution. I have also extended this table in the paper referred to above."

**References:** K. Pearson (1920), Irwin (1925a), Tippett (1925), E. S. Pearson (1926), Hojo (1931), Pearson & Pearson (1931), Davies & Pearson (1934), Pearson & Hartley (1942), Mosteller (1946), Hastings, Mosteller, Tukey & Winsor (1947), Nair (1947, 1948b), Godwin (1949a).

**Citation:** Nair (1949b).

* Gumbel, E. J. (1949). Probability tables for the range. *Biometrika* **36**, 142-148. (MR 11, 527).

**Summary:** The author compares the asymptotic distributions of the range and the midrange for any unlimited distribution of the exponential type with Elfving's distribution of the probability transformation of the normal range. He tabulates the asymptotic cdf $\Psi(R)$ and pdf $\psi(R)$ of the reduced range $R = \alpha(w - 2u)$ to five significant decimal places, where w is the range proper, u is the expected largest value defined by $\Phi(u) = 1 - 1/n$, and the coefficient $\alpha$ is defined by $\alpha = n\phi(u)$, $\phi(x)$ and $\Phi(x)$ being respectively the pdf and the cdf of the distribution under consideration. He states the following conclusions (p. 147): "The asymptotic distributions of the ranges and mid-ranges for an unlimited symmetrical initial distribution of the exponential type are obtained from the convolution of the asymptotic distributions of the extremes. Elfving's approach for the normal range leads to the same results as this direct method which does not require the knowledge of the sample size, nor of the analytical form of the initial distribution, nor of the numerical values of its parameters."

**References:** Pearson (1926), Fisher & Tippett (1928), Gumbel (1935a, 1944, 1946a), Elfving (1947), Gumbel (1947).

* Harris, Theodore E.; Tukey, John W. (1949). Development of Large Sample Measures of Location and Scale which are Relatively Insensitive to Contamination. Memorandum Report No. 31, Stastistical Research Group, Princeton University, Princeton, NJ.

**Summary:** The authors study the relative efficiencies, in large samples, of estimators obtained from the mean and standard deviation by removing the extreme $\gamma i$ at each end of the sample, for varying degrees of contamination.

**Note:** The compiler has been unable to locate a copy of this report, and hence has had to rely entirely on a report on "Statistical Research at Princeton University Supported by the Office of Naval Research 1946-1956" for information concerning it. This statement applies also to Memorandum Reports 25, 32, 33 and 34 [Tukey (1949c,d,e,f)].

**References:** Tukey (1949d).

**Citation:** Tukey (1949d).

* Hartley, H. O. (1949). Tests of significance in harmonic analysis. *Biometrika* **36**, 194-201. (MR 11, 529).

**Summary:** The author writes (Introduction, p. 1): "The classical harmonic analysis and closely related periodogram analysis have, in recent years, been the subject of severe criticisms (Kendall, 1946 . . . ). These have been mainly of two kinds: (a) That the analysis has been widely misused in situations where it is not appropriate and has thereby led to faulty conclusions. (b) That the tests of significance used are based on the assumption of a random series and are therefore hardly ever applicable." In dealing with (b), the author quotes Kendall's summary of his discussion of the Schuster, Walker and Fisher tests, then continues (p. 2): "Kendall's main point . . . is that the 'significance' of an observed intensity may be 'misleading' in that it does not necessarily indicate the reality of the period for which it was observed but, possibly, that of other periods or, indeed, quite a different non-random behaviour of the series. Whilst it would appear to be difficult to develop a test entirely free from the above criticisms, we will show that most of the difficulties can be overcome by using tests in which the periodogram intensities are independent or slightly correlated, and by applying some recent results in analysis of variance test distributions. . . ." Let $y_t$ ($t = 0,1, \ldots, n-1$) be a series of n observations at equidistant intervals $x_t = 2\pi t/n$ of the independent variable. Harmonic analysis of this series consists in fitting, say, $2m+1$ regression coefficients $a_0; a_1, a_2, \ldots, a_m; b_1, b_2, \ldots, b_m$ in the regression function $Y_t = a_0 + \Sigma_{i=1}^{m}(a_i \cos it\gamma + b_i \sin it\gamma)$, where $\gamma = 2\pi/n$ and computing the 'observed intensities' $S_i^2 = a_i^2 + b_i^2$ corresponding to period i. If $m \leq (n-1)/2$, the residual sum of squares is given by $R^2 = \Sigma_{t=0}^{n-1}(y_t - Y_t)^2 = \Sigma_{t=0}^{n-1}(y_t - \bar{y})^2 - (n/2)\Sigma_{i=1}^{m}(a_i^2 + b_i^2)$. In order to test the significance of the harmonic term with the largest intensity $S_{max}^2$, the author converts the criterion of Walker (1914) into an exact test by making use of the residual $R^2$ as an independent estimate of $\sigma^2$ and then 'studentizing' Walker's criterion. The resulting test is one for the maximum variance ratio $F_{max} = (n/4) S_{max}^2(n-2m+1)/R^2$. The author investigates the power of the $F_{max}$ test and its usefulness under various alternative assumptions as to the character of the $y_t$ distributions.

**References:** Walker (1914), Brunt (1917), Fisher (1929), Finney (1941), Hartley (1944), Kendall (1946).

* Howell, John M. (1949). Control chart for largest and smallest values. *Annals of Mathematical Statistics* **20**, 305-309; errata, **21** (1950), 615-616. (MR 10, 724; 12, 347—listing only).

**Summary:** The author writes (Introduction, p. 305): "It may at times be desirable to use a control chart for largest and smallest values (L & S) in place of the conventional charts for averages and ranges ($\bar{X}$ & R). The chart for largest and smallest values has certain advantages: all information may be combined on one chart, computations are simple, and specifications may be placed on the chart. In this paper, constants for the use of this chart are developed and comparison is made with the average and range charts." Corrections to one of the tables are given in the errata notice; the more accurate values given are those of Godwin (1949a).

**References:** Tippett (1925), Hojo (1931), Shewhart (1931), Pearson [& Hartley] (1942), Wilks (1943), Daly (1946), {Godwin (1949a)}.

* Juncosa, M. L. (1949). The asymptotic behavior of the minimum in a sequence of random variables. *Duke Mathematical Journal* **16**, 609-618. (MR 11, 375).

**Summary:** The author writes (Introduction, p. 609): "Let $\xi_n = \min_{k \leq n} X_k$, $n = 1, 2, \ldots$, where $X_1, X_2, \ldots$ is a sequence of mutually independent random variables whose respective distribution functions, assumed to be non-unitary throughout this work, are $F_1(x), F_2(x), \ldots$. If sequences of real constants (norming parameters) $\{a_n > 0\}$ and $\{b_n\}$, $n = 1, 2, \ldots$, exist such that (1) $\lim_{n\to\infty} \Pr\{\xi_n > a_n x + b_n\} = \lim_{n\to\infty} \Pi_{k=1}^{n}[1 - F_k(a_n x + b_n)]$ exists and is of the form $1 - V(x)$ where $V(x)$ is a non-unitary distribution function, then one says that $\{\xi_n\}$ has a limiting distribution. It is known that the limit class $V(ax + b)$, if it exists, is unique. Furthermore, for each non-unitary limit $V(x)$, the sequences $\{a_n > 0\}$ and $\{b_n\}$ are asymptotically unique (see Gnedenko [(1943)]). We shall give some conditions under which $\{\xi_n\}$ has a

limiting distribution. Completing previous work by Fréchet [(1927)], Fisher & Tippett [(1928)], and de Mises [von Mises (1936)] in the case of identically distributed variables $\{X_k\}$, Gnedenko [(1943)] has shown that there exist only three classes of limiting distributions and has given necessary and sufficient conditions for convergence to each class. One of the chief purposes of this work is to point out . . . that much more general classes exist if we drop the restriction that the random variables $\{X_k\}$ are identically distributed. . . ."

**Comments:** The $\xi_n$ are not order statistics, according to our definition, except in the special case of identically distributed variables $\{X_k\}$. Complete results for this case were given by Gnedenko (1943), so this paper adds nothing new that is directly relevant to a study of order statistics.

**References:** Fréchet (1927), Fisher & Tippett (1928), de Mises [von Mises] (1936), Gnedenko (1943).

* Kac, M. (1949). On deviations between theoretical and empirical distributions. *Proceedings of the National Academy of Sciences* (USA) **35**, 252-257. (MR 10, 614).

**Summary:** The author writes (pp. 252-253): "The determination of a theoretical distribution function $\sigma(u)$ by drawing a random sample is a familiar technique. The reliability of this procedure is based on considerations of the least upper bound of the absolute value of the difference between $\sigma(u)$ and the empirical distribution $\sigma_n(u)$, defined as 1/n times the number of sample elements falling below u. The theory requires delicate and powerful mathematical tools. Considerable simplification can, however, be achieved by taking samples whose size is not fixed but drawn from a Poisson population. This note is devoted mainly to exposition of this idea. We also consider different measures of deviations and compare the theories for samples of fixed and random sizes. Let $x_1, x_2, \ldots$ be independent random variables having the same continuous distribution functions $\sigma(u)$. Let furthermore $\psi_n(x)$ be 0 or 1 according as $x > u$ or $x < u$. The empirical distribution function $\sigma_n(u; x_1, \ldots, x_n) \equiv \sigma_n(u)$ is defined by the formula $\sigma_n(u) = (1/n)\sum_{j=1}^{n} \psi_u(x_j)$. (1). Kolmogoroff [(1933)] has shown that $\lim_{n\to\infty}$ Prob. $\{1.u.b_{-\infty<u<\infty}|\sigma_n(u)-\sigma(u)| < \alpha/\sqrt{n}\} = P(\alpha)$, (2) where $P(\alpha) = (\sqrt{2\pi}/\alpha)\sum_{k=0}^{\infty}\exp[-(2k+1)^2\pi^2/8\alpha^2]$. (3). An elementary proof of this and some related results was subsequently given by Feller [(1948)]. Quite recently Doob [(1949)] derived Kolmogoroff's result using the theory of stochastic processes. He has pointed out that $P(\alpha)$ = Prob. $\{\text{Max.}_{0\le u\le 1}|y(u)|<\alpha\}$, (4) where $y(u)$ $(0 \le u \le 1)$ is a Gaussian process satisfying the conditions $y(0) = y(1) = 0$, $E\{y(u)y(v)\} = \text{Min.}(u,v) - uv$. (5). Since the process $y(u)$ is related in a simple way to the classical Wiener process the calculation of (4) can be reduced to a more or less standard calculation in the diffusion theory. Doob's justification of (4) is heuristic and he states that a rigorous proof would be quite delicate. Independently of Doob, the present author also used the reduction to the process $y(u)$ in connection with a related problem (see § 4 of this note)."

**References:** Kolmogoroff (1933), Feller (1948), Doob (1949).

**Citations:** Chung (1949), Doob (1949).

* Kimball, Bradford F. (1949). An approximation to the sampling variance of an estimated maximum value of a given frequency based on fit of doubly exponential distribution of maximum values. *Annals of Mathematical Statistics* **20,** 110-113. (MR 10, 466).

**Summary:** The author writes (Introduction, p. 110): "Given the doubly exponential distribution of maximum values (1) $F(x) = \exp(-e^{-y})$, $y = \alpha(x-u)$, where $\alpha$ and u are unknown parameters, with a prescribed frequency $F_0$ the 'reduced variate' y is fixed, say at $y = y_0$. Thus with $F_0 = .99$, $y_0 = 4.60015\ldots$. Given a sample of n maximum values $x_i$, we are interested in the sampling variance of (2) $\hat{x} = g(\hat{u},\hat{\alpha}) = \hat{u} + y_0/\hat{\alpha}$ due to sampling variations of the estimates $\hat{u}$ and $\hat{\alpha}$. H. Fairfield Smith has recently pointed out to me that the examples of applications of sufficient statistical estimation functions to this problem given in a previous paper (see [Kimball (1946a), pp. 307-309]) give too large a range for $\hat{x} = g(\hat{u},\hat{\alpha})$ because the

sample points $(\hat{u},\hat{\alpha})$ within the confidence region of the constant probability ellipse apply to optimum estimates of $(\hat{u},\hat{\alpha})$ rather than to that of $g = g(\hat{u},\hat{\alpha})$. What the problem calls for is the determination of the positions of curves $\bar{g}(u,\alpha)$ and $g(u,\alpha)$ such that the integral of the pdf of the estimation functions over all sample values $(\hat{u},\hat{\alpha})$ which lie between these two curves is equal to the confidence level (taken as .95 in previous paper). Further considerations of this being the shortest interval $\bar{g} - g$, also come into play. As so often happens in research, the previous analysis, although not giving the final answer, suggests the next step. If we change our parameters to (3) $g(u,\alpha) = u + y_0/\alpha$, $\alpha' = \alpha$ and are able to carry through the inverse of the maximum likelihood solution for fitting of (1) to n sample values $x_i$, then we shall be in a position to find the asymptotic marginal distribution of $\sqrt{n}(\hat{g} - g)$, which will give the answer to our problem (see [Wilks (1943)]). . . ." The author reworks the example of his earlier paper [Kimball (1946a)] and obtains the 95% confidence interval $365.2 < g < 474.2$, which is symmetrical about $\hat{g} = 419.7$ and is shorter than the interval $362.8 < g < 507.4$, which was not symmetrical about $\hat{g}$, obtained in the previous paper.

**References:** Wilks (1943), Kimball (1946a).

* Kontorova, T. A.; Timoshenko, O. A. (1949). Generalization of the statistical theory of strength for nonuniformly stressed bodies. (Russian). *Zhurnal Tekhnicheskoi Fiziki* [*Journal of Technical Physics (USSR)*] **19,** 355-370. (AMR 3, 478).

**Summary:** The authors extend the results on the size effect in brittle fracture, given in several earlier papers by Kontorova, to the bending of bars of rectangular cross section and the torsion of rods of circular cross section. They assume a Gaussian distribution of strengths in an assembly of specimens of equal size, and a proportionality between the number of flaws, the most hazardous (weakest) of which causes failure, and the volume of the specimen, though the reviewer (E. Orowan) believes that, while these assumptions are justified for materials like plaster of Paris, the number of flaws for vitreous materials would be proportional to the surface area rather than the volume. The results show that, with the authors' assumptions, the mean fracture stress of specimens of given volume would be lower in tension than in bending or torsion, which conclusion agrees with earlier measurements on specimens of plaster of Paris.

**References:** Kontorova (1940), Kontorova & Frenkel (1941), Kontorova (1943).

* Lane, E. W.; Lei, Kai (1949). Stream flow variability. *Proceedings of the American Society of Civil Engineers* **75**, 935-949; discussion, 1508-1510. *Transactions of the American Society of Civil Engineers* **115** (1950), 1084-1098; discussion, 1099-1134.

**Summary:** The authors write (Synopsis, p. 935): "An index of stream flow variability is suggested, in the form of the standard deviation of the logarithm of the stream discharge. The method of determining the index is described and the values from the flow records of a large number of streams in the eastern part of the United States were obtained. The possibilities of using this index in estimating the duration curve of the stream flow, where flow records are short or nonexistent, is indicated. . . ." The index of flow variability used is based on the duration curve of stream discharge, which is a curve showing the percentage of the total time covered by the flow record (abscissa) during which the stream discharge (ordinate) was greater than the various discharge values. The authors write (pp. 936-937): "When ordinary coordinates are used for duration curves . . . , in order to fit them on a sheet of reasonable size, the discharge scale is usually such that the low flows cannot be read accurately. This can be remedied by plotting the discharge on a logarithmic scale . . . . A still better presentation has been suggested by the late Allen Hazen . . . , who plotted the curve on his logarithmic probability paper and secured a curve which, for the greater part of the total time, closely followed a straight line . . . . It departs from the straight line at the ends, but only for a small part of the total time, although, because of the distortion of the scale in the probability plotting, this small part of the time covers a large proportion of the space on the graph. A stream with a very uniform flow will produce a curve with little slope . . . , and a variable

stream will give rise to a curve of considerable slope . . . . The steepness of the slope can be used as a measure or index of the flow variability." The discussion includes comments by Ven Te Chow, Dana M. Wood, H. Alden Foster and others, with a rejoinder by the first author. Foster (p. 1107) gives a curve showing the ratio of the median to the mean discharge, for various values of the variability index, which Lane says will be of considerable assistance in applying the methods presented in the paper. Lane writes (p. 1132): "The shape of the duration curve can be obtained by drawing a straight line duration curve on logarithmic probability paper with a slope such that the ratio of the discharge exceeded 15.87% of the time to the discharge exceeded 50% of the time is equal to the antilogarithm of the variability index selected." This statement is a correction of one made on p. 1097 of the paper, where the words "the antilogarithm of" were omitted.

**Comments:** The duration curve is similar to the empirical cdf curve (ogive), but differs from it in two ways: (1) the probabilities are cumulated downward instead of upward; (2) the ordinate and the abscissa are reversed.

**References:** Hazen (1914), Hall (1921), Foster (1924), Goodrich (1927), Hazen (1930), Foster (1934), Slade (1934)[1936], Creager (1939), Beard (1942)[1943], Powell (1943), Gumbel (1945b), Potter (1949a).

* Langbein, W. B. (1949). Annual floods and the partial-duration flood series. *Transactions of the American Geophysical Union* **30**, 879-881; discussion, **31** (1950), 939-941.

**Summary:** The author writes (Abstract, p. 879): "Flood data are ordinarily listed either in annual-flood series [highest peak discharge in each water year] or in a partial-duration series [all floods above a selected base]. If the expectancy of a flood in the duration series $\varepsilon$ is known, then the probability of that flood being an annual flood is shown to be $e^{-\varepsilon}$. From this relationship it is possible to transform the recurrence intervals in the partial-duration series to those in the annual-flood series. It is shown that for equivalent floods, the recurrence intervals in the partial-duration series are smaller than in the annual-flood series, but that the difference becomes inconsequential for floods greater than about the five-year recurrence interval." The discussion includes comments by Ven Te Chow and a closure by the author. Chow writes (pp. 939-940): "In treating hydrologic data for the study of their frequency of recurrence, two phases of study are generally met. One is the frequency of recurrence for the observed distributions, and the other is the corresponding frequency of recurrence for the theoretical distributions of best fit. The former is usually required for the purpose of plotting observed data, and hence is called the 'plotting positions', a term used by FOSTER [1934, p. 1217]. The latter is treated by the mathematical theory of probability, and generally serves as a theoretical interpretation of the observed phenomena. A correct determination of plotting positions has . . . caused a great deal of discussion. Many methods for computing plotting positions have been proposed. . . . If $T_p$ is the recurrence interval of hydrological events in partial duration series, and $T_a$ is the recurrence interval . . . in annual maximum series, then, according to the paper, $T_p = 1/\varepsilon$ (1) and $T_a = 1/(1-e^{-\varepsilon})$ (2). The difference between these two recurrence intervals with respect to $T_a$ is $(T_a-T_p)/T_a = (e^{-\varepsilon}+\varepsilon-1)/\varepsilon$ (3) where $\varepsilon$ is related to $T_p$ and $T_a$ by (1) and (2) respectively. . . . From a theoretical point of view, a recognized distinction between the two methods is that the annual maximum series forms a complete series, but the partial duration series does not; hence, a mathematical analysis of the latter cannot be made. GUMBEL [1941(a)] has utilized the theory of largest values [FISHER and TIPPETT, 1928] to interpret the theoretical distribution of annual maxima. By this method the recurrence interval of the annual maximum series can be expressed by $T_a = 1/[1-\exp(-e^{-x/c})]$ (4) in which $x = a + y$, a and c are parameters, and y is the value of the annual maximum. Comparing (4) with (2), it is obvious that the expectancy of recurrence is $\varepsilon = e^{-x/c}$ (5). Substituting (5) in (1), the recurrence interval in partial duration series becomes $T_p = e^{x/c}$ (6). By substituting $x = a + y$ in (6) and modifying, $y = c \log_e T_p - a$ (7) which will give a straight-line relationship of y and $T_p$ when plotted on a semi-logarithmic graph paper. Therefore, if Gumbel's method of using the theory of largest values in annual maximum series is mathematically adequate, then the conclusion shown by (7) should explain the legitimate use of semi-logarithmic plotting in the empirical method of partial duration series."

**References:** Fisher & Tippett (1928), Foster (1934), Gumbel (1941a).

* Lévi, Robert (1949). Calculs probabilistes de la sécurité des constructions. *Annales des Ponts et Chausées* **119**, 493-539.

**Summary:** The author points out that, just as the strength of a chain is that of its weakest link, the strength of a long bar is that of the weakest of the pieces into which it could be divided by cross-sectional cuts. Since the distribution of the smallest of n values ($n \geq 2$) drawn at random from a normal distribution is not normally distributed, the author [correctly] concludes that if the strength of the links of a chain (or short segments of a long bar) is normally distributed, that of the chain (or bar) is not. He endeavors to use this fact [incorrectly] to justify the use of the lognormal distribution for the strength of bars or chains.

* Maniya, G. M. (1949). Generalization of the criterion of A. N. Kolmogorov for an estimate for the law of distribution for empirical data. (Russian). *Doklady Akademii Nauk SSSR* (N. S.) **69**, 495-497. (MR 11, 261).

**Summary:** The author recalls that Kolmogorov (1933) proved that, for each $\lambda \geq 0$ and random continuous distribution function $F(x)$, $\lim_{n\to\infty} P(D_n \leq \lambda/\sqrt{n}) = \Phi(\lambda) = 1 - 2\sum_{\nu=1}^{\infty}(-1)^{\nu-1}\exp(-2\nu^2\lambda^2)$, while Smirnov (1939a) proved that $\lim_{n\to\infty} P(D_n^+ \leq \lambda/\sqrt{n}) = 1 - e^{-2\lambda^2}$, where $D_n = \sup_{-\infty<x<\infty}|S_n(x)-F(x)|$ and $D_n^+ = \sup_{-\infty<x<\infty}\{S_n(x)-F(x)\}$, $S_n(x)$ being the empirical cdf. The author generalizes the results of Kolmogorov and Smirnov by defining the two random values $D_n^+(\theta_1, \theta_2) = \sup_{\theta_1\leq F(x)\leq\theta_2}\{S_n(x)-F(x)\}$ and $D_n(\theta_1, \theta_2) = \sup_{\theta_1\leq F(x)\leq\theta_2}|S_n(x)-F(x)|$ and stating the following two theorems whose proofs are said to be based on the theorems of continuity of random functions and the Laplace transform: Theorem 1. Let $F(x)$ be the continuous distribution function of each of the independent [unordered sample] values $x_i$ ($i = 1, 2, \ldots, n$), $\theta_1^{(n)} = m_1/n = \theta_1 + o(1/\sqrt{n})$, $\theta_2^{(n)} = m_2/n = \theta_2 + o(1/\sqrt{n})$, $0 < \theta_1 < \theta_2 < 1$. *Then* $\lim_{n\to\infty} P\{D_n^+(\theta_1^{(n)}, \theta_2^{(n)}) < \lambda/\sqrt{n}\} = \Phi^+(\theta_1, \theta_2; \lambda)$, where $\Phi^+(\theta_1, \theta_2; \lambda) = \{\int_{-\infty}^{a}\int_{-\infty}^{b}\exp[-\bar{\theta}(z_1, z_2)/2]dz_1dz_2 - \exp(-2\lambda^2)\int_{-\infty}^{a'}\int_{-\infty}^{b'}\exp[-\theta(z_1, z_2)/2]dz_1dz_2\}/2\pi\sqrt{1-R^2}$, $a = \lambda/\sqrt{\theta_1(1-\theta_1)}$, $b = \lambda/\sqrt{\theta_2(1-\theta_2)}$, $a' = (\lambda - 2\lambda\theta_1)/\sqrt{\theta_1(1-\theta_1)}$, $b' = [\lambda - 2\lambda(1-\theta_2)]/\sqrt{\theta_2(1-\theta_2)}$, $R = \sqrt{\theta_1(1-\theta_1)/\theta_2(1-\theta_2)}$, $\theta(z_1, z_2) = [z_1^2 + 2Rz_1z_2 + z_2^2]/(1-R^2)$, and $\bar{\theta}(z_1, z_2) = [z_1^2 - 2Rz_1z_2 + z_2^2]/(1-R^2)$. Theorem 2. Under the conditions in Theorem 1, $\lim_{n\to\infty} P\{D_n(\theta_1^{(n)}, \theta_2^{(n)}) \leq \lambda/\sqrt{n}\} = \Phi(\theta_1, \theta_2; \lambda)$, where $\Phi(\theta_1, \theta_2; \lambda) = \{\int_{-a}^{a}\int_{-b}^{b}\exp[-\bar{\theta}(z_1, z_2)/2]dz_1dz_2 - 2\sum_{k=1}^{\infty}(-1)^{k-1}\exp(-2k^2\lambda^2)\int_{-\alpha_k}^{\alpha_k}\int_{-\beta_k}^{\beta_k}\exp[-\theta(z_1, z_2)/2]dz_1dz_2\}/2\pi\sqrt{1-R_2}$, where $\alpha_k = (\lambda - 2k\lambda\theta_1)/\sqrt{\theta_1(1-\theta_1)}$ and $\beta_k = \lambda - 2k\lambda(1-\theta_2)/\sqrt{\theta_2(1-\theta_2)}$.

**Note:** The reviewer (J. Wolfowitz) has pointed out an apparent error in Theorem 2, omission of the bar over $\theta$ in the first integrand of the expression for $\Phi(\theta_1, \theta_2; \lambda)$. The compiler has corrected both this error and a similar one in Theorem 1, omission of the plus sign in the first $\Phi^+$.

**References:** Kolmogorov [Kolmogoroff] (1933), Smirnov (1939a), Feller (1948).

* Mather, K. (1949). The analysis of extinction time data in bioassay. *Biometrics* **5**, 127-143.

**Summary:** The relevant portion of this paper has been summarized by Gumbel (1954b), who writes (pp. 43-44): "K. Mather has given an interesting application of the theory of extreme values (without, however, using this term) to the problem of finding the time necessary to kill bacteria by a disinfectant. . . . Mather used the probability of *largest* values as a life table function. Thus the cumulative frequency of extinction can be written $1 - \ell(x) = \exp(-e^{-\alpha(x-u)}) = \exp(-e^{-y})$. . . . Mather . . . minimizes the distance between the observed and theoretical reduced values y. . . . He estimates the parameters by the method of maximum likelihood, uses . . . probability paper . . . , and makes three approximations taking into account the relative weight of each observation. . . . The [results of the analysis] show that the theory of largest values is an adequate tool for the analysis of extinction time of bacteria subject to a disinfectant. However, the philosophical question of why these data follow the law of largest values instead of the smallest, as one might expect, remains open."

**Comments:** The time to extinction is the largest value of the life (after treatment) of any of the bacteria, so it is not clear to the compiler why one should expect, as stated by Gumbel, that the data should follow the law of smallest (rather than largest) values.

**Reference:** Fisher (1935) [1947].

* McMillan, Brockway (1949). Spread of minima of large samples. *Annals of Mathematical Statistics* **20**, 444-447. (MR 13, 479).

**Summary:** The author writes (pp. 444-446): "Let x have the continuous cumulative distribution function $F(x)$. Let $(x_1, \ldots, x_N)$ be a sample of N independent values of x and $y = \inf(x_1, \ldots, x_N)$. Then y is a random variable with the cumulative distribution function (1) $G_N(y) = 1-(1-F(y))^N$. Let K values of the new variable y be drawn, $(y_1, \ldots, y_K)$ and let the spread $w = \sup(y_1, \ldots, y_K) - \inf(y_1, \ldots, y_K)$. Fixing K, we consider the cumulative distribution function of w, $P_N(w)$, as $N \to \infty$. That is, we have K large samples of x and wish to examine the spread among their minima. It is evident intuitively that if $F(x) = 0$ for some finite x, these minima are bounded from below and will cluster near the vanishing point of $F(x)$, making $w \to 0$ statistically as $N \to \infty$. Our theorems also show that even when $y \to -\infty$ statistically, i.e., when $F(x) = 0$ for no finite x, the spread $w \to 0$ statistically if the tail of $F(x)$ is sufficiently small (e.g. Gaussian). On the other hand, if $F(x) = O(e^{kx})$ as $x \to -\infty$, the distribution $P_N(w)$ does not peak as $N \to \infty$, while for larger tails (e.g. algebraic) $w \to +\infty$ statistically. Two simple theorems are I. If $\lim_{x \to -\infty} F(x)/F(x+s) = 1$, then $\lim_{N \to \infty} P_N(s) = 0$. II. Let $s > 0$. If $F(x_0) = 0$ for some $x_0 > -\infty$, or if $\lim_{x \to -\infty} F(x)/F(x+s) = 0$, then $\lim_{N \to \infty} P_N(s) = 1$. Theorem I is directly applicable to distributions with algebraic tails, theorem II to Gaussian tails. We prove them both as corollaries of the more general results: III. If $\liminf_{x \to -\infty} F(x)/F(x+s) = \ell$ then $\limsup_{N \to \infty} P_N(s) \leq (1-\ell)^{K-1}$. IV. Let $s > 0$. If $F(x) = 0$ for no finite x and $\limsup_{x \to -\infty} F(x)/F(x+s) = L$, then $\liminf_{N \to \infty} P_N(s) \geq [e^{-\alpha L} - e^{-\alpha}]^K$ for any $\alpha > 0$. Theorems III and IV together show that an exponential tail ($F(x) = O(e^{kx})$) leads to a $P_N(w)$ which, asymptotically, is bounded away from 0 for any $w > 0$ and bounded away from 1 for w sufficiently small."

* van Meer, H. P.; Plantema, F. J. (1949). Vermoeiing van constructies en constructiedelen. Rapport S. 357, Nationaal Luchtvaartlaboratorium, Amsterdam. (AMR 3 #1270).

**Summary:** This report on a survey of the literature concerning fatigue of materials and structures contains a section on the statistical theory of fatigue failure in which the authors discuss the "weakest link theory" [Weibull (1939a,b)], the "strength summation theory" [Tucker (1941)] and the combination of these two theories proposed by Gause in his contribution to the discussion of Tucker's paper.

**References:** Faulhaber, Buchholtz & Schultz (1933), Weibull (1939a,b), Tucker (1941), Moore & Morkovin (1942-44).

* Meĭzler, D. G. (1949). On a problem of B. V. Gnedenko. (Russian). *Ukrainskii Matematicheskii Zhurnal* **1** (2), 67-84. (MR 14, 186).

**Summary:** The reviewer (J. L. Doob) writes (MR 14, 186): "Let G be the class of distributions each of which can be obtained as the limiting distribution of $X_n/a_n$, where $X_n$ is the maximum of the first n terms of a sequence of mutually independent random variables with distribution functions $F_1, F_2, \ldots$, and $a_1, a_2, \ldots$ are positive constants. It is supposed that, for each x, $\lim_{n \to \infty} F_i(a_n x) = 1$ uniformly in $i \leq n$. . . . [Several] theorems are proved. For the most general treatment of the special case in which $F_n$ does not depend on n see Gnedenko [(1943)]. . . . Juncosa [(1949)] has found conditions on $F_i$ sufficient to get various limiting $\Phi$ . . . ."

**Notes:** (1) *The compiler has been unable to obtain a copy of either the original or a translation of this paper, so he has had to rely entirely on the review by Doob for information concerning its contents.* (2) See note on the paper by Juncosa (1949).

**Reference:** Gnedenko (1943).

* Nair, K. R. (1949a). A further note on the mean deviation from the median. *Biometrika* **36**, 234-235. (MR 11, 608).

**Summary:** The author extends the results given in his earlier paper [Nair (1947)] on the mean deviation from the median and from the mean, m′ and m, respectively. In particular, he compares their coefficients of variation for samples of size n = 2(1)10 from a normal population. When n = 2, m and m′ are identical, and hence so are their coefficients of variation. When n = 3, m and m′ are not identical but their coefficients of variation are identical, the exact common value being $(1/\sqrt{3})\sqrt{(2\pi/3 + \sqrt{3} - 3)} = 0.52486$. The author tabulates approximate numerical values (2DP) of the coefficients of variation of m and m′ for n = 2(1)10. The difference is never greater than a unit in the second decimal place, and the author states that this fact reinforces the strong practical grounds (greater simplicity in calculation) for using m′.

**References:** Hastings, Mosteller, Tukey & Winsor (1947), Nair (1947), Jones (1948), Nair (1948b).

* Nair, K. R. (1949b). The efficiency of Gini's mean difference. *Bulletin of the Calcutta Statistical Association* **2**, 129-130.

**Summary:** The author points out that Godwin (1949b) derived expressions for the most efficient linear systematic statistic d for estimating the standard deviation $\sigma$ of a normal population and compared its efficiency with that of the maximum likelihood estimator for samples of size n = 2(1)10.Godwin found that d is much more efficient than other linear systematic statistics in common use, including the mean deviation from the mean, the mean deviation from the median, and the range; he did not consider Gini's mean difference g. Using the exact formula for the standard error of g given by U. S. Nair (1936), the author has computed the percentage efficiency of g and tabulated it, along with Godwin's corresponding values for d, for n = 2(1)10. A comparison shows that g is almost as efficient as d, the efficiency dropping from 100% for both statistics when n = 2 to 99.0% for d and 98.07% for g when n = 10.

**References:** U. S. Nair (1936), Godwin (1949b).

* Noether, Gottfried E. (1949). Confidence limits in the non-parametric case. *Journal of the American Statistical Association* **44**, 89-100. (MR 10, 554).

**Summary:** The author writes (Abstract, p. 89): "The purpose of this article is to give a survey of certain methods available for finding confidence limits when nothing is assumed about the population from which a sample has been drawn except possibly that it has a continuous distribution. The following three cases are treated: a confidence band for the unknown cumulative distribution function, a confidence interval for the proportion of a population for which the variate is smaller (or larger) than a given value, and confidence intervals for quantiles. The results have been known for many years, but have often been accessible only to those who were able to follow rather involved mathematical arguments. It is the purpose of this paper to state certain important results without referring to any mathematical proofs."

**Note:** The confidence limits in the first two cases discussed by the author depend upon the sample cdf, which is a step function with steps (of size 1/n) at the order statistics, and in the third case they depend directly upon the order statistics.

**References:** Smirnoff [Smirnov] (1939a), Wald & Wolfowitz (1939), Nair (1940b), Kolmogoroff (1941), Thompson (1941), Scheffé & Tukey (1945), Cramér (1946), Smirnoff [Smirnov] (1948).

* Paulson, Edward (1949). A multiple decision procedure for certain problems in the analysis of variance. *Annals of Mathematical Statistics* **20**, 95-98. (MR 10, 467).

**Summary:** The reviewer (A. M. Mood) gives the following summary (MR 10, 467): "Let $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_k$ be varietal means in increasing order, each based on r observations, from the same experiment, which have error variances $\sigma^2$ estimated by $s^2$. The varieties may be classified into two groups by the criterion: put $\bar{x}_i$ in the superior group if $\bar{x}_i > \bar{x}_k - \lambda s/\sqrt{r}$, where $\lambda$ is a constant determined so that the probability of not including all varieties in the superior group has a specified (small) value when all varieties are the same; $\lambda$ is readily found with the aid of tables of the Studentized range. An expression is given for the power of the criterion in the case all varieties but one are the same." In a special case where the error variance $\sigma^2$ is known *a priori*, s is replaced by $\sigma$ in the procedure described above, and $\lambda$ is determined with the aid of tables of the range [Pearson & Hartley (1942)] instead of tables of the studentized range [Pearson & Hartley (1943)].

**References:** Pearson & Hartley (1942, 1943).

* Peterson, R. E. (1949a). Approximate statistical method for fatigue data. *ASTM Bulletin*, No. 156, 50-52; discussion, 62.

**Summary:** The author notes that in a number of recent papers [Weibull (1939b), Fowler (1945), Freudenthal (1946), Davidenkov, Shevandin & Wittmann (1947), Epstein (1948a), and Hill & Schmidt (1948)], size effect is considered to be a consequence of probability relations based on the weakest link concept, and that in most of these analyses the standard deviation $\sigma$ is a criterion of the magnitude of the size effect. He plots failure stresses on normal probability paper, using plotting positions $(2i-1)/2n$ $(i = 1, 2, \ldots, n)$, and estimates the standard deviation to be the 84% point minus $\overline{X}$ (the mean) or $\overline{X}$ minus the 16% point.

**References:** Dodge et al. (1933) [1943], Weibull (1939b), Fowler (1945), Freudenthal (1946), Davidenkov, Shevandin & Wittmann (1947), Epstein (1948a), Hill & Schmidt (1948).

**Citation**: Peterson (1949b).

* Peterson, R. E. (1949b). Application of fatigue data to machine design. *Properties of Metals in Materials Engineering* (ed. R. L. Templin et al.), pp. 60-78. American Society for Metals, Cleveland, Ohio.

**Summary**: The author presents the following statistical approach to the "size effect" problem (pp. 68-73): "As pointed out in the main body of the paper, the test data available to the designer usually are obtained from standard test pieces of rather small diameter. We would like to have a method of estimating the properties for large diameter pieces. While some tests are being made where size is the variable, it is desirable that parallel studies be made to seek a fundamental basis for interpretation of the results. Otherwise the amount of testing would be endless. Furthermore, it is clearly out of the question to run fatigue tests of sizes represented in Fig. 1 [ratio of diameters 100 to 1]; our only hope is to develop relations which will enable us to extrapolate with more confidence. In recent years, statistical analysis has been used to explain the lower strength obtained in specimens having a larger volume. In the engineering field, Weibull [1939b)] seems to have been the first to have demonstrated this possibility, although evidently unknown to Weibull, a similar approach was used by Peirce [(1926)] in explaining the differences in strength obtained from textile threads of different lengths. Epstein [(1948a)] has shown that these and other papers are all based on the so-called 'weakest link theory' which derives from a mathematical study of the distribution of the lowest values as a function of sample size. Hill and Schmidt [(1948)] have prepared a convenient chart (Fig. 6) from which the breakdown voltage of large insulation areas can be estimated, starting with data from a small area. It is believed that this chart, which is based on a Gaussian 'normal' distribution, can also be used for certain fracture problems. . . . Whether the above general method can be extended to fatigue problems is open to question. It is hoped that the present attempt will focus attention on a method of attack and will lead to other contributions in this field. In some respects a fatigue fracture resembles a static fracture of a brittle material—yielding in the ordinary sense does not occur (broken pieces held together usually fit closely); cracking follows the normal stress field. Since fatigue starts locally and propagates by means of a progressively growing

crack, it is not too unreasonable to expect that 'weakest link' theory might be applicable to the starting conditions of such a crack. Strictly speaking, comparisons should be made on the basis of a small detectable crack of given size, since the time of propagation through pieces of various sizes will vary."

**References:** Peirce (1926), Weibull (1939b), Epstein (1948a), Hill & Schmidt (1948), Peterson (1949a).

* Potter, W. D. (1949a). Simplification of the Gumbel Method for Computing Probability Curves. SCS-TP-78, Soil Conservation Service, U. S. Department of Agriculture, Washington, D. C.

**Summary**: The author writes (pp. 1-3): "To determine peak rates of runoff or amounts or intensity of rainfall that might be expected to be equaled or exceeded in various recurrence intervals by the Gumbel method involves the following procedure: 1. Arrange highest annual values in ascending order of magnitude. 2. Determine frequency [for Type 1 asymptotic distribution of largest values] of highest and lowest values in accordance with Gumbel's formulas in which frequencies are expressed as functions of N (table 1, p. 2). 3. Determine frequency for all intermediate values by prorating difference between frequency for highest and lowest values. 4. Plot frequency versus Y (peak rates of runoff, amounts, or intensity of rainfall) on special [extreme-value (Gumbel)] probability paper. 5. For each plotted point, determine corresponding value of X on reduced variate scale. 6. Determine least squares curve (straight line) by solving for a and b in the following equations: (1) $\Sigma xy - a\Sigma x^2 - b\Sigma x = 0$, (2) $\Sigma y - a\Sigma x - nb = 0$. Substitute values of a and b in the equation for a straight line, $y = ax + b$. 7. Determine values of X corresponding to the desired recurrence intervals and substitute in the equation for the least squares curve." On page 3, the author gives conditions justifying the assumption that "the value of Y for any recurrence interval divided by the mean Y should vary as some function of the standard deviation divided by mean Y for like values of N [years of record]." He continues (p. 4): "To test the accuracy of this assumption and to determine the relationship between $Y/\overline{Y}$ and the coefficient of variation ($C_v$) for various values of N, 370 Gumbel probability curves were computed by the method outlined on pages 1 and 3. . . . To determine the relationship between $Y/\overline{Y}$ and $C_v$ for values of N ranging from 5 to 100 years, additional graphs were prepared. . . . Since these curves were fitted by eye, it was now necessary to balance them so that the theoretical or curve values of $Y/\overline{Y}$ would satisfy the $Y/\overline{Y}$ versus $C_v$ relationship defined by the experimental data and also would be such that smooth curves could be drawn when these theoretical values were plotted against corresponding values of N for like values of $C_v$. This was accomplished by trial and error. Based on these balanced curves, figures 13 and 14, pages 21 and 22, are a series of $Y/\overline{Y}$ versus $C_v$ curves for values of $C_v$ ranging from 0 to 150 and for values of N of from 5 to 100 years for recurrence intervals of 2, 5, 10, 25, and 50 years." The author proposes the following simplified method (pp. 4-7): "Using the curves in figures 13 and 14, the procedure for determining values of Y for the several recurrence intervals is as follows: 1. Determine the coefficient of variation ($C_v$). $C_v$ = standard deviation divided by the mean $Y = \ldots = \sqrt{N\Sigma Y^2 - (\Sigma Y)^2}/\Sigma Y$. 2. From curves in figures 13 and 14, pages 21 and 22, select values of $Y/\overline{Y}$ corresponding to the computed $C_v$ for the recurrence intervals desired. 3. Multiply the selected values of $Y/\overline{Y}$ by $\overline{Y}$ to obtain desired values of Y." An illustrative example is given on pp. 7-8, and graphs culminating in figures 13 and 14 are given on pp. 9-22.

**Reference:** Gumbel (1945a).

**Citation**: Lane & Lei (1949).

* Potter, W. D. (1949b). Effect of rainfall on magnitude and frequency of peak rates of surface runoff. *Transactions of the American Geophysical Union* **30,** 735-751.

**Summary:** The author writes (Abstract, p. 735): "This study establishes a relationship between rainfall intensity and peak rates of surface runoff, and uses that relationship to estimate the peak rates of runoff that might be expected for various recurrence intervals. The procedures thus developed are utilized to

determine the effect of intensity and amount of rainfall and number of excessive storms on the magnitude of peak rates of runoff for various frequencies. The relationships developed in the study are also utilized to adjust peak rates of runoff from watersheds of like physiography and land use for differences in rainfall." The author finds a fairly good agreement between his results at the maximum watershed condition (at which maximum peak rates of runoff could be expected) and the peak rates of runoff determined from a Gumbel probability curve [Gumbel (1945a)], even though the former considers all runoffs exceeding some arbitrary value and the latter considers only annual maxima.

**Reference:** Gumbel (1945a).

* Press, Harry (1949). The Application of the Statistical Theory of Extreme Values to Gust-Load Problems. Technical Note 1926, National Advisory Committee for Aeronautics, Washington, D.C. Superseded by NACA Report 991 (same title), 1950.

**Summary:** The author writes (Summary, p. 1 of technical note and p. 711 of report): "An analysis is presented which indicates that the statistical theory of extreme values is applicable to the problems of predicting the frequency of encountering the larger gust loads and gust velocities for both specific test conditions as well as commercial transport operations. The extreme-value theory provides an analytic form for the distributions of maximum values of gust load and velocity. Methods of fitting the distribution are given along with a method of estimating the reliability of the predictions. The theory of extreme values is applied to available load data from commercial transport operations. The results indicate that the estimates of the frequency of encountering the larger loads are more consistent with the data and more reliable than those obtained in previous analyses."

**References:** Fisher & Tippett (1928), Gumbel (1935a, 1941a, 1942c, 1945a), Kendall (1946) [1948], Kimball (1946a), Gumbel (1948).

* Prot, Marcel (1949a). La sécurité. *Annales des Ponts et Chaussées* **119,** 19-49.

**Summary:** In a section (pp. 40-41) on the influence of the dimensions of test specimens, the author points out that if one performs traction tests on specimens of wire of length $3\ell$, one should not expect the same results as for single specimens of length $\ell$, but rather the same as for [the minimum breaking strength of] groups of three specimens of length $\ell$. When the specimens are homothetic, the influence of the dimensions is less evident, but it is not nonexistent. In this case, the effect on the dispersion tends to be greater than that on the average—the smaller the specimens, the greater the dispersion.

**Citation:** Prot (1949c).

* Prot, Marcel (1949b). Essais statistiques sur mortiers et bétons. *Annales de l'Institut Technique du Bâtiment et des Travaux Publics* (Paris) (N.S.) **2** (81), 1-40.

**Summary:** The author writes (English summary, p. 1): "The mechanical properties of any material can only be said to be well known after a certain number of tests have been carried out in order to obtain a fairly precise average value on the one hand and a dispersion index on the other. This fact is now accepted almost without dispute. The question now arises as to how many tests are to be carried out in practice. The tests of which the results are given in this report answer the question as fully as possible, i.e. they indicate what degree of precision can be reached in given test conditions when the number of tests is varied. These tests were carried out using specimens of different sizes, so that attention is drawn to the effect of the size of the test pieces on the results obtained—a fact which is not well known; the dispersion index, in particular, appears to be a somewhat variable function of the similarity of test pieces. The practical conclusion of these experiments is that, if the traditional safety factors are raised to no purpose for large-sized pieces, they may on the other hand be inadequate for pieces of small thickness [since the dispersion as well as the average of the strength increases as the size decreases]." The

author reports median values and maximum positive, negative and/or absolute deviations of some of his test measurements.

**Citation:** Prot (1949c).

* Prot, Marcel (1949c). Statistique et sécurité. *Revue de Métallurgie* **46,** 716-718; discussion, 718.

**Summary:** The author repeats most of the statements made in an earlier paper [Prot (1949a)] concerning the size effect. He adds that the small test specimens tend to have slightly larger average strength than larger ones. In answer to a question by G. Valensi, he states that the problem of the distribution of the smallest of n values can be solved completely for various forms (including normal) of the initial distribution.

**References:** Prot (1949a,b).

* Purcell, W. R. (1949). Saving time in testing life. *Electrical Engineering* **68,** 617-620.

**Summary:** The author writes (Synopsis, p. 617): "By using only the median and the shortest life in statistical analysis, the time required to predict performance from a test is accelerated. This article describes how the method was used successfully for saving time in testing life of incandescent lamps; however, this method is just as applicable to many other products."

**Note:** Parts of this paper are identical with the author's earlier paper [Purcell (1947)] by the same title, to which he does not refer. However, a different example is given, along with a table of means, standard deviations and correlation coefficients for certain order statistics which was not included in the earlier paper.

**References:** Hastings, Mosteller, Tukey & Winsor (1947), Jacobson (1947).

* Scheffé, Henry (1949). Operating characteristics of average and range charts. *Industrial Quality Control* **5** (5), 13-18. (MR 11, 43).

**Summary:** The author gives theoretical results and graphs for the OC (operating characteristic) curves of $\overline{X}$ and R charts for (1) control with no standard given and (2) control with respect to a given standard. The mathematical results are relegated to an appendix.

**Reference:** Pearson [& Hartley] (1942).

* Shone, K. J. (1949). Relations between the standard deviation and the distribution of range in non-normal populations. *Journal of the Royal Statistical Society B* **11,** 85-88. (MR 11, 260).

**Summary:** The author writes (p. 85): "In B.S. 600 . . . [Pearson (1935)], the range of small samples taken from a larger parent population is used as a method for the estimation of the standard deviation of the parent population. The ratio between range and standard deviation is given for normal populations. In this publication it is indicated that the effect of non-normality is unknown. The problem is to estimate the effect of non-normality of the parent population. In the present work the following formulae are deduced for non-normal populations with a single mode: $2\sigma_p^2 = \bar{r}^2 + \sigma_r^2$ where r is range of 2 in a sample, $\bar{r}/\sigma_p = 2.10 - 0.81\sigma_r/\bar{r}$ where r is range of 3 in a sample, $\bar{r}/\sigma_p = 2.29 - 0.69\ \sigma_r/\bar{r}$ where r is range of 4 in a sample, $\bar{r}/\sigma_p = 2.41 - 0.46\ \sigma_r/\bar{r}$ where r is range of 5 in a sample and where $\sigma_p$ is the population standard deviation, $\sigma_r$ is the standard deviation of the range distribution . . . and $\bar{r}$ is the mean range." The above relation for samples of size 2 is exact, while the others are only approximate. The author states the following conclusion (p. 88): "The correcting term $\sigma_r/\bar{r}$ does seem to enable a more accurate value of $\bar{r}/\sigma_p$ to be calculated, and hence to reduce the bias that would be introduced by assuming the population to be normal. No theoretical basis for these results which are empirical has been deduced despite some effort in that direction. . . . The writer's conclusion is that though good agreement has been shown to exist,

there is no reason to expect this to be perfect; in fact, his opinion is that the formulae given are empirical only, and do not represent a near approximation to a mathematical identity. The work was started to test the validity of the use of $\bar{r}$ for determining $\sigma_p$. It has been written up for the interest of any who may be able to expound the reason for the results."

**References:** K. Pearson (1931), E. S. Pearson (1935).

* Smirnov, N. V. (1949a). On the Cramér-Mises criterion. (Russian). *Uspekhi Mathematicheskikh Nauk* (N. S.) **4** (4), 196-197. (MR 11, 261).

**Summary:** The author writes (translation of pp. 196-197): "This report gives a new approach to deriving the limiting distribution of criterion $\omega_n^2 = n\int_{-\infty}^{\infty}[S_n(x) - F(x)]^2 dF(x) = 1/12n + \Sigma_{v=1}^{n}[F(x_v - \overline{2v-1}/2n = n\int_0^1 [S'_n(y) - y]^2\, dy$, where $S_n(x)$ is the empirical distribution function constructed from n independent observations of the random variable $\xi$, which follows the continuous distribution law $F(x)$, $x_1 < x_2 < \ldots < x_n$ is the series of observed values, and $S'_n(y)$ is the empirical distribution function of value $F(\xi)$, uniformly distributed in the segment (0,1). The random function $y_n(t) = \sqrt{n}[S'_n(t) - t]$, $0 < t < 1$ is asymptotically normally distributed for large n. The characteristic function $\phi_s(u_1, u_2, \ldots, u_s)$ of the s-dimensional distribution law for random values of function $y_n(t)$ at s assigned points $0 < t_1 < t_2 < \ldots < t_s < 1$ is as follows: $\phi_s(u_1, u_2, \ldots, u_s) = \exp(-\Sigma_{j=1}^{s}\Sigma_{k=1}^{s} B_{jk}u_j u_k/2)$, where $B_{jj} = t_j(1 - t_j)$, $B_{jk} = t_j(1 - t_k)$ $(j \le k)$, $B_{jk} = t_k(1 - t_j)$ $(j \ge k)$. The random function $y_n(t)$ can be expanded into a series with respect to the inherent functions $\omega_p(t) = \sin p\pi t$ $(p = 1, 2, \ldots)$ of the kernel $B(t,s) = t(1 - s)$ $(t \le s)$, $B(t,s) = s(1 - t)$ $(t \ge s)$, $y_n(t) = \Sigma_{p=1}^{\infty} a_p^{(n)}\omega_p(t)$, converging with a probability of one. The random values $a_p^{(n)}$, $p = 1, 2, \ldots$ are maximally independent and follow the normal distribution law; here $M(a_p^{(n)}) \to 0$, $M[(a_p^{(n)})^2] \approx \lambda_p = 1/p^2\pi^2$ and $\omega_n^2 = \int_0^1 y_n^2 dt = \Sigma_{p=1}^{\infty}\{a_p^{(n)}\}^2$. Whence, it is easy to obtain $\psi_n(t) = M(e^{it\omega_n}) = \psi(t) = 1/\sqrt{\Pi_{p=1}^{\infty}(1 - 2it/p^2\pi^2)} = 1/\sqrt{\sin\sqrt{2it}/\sqrt{2it}}$."

* Smirnov, N. V. (1949b). Limit distributions for the terms of a variational series. (Russian). *Trudy Matematičeskogo Instituta imeni V. A. Steklova* **25,** 60 pp.; English translation, *American Mathematical Society Translation* No. 67 (1952), pp. 1-64. (MR 11, 605).

**Summary:** The reviewer (J. Wolfowitz) writes (MR 11, 605): "Let $Y_{(k,n)}$ be the $k^{th}$ in order of size of n independent chance variables with the same distribution function $F(x)$. This paper deals at great length with the asymptotic distribution $\Phi(x)$ of $(Y_{(k,n)} - b_n)/a_n$, where $a_n > 0$ and $b_n$ are suitable constants. Most of the results fall into two classes: (a) $n \to \infty$, k constant, (b) $n \to \infty$, $k/n \to \lambda$ (a constant), $n^{1/2}(k/n - \lambda) \to 0$. It is impossible to describe the results within the bounds of a brief review. A sample is given by theorem 7 [of Part I; see page 27 of translation]: for the case (b) $\Phi(x)$ can only be one of the following types: (1) $\Phi_\alpha^1(x) = (2\pi)^{-1/2}\int_{-\infty}^{cx^\alpha} e^{-x^2/2}dx$, $x \ge 0$, $c > 0$; $\Phi_\alpha^1(x) = 0$, $x < 0$; (2) $\Phi_\alpha^2(x) = (2\pi)^{-1/2}\int_{-\infty}^{c|x|^\alpha} e^{-x^2/2}\,dx$, $x \ge 0$, $c > 0$; $\Phi_\alpha^2(x) = 1$, $x > 0$; (3) $\Phi_\alpha^3(x) = (2\pi)^{-1/2}\int_{-\infty}^{c_1|x|^\alpha} e^{-x^2/2}dx$, $x < 0$, $c_1 > 0$; $\Phi_\alpha^3(x) = 1/2 + (2\pi)^{-1/2}\int_0^{c_2x^\alpha} e^{-x^2/2}dx$, $x > 0$, $c_2 > 0$; (4) $\Phi_\alpha^4(x) = 0$, $x \le -1$; $\Phi_\alpha^4(x) = 1/2$, $-1 < x \le 1$; $\Phi_\alpha^4(x) = 1$, $x > 1$." Part I deals with sequences of central terms [class (b) of Wolfowitz's review]. Part II deals with sequences of extreme terms with constant rank number [class (a) of the review]. The author writes (p. 48 of the translation): "We pass now to an investigation of limit distributions for sequences of extreme terms in a variational series. We shall restrict ourselves here to consideration of extreme terms with fixed left rank number k or right rank number $n-k+1$. The more general case of limiting rank equal to zero $(k/n \to 0)$ or one $(k/n \to 1)$ as $n \to \infty$ will be considered in another work. The cases $k = 1$ and $n-k+1 = 1$, corresponding to minimal and maximal terms of variational series were exhaustively studied in a work by B. V. Gnedenko [(1943)]. We shall show that the results of this work can be generalized without fundamental difficulty if with increasing n the left or right rank number keeps a constant value." He establishes the fundamental theorem (Theorem 5, page 56) allowing classification of the limit types for this case: "The proper limit laws for sequences of extreme terms with constant rank k are enumerated in the following three types: 1. $\psi_\alpha^{(k)}(x) = 0$ when $x < 0$; $\psi_\alpha^{(k)}(x) = [1/(k-1)!]\int_0^{x^\alpha} e^{-x}x^{k-1}dx$, $x > 0$, $\alpha > 0$. 2. $\Phi_\alpha^{(k)}(x) = [1/(k-1)!]\int_0^{|x|^\alpha} e^{-x}x^{k-1}dx$ when $x < 0$, $\alpha > 0$; $\Phi_\alpha^{(k)}(x) = 1$ when $x > 0$. 3. $\lambda^{(k)}(x) = [1/(k-1)!]\int_0^{e^x} e^{-x}x^{k-1}dx$, $-\infty < x < \infty$."

**References:** Gnedenko (1941, 1943), Cramér (1946).

* Tukey, John W. (1949a). Comparing individual means in the analysis of variance. *Biometrics* **5**, 99-114. (MR 11, 43).

**Summary:** This paper is the same as the author's earlier report [Tukey (1948c)] by the same title, with minor exceptions. The material on the largest gap given in the appendix of the report has been condensed somewhat and included in the body of the paper.

**References:** Fisher (1935)[1947], Snedecor (1937)[1946], Fisher & Yates (1938)[1948], Newman (1939), Duncan (1947), Nair (1948b).

* Tukey, John W. (1949b). The Simplest Signed-Rank Tests. Memorandum Report No. 17, Statistical Research Group, Princeton University, Princeton, NJ.

**Summary:** "The signed-rank test [based only on ranks, not on order statistics] for the difference in medians of two samples with paired observations was introduced by Wilcoxon . . . . (One sample can, of course, be compared with theory in the same way). Recent work in two directions by Walsh (1949a,b) throws new light on these tests, since these two different approaches turn out to be equivalent. The present memorandum presents the practical and theoretical conclusions which result from the combination of these ideas. The first part is directed toward the applications, and presents, successively, a table of critical values for the usual significance levels, examples of its use (with and without interpolation) in connection with both significance testing and the setting of confidence intervals. This is followed by a discussion of the merits of the test based on the conclusions of the second part. The second part establishes the equivalence of the two approaches, develops the distribution theory from the Wilcoxon approach, applies Walsh's idea of bounded significance level, and discusses the asymptotic and moderate sample behavior of the significance levels. The third part tabulates the exact distribution for small samples, and bounds for the significance levels of critical values given in the first part." (Author's summary, pp. 1-2). The author states that Wilcoxon's approach is simplest for significance testing, while Walsh's approach is simplest for setting confidence intervals.

**References:** Pearson [& Adyanthāya] (1929), Fisher (1935)[1947], Snedecor (1937)[1946], Kendall (1943), Daly (1946), Lord (1947), Walsh (1949a,b).

* Tukey, John W. (1949c). Scaling by and for Power-Means—The Asymptotic Means. Memorandum Report No. 25, Statistical Research Group, Princeton University, Princeton, NJ.

**Summary:** In this report and two subsequent ones [Tukey (1949e,f)], the author studies the relative efficiencies and effectivenesses (in large samples) of various estimation procedures which have scaling-like purposes when the distribution differs from normality in the direction of long tails (actually, when the underlying distribution is a contaminated distribution, a mixture of two normals with the same center and different scales).

**Note:** See note on the report by Harris & Tukey (1949).

**Reference:** Tukey (1946).

* Tukey, John W. (1949d). The Truncated Mean in Moderately Large Samples. Memorandum Report No. 32, Statistical Research Group, Princeton University, Princeton, NJ.

**Summary:** The author studies the relative efficiency, in moderately large samples, of estimators obtained from the mean and standard deviation by removing the extreme $\gamma\%$ at each end of the sample, for varying degrees of contamination.

**Note:** See note on the report by Harris & Tukey (1949).

**Reference:** Harris & Tukey (1949).

**Citation:** Harris & Tukey (1949).

* Tukey, John W. (1949e). Scaling by and for Percentiles and Exponential Averages. Memorandum Report No. 33, Statistical Research Group, Princeton, NJ.

**Summary:** See summary of the related report by Tukey (1949c).

**Note:** See note on the report by Harris & Tukey (1949).

**Reference:** Tukey (1946).

* Tukey, John W. (1949f). Skeleton Tables Related to Contaminated Distributions at Scale 3. Memorandum Report No. 34, Statistical Research Group, Princeton University, Princeton, NJ.

**Summary:** See summary of the related report by Tukey (1949c).

**Note:** See note on the report by Harris & Tukey (1949).

**Reference:** Tukey (1946).

* Walsh, John E. (1949a). Some significance tests for the median which are valid under very general conditions. *Annals of Mathematical Statistics* **20,** 64-81. (MR 10, 554).

**Summary:** The author writes (Summary, p. 64): "Order statistics are used to derive significance tests for the population median which are valid under very general conditions. These tests are approximately as powerful as the Student t-test for small samples from a normal population. Also the application of a test requires very little computation. Thus the tests derived compare very favorably with the t-test for small sets of observations. Applications of these order statistic tests to certain well known statistical problems are given in another paper [Walsh (1949c)]." The author considers n independent observations, one from each of n populations which are continuous, symmetrical, and have the same median $\phi$. He emphasizes that no two of the observations are necessarily drawn from the same population. He derives significance tests to compare $\phi$ with a given constant value $\phi_0$ (tests of the hypothesis H: $\phi = \phi_0$).

**Comments:** The ordered observations used by the author are not order statistics in the usual sense, since they may all come from different populations, about which nothing is known except that they are all continuous and symmetric about the same median.

**References:** Yule (1911) [Yule & Kendall (1947)], Fisher (1935)[1942], Scheffé & Tukey (1945), Walsh (1949c).

**Citations:** Tukey (1949b), Walsh (1949b,c,d).

* Walsh, John E. (1949b). On the range-midrange test and some tests with bounded significance levels. *Annals of Mathematical Statistics* **20**, 257-267. (MR 11, 191).

**Summary:** The author writes (Summary, p. 257): "This paper is divided into two parts. The significance tests investigated in Part I concern the population mean and are based on the quantity [D =] [(sample midrange) − (hypothetical mean)]/(sample range). The case in which the observations are a sample from a normal population is considered in detail. The tests investigated are summarized in Table 1. These tests are found to be very efficient for small samples . . . . An investigation of several extremely non-normal populations using the values of $D_\alpha$ [critical values of D at significance level $\alpha$] obtained for normality indicates that the significance level of the range-midrange test is not very sensitive to the requirement of normality for small samples . . . . Also the tests of Table 1 can be applied without computation through the use of an easily constructed graph . . . . These properties suggest that the range-midrange test is preferable to the Student t-test and the analogue of the Student t-test using the sample range (see [Daly (1946)] and [Lord (1947)]) whenever the sample size is sufficiently small. Use of the range-midrange test for the case of normality was proposed by E. S. Pearson [& Adyanthāya (1929)]. . ., where properties of the test were experimentally investigated for the normal and certain non-normal populations. In Part II several significance tests for the mean are developed which have a specified significance level for the case of a sample from a normal population but whose significance level is bounded near the specified value under very general conditions, one of which is that the observations are from continuous symmetrical populations. Some of these tests are range-midrange tests. Table 2 contains a summary of the tests and their properties . . . ."

**References**: Pearson [& Adyanthāya] (1929), Daly (1946), Lord (1947), Walsh (1949a).

**Citations:** Tukey (1948b), Wilks (1948), Tukey (1949b), Walsh (1949c).

* Walsh, John E. (1949c). Applications of some significance tests for the median which are valid under very general conditions. *Journal of the American Statistical Association* **44**, 342-355.

**Summary:** The author writes (Abstract, p. 342): "In two other papers [Walsh (1949a,b)] order statistics were used to derive some tests for the population median which have significance levels which are either exact or bounded under some very general conditions. These order statistic tests were found to be very efficient for small samples from a normal population; also they can be applied with very little computation. This paper contains applications of these tests to several well known statistical problems. Also a graphical method of applying the tests of [Walsh (1949a)] is outlined." He continues (Introduction, pp. 342-344): "The significance tests for the population median derived in [Walsh (1949a)] are valid if the n observations on which a test is based are independent and are drawn from n populations satisfying the conditions (A) 1) Each population is continuous (i.e. its cumulative distribution is continuous). 2) Each population is symmetrical. 3) The median of each population has the same value $\phi$. These tests compare $\phi$ with a given constant value $\phi_0$. The tests for comparing $\phi$ with $\phi_0$ derived in [Walsh (1949b)] are based on the assumption of a sample from a normal population. The significance level of these tests, however, is bounded near the value for normality if the n independent observations are from populations necessarily satisfying only conditions (A). An important feature of conditions (A) is that no two of the observations are necessarily drawn from the same population. Thus the order statistic tests can be applied to a wide variety of situations. The motivation for introducing tests with bounded significance levels in addition to the tests of [Walsh (1949a)] was to obtain a wider variety of suitable significance levels for small values of n without greatly weakening the generality of application. The purpose of this paper is to use these two types of tests to obtain generalized solutions for the cases of quality control slippage, and the sign test. Also direct applications of the tests are considered. . . ." With regard to the sign test, the author shows that the usual assumption that the dummy observations $x_1 - y_1, \ldots, x_n - y_n$ satisfy the condition (1) $\Pr(x_i - y_i > 0) = \Pr(x_i - y_i < 0) = 1/2$ $(i = 1, \ldots, n)$ is almost equivalent to the assumption that the dummy observations satisfy conditions (A) with zero median, which suggests that the tests of Walsh (1949a) can

be used in practical situations where the sign test is usually used. The author asserts that the tests of Walsh (1949a) are preferable to the sign test from the viewpoint of suitable significance levels and efficiency for normality.

**References**: Walsh (1949a,b).

**Citation**: Walsh (1949a).

* Walsh, John E. (1949d). Some comments on the efficiency of significance tests. *Human Biology* **21**, 205-217.

**Summary**: The author writes (Summary, pp. 205-206): "A method sometimes used to measure the efficiency of a significance test consists in associating a statistic with the test and defining the efficiency of the test to be the efficiency of the statistic considered as an estimate. This paper investigates the power function implications of the estimate method of defining the efficiency of a test for several special cases where uniformly most powerful tests exist. Of the various ways of defining the efficiency of an estimate, consideration will be limited to one definition for finite sample sizes and to two definitions for the asymptotic case. For finite sample sizes, the efficiency of an estimate is defined to be the minimum variance for such estimates divided by the variance of the given estimate. . . . Examples are presented which show that an estimate efficiency of 100 E% does not necessarily imply that the corresponding most powerful test based on 100 E% as many sample values has approximately the same power function as the given test for the admissible set of alternative hypotheses. . . ." As an example, the author considers n sample values $x_1, x_2, \ldots, x_n$ from a normal population with unknown mean $\mu$ and known variance $\sigma^2$. He denotes by $y_1, y_2, \ldots, y_n$ the values of $x_1, x_2, \ldots, x_n$ arranged in increasing order of magnitude. He compares the efficiency of the one-sided sign test of $\mu > 0$: Accept $\mu > 0$ if $y_i > 0$, where the value of i depends upon n and the desired significance level $\alpha$, with that of the most powerful one-sided test: Accept $\mu > 0$ if $\sqrt{n}\,\bar{x}/\sigma > K_\alpha$ where $K_\alpha$ is the value of the standardized normal deviate exceeded with probability $\alpha$.

**References:** Wilks (1943), Cramér (1946), Hastings, Mosteller, Tukey & Winsor (1947), Walsh (1949a).

* Weibull, Waloddi (1949). A statistical representation of fatigue failure in solids. *Transactions of the Royal Institute of Technology* (Stockholm), separate no. 27. (AMR 3#1098).

**Summary:** The author writes (Introduction, pp. 4-5): "When some ten years ago the author [Weibull (1939a,b)] published his statistical theory of the strength of materials, he was quite aware of the possibility of applying the theory to other problems than the ultimate strength of materials at statical loads. It appeared particularly promising to try the idea on fatigue problems . . . . When it first became evident—which was not very long ago—that a small stress, sufficiently repeated, was able to cause a failure, the designer based his calculations on the simple rule that the stress should be well below the endurance limit, i.e. the stress below which it is known that an infinite number of stress cycles can be borne. It seemed too hazardous to allow stresses giving a finite life. The increasing demand for more economical or less heavy constructions has actualized the question of the relation between the load and the life. It seems as though this problem was first introduced in the ball bearing design, where an infinite life would give economically prohibitive dimensions and later on in the airplane constructions, where the weight is of vital importance. Now it is not sufficient to know the load, at which failure never will occur, but also the number of cycles endured at high loads. In this way, there has been a continuous transition from statical to dynamical theory of strength. Furthermore, at all real, not idealized materials there is a dispersion of the properties of such a magnitude that statistical points of view have to be introduced. . . ." Section 1 (pp. 6-12) deals with the statistical aspects, in which order statistics play an important role. The author derives the c.d.f. of the $\nu^{th}$ order statistic of a sample of size n from a uniform distribution on the interval (0,1), plots it for $n = 1, 2, 5$; $r = 1, \ldots, n$, shows that the expected value is $\nu/(n+1)$, and discusses the sampling variation. Section 5 (pp. 31-37) deals with the size effect, employing the weakest-link theory, which is valid, the author says, for determining the effect of the length of the

specimen, but not necessarily that of its diameter or its volume. Assuming a Weibull distribution of strength, he derives an equation for the volume effect. This relation is based on the further assumption that the stress is uniform throughout the entire volume, which he admits is not satisfied for bending or torsional loads.

**References:** Weibull (1939a,b), Kontorova & Frenkel (1941) [Frenkel & Kontorova (1943)], Freudenthal (1946).

* Wolfowitz, J. (1949). Non-parametric statistical inference. *Proceedings of the Berkeley Symposium on Mathematical Statistics and Probability* (ed. J. Neyman), pp. 93-113. University of California Press, Berkeley-Los Angeles. (MR 10, 387).

**Summary**: The author surveys various methods of non-parametric inference. Many of these methods depend only on the ranks of the observations in a sample or samples, but others make use of the values of the ordered observations (order statistics) as well. The author discusses, in particular, the following: point estimation of the population cdf by the sample cdf (a step function with jumps at the order statistics); use of the theory of Kolmogoroff (1933), Smirnov (1939b), and Wald & Wolfowitz (1939) to determine confidence regions for the population cdf; the work of Thompson (1936) on the use of order statistics to set confidence intervals on the median and other percentiles; the work of Wilks (1941, 1942) and Wald (1943) on the use of order statistics in setting tolerance intervals; Kolmogoroff-Smirnov tests of hypotheses in the one-sample and two-sample cases; and the median test of Mathisen (1943), which Bowker (1944) proved not consistent.

**References:** Kolmogoroff (1933), Fisher (1935), Thompson (1936), Neyman (1937b), Smirnov (1939b), Wald & Wolfowitz (1939), Wilks (1941, 1942), Mathisen (1943), Scheffé (1943), Wald (1943), Bowker (1944).

* Yamanouchi, Ziro (1949). Estimates of the mean and standard deviation of a normal distribution from linear combinations of some chosen order statistics. (Japanese. English summary). *Bulletin of Mathematical Statistics* **3** (1-2), 52-57.

**Summary:** According to Ogawa (1951), the author makes great contributions to the theory of estimation of the mean and standard deviation of a univariate normal population by the use of systematic statistics. His results are essentially that best linear unbiased estimates are considerably more efficient than those used by Mosteller (1946).

**Note:** The compiler has been unable to obtain a copy of this paper, and hence has had to rely entirely upon Ogawa (1951) for information concerning its contents.

* Youden, W. J. (1949). The fallacy of the best two out of three. *National Bureau of Standards Technical News Bulletin* **33** (7), 77-78.

**Summary:** The author writes (pp. 77-78): "In scientific and engineering investigations a single measurement of an unknown quantity is seldom considered sufficient. Two or more measurements are usually made in order to establish a check on instrument errors, operator's errors in making readings, and reliability of the sample. These multiple measurements have two principal advantages: they reveal by their concordance the precision of the measuring process, and they make possible the use of an average of several measurements which will, in general, have a higher precision than one measurement alone. If three measurements are made, it is fairly common practice for students to take the 'best two out of three'—averaging the two values closest together and discarding the other. Recently, however, . . . Youden . . . has shown that this procedure very often leads to less precise results than the averaging of all three measurements together. . . . Intuition suggests that if two of the three measurements are in close agreement while the third stands apart considerably removed from either of the others, then there may

be grounds for suspecting and perhaps rejecting the third value. In terms of the difference between the two in good agreement, how different may be the third measurement be before it should be suspected? . . . An approximate solution has been obtained in the Bureau's statistical engineering laboratory through an empirical study of triads drawn at random from a large group of measurements constructed to conform to the characteristics of a normally distributed set entirely free from any gross errors. . . . The mathematical solution of this problem has been obtained by J. Lieblein and the frequency of occurrence for various ratios of the two differences has been calculated. . . . These results reveal that in an average of one out of every twelve sets, one of the measurements will be at least 19 times farther away from its neighbor than the difference separating the two closest. Since in every 12 sets 1 shows such a spacing for measurements with no gross observational errors, it appears that measurements that should be retained are often dropped from the record. . . ."

## REFERENCES TO POST-1949 LITERATURE

(To be Summarized in Later Volumes)


1. Lord, E. (1950). Power of the modified t-test (u-test) based on range. *Biometrika* **37,** 64-77. (MR 12, 117).

2. Ogawa, Junjiro (1951). Contributions to the theory of systematic statistics, I. *Osaka Mathematical Journal* **3,** 175-213. (MR 13, 762).

3. Bennett, Carl Allen (1952). Asymptotic Properties of Ideal Linear Estimators. Ph.D. dissertation, University of Michigan, Ann Arbor. Publication No. 3716, University Microfilms, Ann Arbor, Michigan.

4. Goodman, Leo A. (1952). Serial number analysis. *Journal of the American Statistical Association* **47,** 622-634.

5. Gumbel, E. J. (1958a). *Statistics of Extremes*. Columbia University Press, New York. (MR 20 #2826).

6. Plackett, R. L. (1958b). Studies in the history of probability and statistics. VII. The principle of the arithmetic mean. *Biometrika* **45,** 130-135.

7. Eisenhart, Churchill (1961). Boscovich and the combination of observations. *R. J. Boscovich, F. R. S.: Studies of his Life and Work* (ed. Lancelot Law Whyte), Chapter 9. Allen and Unwin, Ltd., London.

8. Kendall, M. G. (1961). Studies in the history of probability and statistics. XI. Daniel Bernoulli on maximum likelihood. *Biometrika* **48,** 1-2; C. G. Allen's English translation of papers by D. Bernoulli (1778) and by Euler (1778), 3-18. (MR 23 #A2295; STMA 4/563).

9. Eisenhart, Churchill; Deming, Lola S.; Martin, Celia S. (1963). Tables Describing Small-Sample Properties of the Mean, Median, Standard Deviation, and Other Statistics in Sampling from Various Distributions. National Bureau of Standards Technical Note No. 191. (MR 27 #2020; STMA 5/522).

10. Eisenhart, Churchill (1964). The meaning of "least" in least squares. *Journal of the Washington Academy of Sciences* **54,** 24-33. Reprinted 1969 in *Precision Measurement and Calibration*, Vol. 1: *Statistical Concepts and Procedures* (ed. Harry H. Ku), pp. 265-274. National Bureau of Standards Special Publication No. 300, U.S. Government Printing Office, Washington.

11. Sheynin, O. B. (1966a). Origin of the theory of errors. *Nature* **211,** 1003-1004. (STMA 9/872).

12. Edwards, A. W. F. (1974). A history of likelihood. *International Statistical Review* **42,** 9-15.


## ABBREVIATIONS OF NAMES OF ABSTRACTING AND REVIEWING JOURNALS

AMR = *Applied Mechanics Reviews*
JFM = *Jahrbuch über die Fortschritte der Mathematik*
MR = *Mathematical Reviews*
STMA = *Statistical Theory and Method Abstracts*

## SECTION II

## AUTHOR INDEX (ALPHABETICAL, WITH YEARS OF PUBLICATION)

(Names alphabetized according to first *capital* letter)

d'Addario, Raffaele (1930, 1931, 1932, 1934a,b, 1936, 1939)
Adrain, Robert (1808, 1818)
Adyanthāya, N. K.—See Pearson, Egon Sharpe
Afanasev, N. N. (Afanassiev, N. N.)—See Afanassiev, N. N.
Afanassiev, N. N. (1940)
Airy, George Biddell (1856, 1861)
Aitken, Alexander Craig (1937)
Aldanondo, I. (1942)
Alexandrov, A. P.; Žurkov, Serafim Nikolaevich (1933)
Alliaume, M. (1927a,b)
Amoroso, Luigi (1925)
Anderson, Oskar Nikolai (1935)
Anderson, Theodore Wilbur, Jr. (1945, 1946, 1948)
von Andrae, Carl Christopher Georg (1860, 1869, 1872)
Anonymous (1821)
Archibald, Raymond Clare (1925)
Arley, Niels Henrik (1940)
Arley, Niels Henrik; Buch, Kai Rander (1940)
Armand, H.—See Brenet, Jean-Paul
Baeyer, Johann Jacob—See Bessel, Friedrich Wilhelm
Baidaff, B. I.; Barral Souto, Jose (1934, 1935)
Baker, George Allen (1946)
Banerjee, K. S.—See Nair, K. Raghavan
Barnard, George Alfred (1943, 1944)
Barral Souto, Jose (1935, 1938)
Barral Souto, Jose—See Baidaff, B. I.
Barricelli, Nils Aall (1943)
Bartels, August Julius (1928)
Bartlett, Maurice Stevenson (1933, 1938, 1947a,b, 1948, 1949)
Baticle, Edgar (1933a,b, 1935)
Beard, Leo R. (1942)
Becker, Karl, Co-Author of 1910 and later eds. of Cranz, Carl Julius (1896) (Q. V.)
Becker, Ludwig Wilhelm Emil Ernst (1917)
van Beeck Calkoen, Jan Frederik (1816)
Benson, F. (1949)
van den Berg, F. J. (1891)
Bernoulli, Daniel (1778)
Bernoulli, Jean III (1785)
Bernoulli, Johann III (Bernoulli, Jean III)—See Bernoulli, Jean III
Bernoulli, Nicolas I (1709)
Bernoulli, Nicolaus I (Bernoulli, Nicolas I)—See Bernoulli, Nicolas I
Bertrand, Joseph Louis François (1887a,b, 1888a,b,c, 1889)
Bessel, Friedrich Wilhelm; Baeyer, Johann Jacob (1838)
Bickerstaff, Thomas Alton (1947)
Birkeland, B. J. (1928)
Birkeland, B. J.; Frogner, E. (1935)
Birnbaum, Zygmunt William; Zuckerman, Herbert Samuel (1949)
Blümcke, Gustav Adolf (1897, 1898, 1901)
Bohlmann, G. (1909)
Boldrini, Marcello (1942)
Boldrini, Marcello—See Gini, Corrado
Bond, Wilfrid Noel (1935)



PRECEDING PAGE BLANK-NOT FILMED

von Bortkiewicz, Ladislaus (1915,1922a,b, 1931)
Boscovich, Roger Joseph (1757, 1760)
Boscovich, Roger Joseph—See Maire, Christopher
Boscovich, Ruggiero Guiseppe (Boscovich, Roger Joseph)—See Boscovich, Roger Joseph
Bose, S. S. (1938)
Bottema, Oene (1925)
Boudin, E. J. (1865)
Bowker, Albert Hosmer (1944)
Bowley, Arthur Lyon (1901, 1928, 1937)
Bréger, P. (1876, 1881)
Brenet, Jean-Paul; Armand, H. (1938)
Brennan, J. F.—See Housner, George William
Bresciani-Turroni, Costantino (1937, 1939)
Brodie, Henry—See Ruggles, Richard
Brookner, R. J. (1941)
Brooks, Hamilton—See Epstein, Benjamin
Brown, George William; Tukey, John Wilder (1946)
Bruen, Curtis (1938)
Bruns, Ernst Heinrich (1882)
Brunt, David (1917)
Buch, Kai Rander—See Arley, Niels
Buchholtz, Herbert Gustav Alfred—See Faulhaber, Richard
Burgess, Robert Wilbur (1927)
Calichiopulo, Antonio (1937)
Camp, Burton Howard (1937)
Campbell, Norman (1920)
Cantelli, F. P. (1933)
Carlton, Alvar George (1946)
Castellano, Vittorio (1933a,b, 1935)
Cauchy, Augustin Louis (1824, 1831, 1837)
Chadwick, Walter B. (1926)
Chandra Sekar, C.; Francis, Mary G. (1941)
Chandra Sekar, C.—See Pearson, Egon Sharpe
Chaplin, Winfield Scott (1880, 1882)
Charlier, Carl Vilhelm Ludvig (1906, 1910)
Chauvenet, William (1863)
Chebyshev, Pafnuti Lvovich (Tchebychef, Pafnutii Lvovich)—See Tchebychef, Pafnutii Lvovich
Chung, Kai-Lai (1949)
Cisbani, Renzo (1938)
Clifford, William Kingdon (1866)
Cochran, William Gemmell (1941)
Cochran, William Gemmell, Co-Author of 1967 ed. of Snedecor, George Waddell (1937) (Q. V.)
Cohan, G. M. (1947)
Cole, Randal Hudie (1949)
Contarino, Francesco (1914)
Coolidge, Julian Lowell (1925)
Cotes, Roger (1722)
Cournot, Antoine Augustin (1843)
Coutagne, A. (1937, 1939)
Cox, David Roxbee (1948, 1949)
Craig, Allen Thornton (1932a,b, 1933, 1943)
Cramér, Harald (1928, 1946)

Cranz, Carl Julius (1896)
Creager, William P. (1939)
Crowe, Percy Robert (1933, 1936)
Crum, William Leonard (1923)
Curtiss, John Hamilton (1943)
Czuber, Emanuel (1890, 1891a,b, 1899, 1903, 1914, 1918, 1921)
Dalton, Hugh (1920)
Daly, Joseph Francis (1946)
Daniell, Percy J. (1920)
Daniels, Harry Ellis (1941, 1945)
Darmois, Georges (1936)
Davidenkov, Nikolai Nikolaevich; Shevandin, E.; Wittmann, F. (1947)
Davies, Owen Lewis; Pearson, Egon Sharpe (1934)
Davis, Dale Stroble—See Higuchi, Takeru
Davis, Harold Thayer (1941)
Delambre, Jean Baptiste Joseph (1806-10, 1813)
Delauney, Charles Eugène (1888)
*Deming, Lola Shupe—See Eisenhart, Churchill*
Deming, William Edwards (1943)
De Morgan, Augustus (1847, 1864)
Derevitsky, N. (1939)
Dettori, Giovanni (1912)
Devol, Manson Lee—See Ruark, Arthur Edward
Dirichlet, Peter Gustav Lejeune (1836)
Dodd, Edward Lewis (1913, 1914, 1922, 1923, 1938, 1940, 1942)
Dodge, Harold French (1932, 1933)
Domb, Cyril (1947)
Dominedò, Valentino (1934)
Doob, Joseph Leo (1949)
Doolittle, Myrick Hascall (1884)
Douglass, Clemeunt T.—See Wing, S. P.
Dove, Heinrich Wilhelm (1838)
Dudding, B. P.; Jennett, W. J. (1942)
Dufton, A. F. (1928)
Duncan, David Beattie (1947)
Dunkel, Otto (1909)
von Eberhard, Otto Eduard Hermann (1938)
von Eberhard, Otto Eduard Hermann, Co-Author of 1925 ed. and 1936 Erganzungsband of Cranz, Carl Julius (1896) (Q. V.)
Eddington, Arthur Stanley (1933)
Edgett, George Lewis (1931)
Edgeworth, Francis Ysidro (1883a,b, 1885, 1886a,b, 1887a,b,c,d,e,f, 1887-90, 1888, 1893, 1898, 1905, 1911, 1913, 1923)
Eells, Walter Crosby (1930)
Egudin, G. I. (1947)
Eisenhart, Churchill (1939)
Eisenhart, Churchill; Deming, Lola Shupe; Martin, Celia S. (1948a,b)
Eisenhart, Churchill; Hastay, Millard Walter; Wallis, Wilson Allen (1947)
Eisenhart, Churchill; Martin, Celia S. (1948)
Eisenhart, Churchill; Solomon, Herbert (1947)
Elfving, Gustav (1947)
Ellis, Robert Leslie (1844)

Encke, Johann Franz (1832-34)
Epstein, Benjamin (1948a,b, 1949a,b)
Epstein, Benjamin; Brooks, Hamilton (1948)
Estienne, J. E. (1890, 1900a,b, 1926-27, 1927)
Euler, Leonhard (1749, 1778)
Evans, Wilmoth Duane (1942)
Eyraud, Henri (1934a,b, 1935)
Faulhaber, Richard (1933)
Faulhaber, Richard,; Buchholtz, Herbert Gustav Alfred; Schulz, Ernst Hermann (1933)
Faye, Hervé Auguste Étienne Albans (1888)
Fechner, Gustav Theodor (1874,1897)
Feller, William (Feller, Willy)—See Feller, Willy
Feller, Willy (1940, 1948)
Ferber, Martin—See Thibaud, Jean
de Finetti, Bruno (1931a,b, 1932, 1933, 1934)
de Finetti, Bruno; Paciello, U. (1930)
Finney, David John (1941)
Fisher, John Crocker; Hollomon, John Herbert (1947)
Fisher, Ronald Aylmer (1912, 1920, 1922, 1929, 1933, 1934, 1935, 1938, 1939, 1940)
Fisher, Ronald Aylmer; Tippett, Leonard Henry Caleb (1928)
Fisher, Ronald Aylmer; Yates, Frank (1938)
Fogelson, S. (1930)
Foster, Henry Alden (1924, 1934, 1936)
Fourier, Jean Baptiste Joseph (1823, 1824a,b,c, 1826a,b, 1829, 1831)
Fowler, Franklin Harper, Jr. (1945)
de Franchis, Michele (1934)
Francis, Mary G.—See Chandra Sekar, C.
Frank, Philipp (1918)
Frankel, A.—See Kullback, Solomon
Frankel, Jacob Porter (1948)
Fréchet, Maurice René (1927,1935, 1940a,b, 1941, 1942)
Frenkel, J. I.—See Kontorova, T. A.
Freudenthal, Alfred Martin (1946)
Friedrich, Konrad (1927, 1937)
Frogner, E.—See Birkeland, B. J.
Fuller, Weston E. (1914)
Furlan, V. (1911)
Gaede, Kurt (1942)
Galilei, Galileo (1632)
Galton, Francis (1869, 1875, 1881, 1888, 1889, 1896, 1899a,b, 1902, 1907a,b,c)
Galvani, Luigi (1927, 1931, 1932, 1933)
Galvani, Luigi—See Gini, Corrado
Gartštejn, B. N. (1948)
Garver, Raymond (1932)
Garwood, F. (1940)
Gauss, Carl Friedrich (1809, 1816, 1823)
Geary, R. C. (1944, 1948)
Geil, Glenn Wallace—See McAdam, Dunlap Jamison, Jr.
Gensamer, Maxwell; Saibel, Edward; Lowrie, Robert Elmer (1947)
Gensamer, Maxwell; Saibel, Edward; Ransom, John Thompson (1947)
Gerling, Christian Ludwig (1843)
Gibrat, Robert (1930, 1931, 1932a,b)

Gillett, Horace Wadsworth (1940)
Gini, Corrado (1909, 1910, 1912, 1914a,b, 1915a,b, 1916a,b, 1917, 1918, 1926, 1930, 1932, 1936, 1938, 1939)
Gini, Corrado; Boldrini, Marcello; Galvani, Luigi; Venere, A. (1933)
Gini, Corrado; Galvani, Luigi (1929)
Girshick, Meyer A. (1936, 1939)
Glaisher, James Whitbread Lee (1872, 1873, 1874)
de Gleria, Amadio (1929, 1930)
Glivenko, V. (1933)
Gnedenko, Boris Vladimirovich (1941, 1943)
Godwin, Herbert James (1949a,b)
Goedseels, P. J. Ed. (1909, 1910, 1911, 1925)
Goedseels, P. J. Ed.; Mansion, Paul (1893)
Goldhizer, K. (1923)
Goodrich, Ralph Dickinson (1927)
Goodwin, Harry Manley (1913)
Gore, James Howard (1889)
Gosset, William Sealy—See Student (Gosset, William Sealy)
Gottschick, J. (1939)
Gould, Benjamin Apthorp, Jr. (1855)
Grant, Eugene Lodewick (1940, 1946)
Grassberger, Hans (1932, 1933, 1936)
Greenwood, Major (1946)
Griffin, Frank Loxley (1933)
Griffith, Alan Arnold (1920, 1924)
Grodsky, V. A.—See Logan, K. H.
Grubbs, Frank Ephraim; Weaver, Chalmers L. (1947)
Gumbel, Emil Julius (1927, 1928, 1932, 1933a,b,c,d,e,f,g, 1934a,b,c,d,e,f,g, 1935a,b,c,d, 1935-36, 1936a,b,c, 1936-37, 1937a,b,c,d,e, 1937-38, 1938a,b,c, 1939a,b,c,d,e,f, 1940, 1941a,b, 1942a,b,c, 1943a,b,c, 1944, 1945a,b, 1946a,b, 1947, 1948, 1949)
Gumbel, Émile-Jules (Gumbel, Emil Julius)—See Gumbel, Emil Julius
Gurney, C. (1945, 1947)
Gurney, C.; Pearson, S. (1947)
Haag, Jules (1924)
Haar, Alfred (1918)
Hagen, Gotthilf Heinrich Ludwig (1837)
Haines, Joan—See Pearson, Egon Sharpe
Hall, L. Standish (1921)
Halphen, Georges Henri (1873)
Hancock, E. L.—See Slocum, C. E.
Harris, Theodore Edward; Tukey, John Wilder (1949)
Hartley, Herman Otto (1942, 1944, 1949)
Hartley, Herman Otto—See Pearson, Egon Sharpe
Hastay, Millard Walter—See Eisenhart, Churchill
Hastings, Cecil, Jr.; Mosteller, Charles Frederick; Tukey, John Wilder; Winsor, Charles P. (1947)
Hauber, Carl Friedrich (1830, 1830-32)
Hayford, John Fillmore (1902)
Hayford, John Fillmore, Co-Author of 1906 ed. of Wright, Thomas Wallace (1884) (Q. V.)
Hazen, Allen (1914, 1930)
Helmert, Friedrich Robert (1872, 1875a,b, 1876a,b, 1877a,b)
Henry, Franklin Melville (1942)
l'Hermite, Robert Gaston (1939)

Herschel, Clemens (1878)
Herschel, John Frederick William (1850)
Hertzman, Max (1938)
Heydenreich, W. (1903)
Higuchi, Takeru; Leeper, Harold Murray; Davis, Dale Stroble (1948)
Hilferty, Margaret M.—See Wilson, Edwin Bidwell
Hill, L. R.; Schmidt, P. L. (1948)
Hoeffding, Wassily (1948)
Hoel, Paul Gerhard (1946)
Hojo, Tokishige (1931, 1933)
Hollomon, John Herbert—See Fisher, John Crocker
Holmes, George Kirby (1892, 1905)
Hooker, R. H. (1907)
Horst, Paul (1931)
Horton, Robert Elmer (1924, 1936)
Hotelling, Harold (1933, 1936a,b)
Hotelling, Harold; Solomons, Leonard M. (1932)
Housner, George William; Brennan, J. F. (1948)
Howell, John M. (1949)
Hsu, P. L. (1939, 1941a,b)
Hulme, Henry Rainsford; Symms, L. S. T. (1939)
Huzurbazar, V. S. (1948)
Insolera, Filadelfo (1931, 1933, 1934, 1935, 1937-38)
Irwin, J. O. (1925a,b)
Ivory, James (1825)
Jackson, Dunham (1921, 1923, 1924)
Jacobson, Paul Harold (1947)
Jarvis, Clarence S. (1926)
Jeffreys, Harold (1932a,b, 1933, 1934, 1938, 1939)
Jenkins, W. D.—See McAdam, Dunlap Jamison, Jr.
Jennett, W. J. —See Dudding, B. P.
Jevons, William Stanley (1874)
Jones, A. E. (1946)
Jones, D. Caradog (1921)
Jones, Howard Lewis (1948)
Jordan, Charles (1927)
Jordán, Károly (Jordan, Charles)—See Jordan, Charles
Jordan, Wilhelm (1869, 1872, 1877a,b, 1879, 1888, 1890)
Julin, Armand (1921, 1933)
Juncosa, Mario Leon (1949)
Kac, Mark (1949)
Kapteyn, Jacobus Cornelius (1903)
Kendall, Maurice George (1940, 1943, 1946)
Kendall, Maurice George, Co-Author of 1937 and later eds. of Yule, George Udny (1911) (Q. V.)
Keynes, John Maynard (1911, 1921)
Kimball, Bradford Fisher (1938, 1942, 1946a,b, 1947, 1949)
Kincaid, Wilfred MacDonald (1947)
King, Willford Isbell (1912)
Kirchberger, Paul (1903)
Knudsen, Lila F. (1943)
Kolmogoroff, Andrei Nikolaevich (1931, 1933, 1941)

Kolmogorov, Andrei Nikolaevich (Kolmogoroff, Andrei Nikolaevich)—See Kolmogoroff, Andrei Nikolaevich
Kontorova, T. A. (1940, 1943, 1946)
Kontorova, T. A.; Frenkel, J. I. (1941)
Kontorova, T. A.; Timoshenko, O. A. (1949)
Köppen, Wladimir Peter (1928)
Kozák, Josef (1907, 1908-10)
Kozakiewicz, W. (1937)
Kullback, Solomon; Frankel, A. (1940)
Lagrange, Joseph Louis (1774)
Lalande, Joseph Jérôme Le François de (1771)
Lalanne, Leon Louis Chrétien (1879)
Lambert, Johann Heinrich (1760, 1765a,b)
Lane, E. W.; Lei, Kai (1949)
Langbein, Walter B. (1949)
Laplace, Pierre Simon de (1774, 1781, 1786, 1793, 1799, 1810, 1811a,b, 1812)
Láska, W. (1900)
Laurent, Paul Mathieu Hermann (1875)
Lawley, D. N. (1940)
Leeper, Harold Murray—See Higuchi, Takeru
Legendre, Adrien Marie (1805, 1814)
Lehmann, Erich Leo (1947)
Lehmann-Filhes, R. (1887)
Lei, Kai—See Lane, E. W.
Lemoine, Émile Michel Hyacinthe (1873)
Lenz, F.; von Verschuer, Otmar (1928)
Lévi, Robert (1949)
Lévy, Paul Pierre (1925, 1929, 1939)
Lexis, W. (1875)
Lidstone, G. J. (1930)
von Lindenau, Bernhard August (1806)
Linders, F. J. (1933)
Livada, Gregorio (1942)
Lloyd, Humphrey (1855)
Logan, K. H. (1936)
Logan, K. H.; Grodsky, V. A. (1931)
Lord, E. (1947)
Lorenz, Max Otto (1905)
Lowrie, Robert Elmer—See Gensamer, Maxwell
Luyten, Willem Jacob (1932)
Madow, William Gregory (1937)
Maire, Christopher; Boscovich, Roger Joseph (1755)
Maniya, G. M. (1949)
Mansion, Paul (1906, 1913)
Mansion, Paul—See Goedseels, P. J. Ed.
Mantel, W. (1915)
March, Lucien (1930)
Martin, Celia S.—See Eisenhart, Churchill
Mather, Kenneth (1949)
Mathews, H. A. (1936)
Mathieu, Claude Louis (1813-14)
Mathisen, Harold C. (1943)

Mayer, Johann Tobias (1750)
Mazzuoli, A. (1908)
McAdam, Dunlap Jamison, Jr.; Geil, Glenn Wallace; Woodard, D. H.; Jenkins, W. D. (1948)
McKay, A. T. (1935)
McKay, A. T.; Pearson, Egon Sharpe (1933)
McMillan, Brockway (1949)
van Meer, H. P.; Plantema, Frederik Johan (1949)
Mees, R. A. (1875, 1876)
Melzler, D. G. (1949)
Mendeleev, Dmitri Ivanovich (1895)
Merriman, Mansfield (1877, 1884)
von Mises, Richard (1923, 1931, 1936, 1947)
Mitscherlich, Alfred (1903)
de Montessus de Ballore, Robert Fernand Bernard (1930)
Mood, Alexander McFarlane (1941)
Moore, H. F.; Morkovin, D. (1942-44)
Moran, Patrick Alfred Pierce (1947)
Morant, G. M. (1921)
Morkovin, D.—See Moore, H. F.
Mortara, Giorgio (1911, 1934a,b)
Mosteller, Charles Frederick (1946)
Mosteller, Charles Frederick—See Hastings, Cecil, Jr.
Muncke, Georg Wilhelm (1825)
Münzner, Hans (1934)
Murphy, Ray Bradford (1948)
Nair, K. Raghavan (1940a,b, 1947, 1948a,b, 1949a,b)
Nair, K. Raghavan; Banerjee, K. S. (1943)
Nair, K. Raghavan; Shrivastava, M. P. (1942)
Nair, U. S. (1936)
Nanda, D. N. (1948a,b)
Newcomb, Simon (1886, 1912)
Newman, D. (1939)
Neyman, Jerzy (1923, 1937a,b)
Neyman, Jerzy; Pearson, Egon Sharpe (1928, 1936)
Niceforo, Alfredo (1919)
Noether, Gottfried Emanuel (1948, 1949)
Nordquist, John Melville (1945)
Oding, Ivan Avgustovich (1946)
Ogrodnikoff, Kyrill (1928)
Olds, Edwin Glenn (1935)
Paciello, U.—See de Finetti, Bruno
Pareto, Vilfredo (1897)
Paulson, Edward (1940, 1941, 1949)
Pearson, Egon Sharpe (1926, 1932, 1935, 1939, 1942)
Pearson, Egon Sharpe; Adyanthāya, N. K. (1928, 1929)
Pearson, Egon Sharpe; Chandra Sekar, C. (1936)
Pearson, Egon Sharpe; Haines, Joan (1935)
Pearson, Egon Sharpe; Hartley, Herman Otto (1942, 1943)
Pearson, Egon Sharpe—See Davies, Owen Lewis; McKay, A. T.; Neyman, Jerzy
Pearson, Karl (1895, 1902, 1920, 1925, 1931)
Pearson, Karl; Pearson, Margaret V. (1931, 1932)
Pearson, Margaret V.—See Pearson, Karl
Pearson, S.—See Gurney, C.

Peirce, Benjamin (1852, 1878)
Peirce, Charles Sanders (1873)
Peirce, Frederick Thomas (1926)
Persons, Warren Milton (1909, 1919)
van Pesch, A. J. (1882, 1884)
Peterson, Rudolph Earl (1930, 1949a,b)
Petzval, Joseph (1857)
Pietra, Gaetano (1915, 1925, 1932a,b, 1935a,b)
de Pietri-Tonelli, Alfonso (1920)
Pillai, K. C. Sreedharan (1948)
Pitman, Edwin James George (1936, 1937, 1939)
Pizzetti, Ernesto (1940, 1941a,b,c)
Pizzetti, Paolo (1889, 1892)
Plackett, Robert Lewis (1947)
Plantema, Frederik Johan—See van Meer, H. P.
Poincaré, Jules Henri (1896)
Pollard, Harry Strange (1934)
Poncelet, Jean Victor (1835)
Porru, Emanuele (1912)
Potter, W. D. (1949a,b)
Powell, Ralph Waterbury (1943)
Press, Harry (1949)
Price, Reginald C. (1943)
Price, Walter Hugh—See Wing, S. P.
Prony, Gaspard Clair François Marie Riche de (1804)
Prot, Marcel (1949a,b,c)
Ptolemaeus, Claudius (circa 150)
Ptolemy, Claudius (Ptolemaeus, Claudius)—See Ptolemaeus, Claudius
Puissant, Louis (1805)
Purcell, Warren B. (1947, 1949)
Purcell, Warren R. (Purcell, Warren B.)—See Purcell, Warren B.
Quetelet, Lambert Adolphe Jacques (1835, 1846)
Rae, William Norman—See Reilly, Joseph
Rajalakshman, D. V. (1941, 1943)
Rammler, Helmuth Erich—See Rosin, P.
Ransom, John Thompson—See Gensamer, Maxwell
Rao, Calyampudi Radhakrishna (1948)
Reed, W. G.—See Spillman, William Jasper
Reiersöl, Olav (1944)
Reilly, Joseph; Rae, William Norman; Wheeler, Thomas Sherlock (1925)
Reinkober, O. (1931)
Rhodes, E. C. (1930, 1933)
Ricci, Umberto (1916)
Rider, Paul Reece (1929, 1933)
von Riese, C. (1830)
Rietz, Henry Lewis (1924)
Robbins, Herbert (1944a,b)
Robinson, George—See Whittaker, Edmund Taylor
Rodewald, H. (1898)
Rodgers, J. W. (1944)
Rohne, H. (1907)
Roller, Paul S. (1941)

Romanovsky, V. (1933)
Rosin, P.; Rammler, Helmuth Erich (1934)
Ross, Frank Alexander (1930)
Roy, Samarendra Nath (1939, 1940a,b, 1942a,b, 1945, 1946a,b)
Ruark, Arthur Edward (1939)
Ruark, Arthur Edward; Devol, Manson Lee (1936)
Ruggles, Richard; Brodie, Henry (1947)
Runge, Carl David Tolmé (1890)
Saibante, Mario (1926, 1928)
Saibel, Edward—See Gensamer, Maxwell
Sastry, N. Sundara Rama (1935)
Saunder, S. A. (1903)
Saville, Thorndike (1936)
Savorgnan, Franco (1915, 1921)
Savur, S. R. (1937a,b, 1938)
Scarborough, James Blaine (1930)
Scates, Douglas Edgar (1933)
Scheffé, Henry (1943, 1949)
Scheffé, Henry; Tukey, John Wilder (1944, 1945)
von Schelling, Hermann (1941)
Schmidt, P. L.—See Hill, L. R.
Schott, Charles Anthony (1878)
Schreiber, O. (1882)
Schulz, Ernst Hermann—See Faulhaber, Richard
Schützenberger, Marcel-Paul (Schützenberger, Marco P.)—See Schützenberger, Marco P.
Schützenberger, Marco P. (1948)
Secrist, Horace (1917)
Sheppard, William Fleetwood (1899a,b)
Shevandin, E.—See Davidenkov, Nikolai Nikolaevich
Shewhart, Walter Andrew (1931)
Shone, K. J. (1949)
Short, James (1763)
Shrivastava, M. P.—See Nair, K. Raghavan
Simon, Leslie Earl (1941)
Simpson, Thomas (1756, 1757)
Singleton, Robert Richmond (1940)
Slade, James Jeremiah, Jr. (1934, 1936)
Sloane, Charles Swift (1923)
Slocum, C. E.; Hancock, E. L. (1906)
Smirnoff, Nikolai Vasilovich (1935, 1936, 1937a,b, 1939a,b,c, 1941, 1944, 1948, 1949a,b)
Smirnov, Nikolai Vasilovich (Smirnoff, Nikolai Vasilovich)—See Smiroff, Nikolai Vasilovich
Smith, David Eugene (1929)
Smith, Thomas (1934)
Snedecor, George Waddell (1937)
Solomon, Herbert—See Eisenhart, Churchill
Solomons, Leonard M.—See Hotelling, Harold
Sophister (1928)
Spillman, William Jasper; Tolley, Howard Ross; Reed, W. G. (1916)
Spława-Neyman, Jerzy (Neyman, Jerzy)—See Neyman, Jerzy
Stampfer, Simon (1839)
Steffensen, Johan Frederik (1930, 1931)
Stevens, W. L. (1939, 1948)

Stewart, R. M. (1920a,b)
Stone, Edward James (1868, 1873a,b, 1874)
Student (Gosset, William Sealy) (1927, 1931)
Sukhatme, Pandurang Vasudeo (1936, 1937)
Svanberg, Jöns (1805)
Symms, L. S. T.—See Hulme, Henry Rainsford
Tanenhaus, Seaman J. (1947)
Tchebychef, Pafnutii Lvovich (1854)
Thibaud, Jean (1937)
Thibaud, Jean; Ferber, Martin (1937)
Thomas, Harold Allen, Jr. (1948)
Thompson, Catherine M. (1941)
Thompson, William Rae (1935, 1936, 1938)
Thomson, Godfrey H. (1934)
Timoshenko, O. A.—See Kontorova, T. A.
Tintner, Gerhard (1945, 1946)
Tippett, Leonard Henry Caleb (1925)
Tippett, Leonard Henry Caleb—See Fisher, Ronald Aylmer
Tits, L. (1912)
Todhunter, Isaac (1865, 1873)
Tolley, Howard Ross (1916)
Tolley, Howard Ross—See Spillman, William Jasper
Trachtenberg, M. I. (1915)
Trembley, Jean (1804)
Tricomi, Francesco Giacomo (1933)
Tucker, John, Jr. (1927, 1941, 1945a,b)
Tukey, John Wilder (1946, 1947, 1948a,b,c, 1949a,b,c,d,e,f)
Tukey, John Wilder—See Brown, George William; Harris, Theodore Edward; Hastings, Cecil, Jr.; Scheffé, Henry
Turner, Herbert Hall (1887)
van Uven, M. J., Co-Author of 1916 ed. of Kapteyn, Jacobus Cornelius (1903) (Q. V.)
de la Vallée Poussin, Charles Jean Gustave Nicolas (1909, 1911, 1919)
Vallier, E. (1894)
Venere, A.—See Gini, Corrado
Venn, John (1866, 1891)
de Vergottini, Mario (1939, 1940)
von Verschuer, Otmar (1927)
von Verschuer, Otmar—See Lenz, F.
Ville, Jean-Andre (1936)
Vinci, Felice (1918)
Vogeler, R. (1907)
Wald, Abraham (1939, 1940, 1942, 1943)
Wald, Abraham; Wolfowitz, Jacob (1939, 1941)
Walker, Gilbert T. (1914)
Wallis, Wilson Allen—See Eisenhart, Churchill
Walsh, John Edward (1946a,b, 1947, 1949a,b,c,d)
Waschakidse, D. (1938)
Watkins, G. P. (1905, 1908, 1909, 1933)
Weaver, Chalmers L.—See Grubbs, Frank Ephraim
Weibull, Ernst Hjalmar Waloddi (1939a,b, 1949)
Weinberg, W. (1916)
Weinberg, Wilhelm (1933)

Wellisch, Siegmund (1909, 1913)
Wheeler, Thomas Sherlock—See Reilly, Joseph
Whittaker, Edmund Taylor; Robinson, George (1924)
Whitworth, William Allen (1867, 1897)
Wilks, Samuel Stanley (1938, 1941, 1942, 1943, 1948)
Wilson, Edwin Bidwell (1923, 1940)
Wilson, Edwin Bidwell; Hilferty, Margaret M. (1929)
Wilson, J. M. (1877)
Wing, S. P.; Price, Walter Hugh; Douglass, Clemeunt T. (1944)
Winkler, Wilhelm (1924)
Winlock, Joseph (1856)
Winsor, Charles P.—See Hastings, Cecil, Jr.
Winsten, C. B. (1946)
Wishart, John (1938)
Wittmann, F.—See Davidenkov, Nikolai Nikolaevich
Wold, Herman (1935, 1937)
Wölffing, E. (1899)
Wolfowitz, Jacob (1949)
Wolfowitz, Jacob—See Wald, Abraham
Woodard, D. H.—See McAdam, Dunlap Jamison, Jr.
Wright, Thomas Wallace (1884)
Yamanouchi, Ziro (1949)
Yang, Simon (1933)
Yates, Frank—See Fisher, Ronald Aylmer
Yntema, Dwight Baldwin (1933)
Youden, William John (1949)
Yule, George Udny (1896a,b, 1907, 1911)
von Zach, Franz Xaver (1805)
Zachariae, Georg Carl Christian (1871)
Zappa, Guido (1939, 1940)
Zhurkov, Serafim Nikolaevich (Žurkov, Serafim Nikolaevich)—See Žurkov, Serafim Nikolaevich
Zuckerman, Herbert Samuel—See Birnbaum, Zygmunt William
Žurkov, Serafim Nikolaevich—See Alexandrov, A. P.

## SECTION III

## SUBJECT INDEX

## (PERTINENT PUBLICATIONS LISTED CHRONOLOGICALLY)

ABSOLUTE DEVIATIONS, MEAN OF—SEE AVERAGE (ABSOLUTE) DEVIATION

ABSOLUTE DEVIATIONS, MEDIAN OF—SEE MEDIAN OF ABSOLUTE DEVIATIONS

ACCIDENT STATISTICS, APPLICATIONS TO
Sukhatme (1936), Grant (1946), Kendall (1946)

ACCURACY, INTRINSIC—SEE INTRINSIC ACCURACY

AERONAUTICS, APPLICATIONS TO
Press (1949)

AGE, LIMITING—SEE LIMITING AGE

AGE, OLDEST—SEE LIMITING AGE

AGRICULTURE, APPLICATIONS TO
Rodewald (1898), Mitscherlich (1903), Charlier (1906), Yule (1911), Gini (1914a), Czuber (1918),Gini (1918), von Bortkiewicz (1922b), Sloane (1923), Rietz (1924), Student (1927), Gibrat (1930), March (1930), de Montessus de Ballore (1930), Rhodes (1930), Student (1931), Pietra (1932a,b), Fisher (1935), Madow (1937), Neyman (1937b), Savur (1937b), Snedecor (1937), Bruen (1938), Fisher & Yates (1938), Wishart (1938), Derevitsky (1939), Gini (1939), Newman (1939), Pearson (1939), Boldrini (1942), Kendall (1943), Cramér (1946), Kendall (1946), Tintner (1946), Bartlett (1947b), Duncan (1947), Nair (1948a,b), Tukey (1948c), Paulson (1949), Tukey (1949a)

AIRCRAFT INDUSTRY, APPLICATIONS TO
Grant (1946), Weibull (1949)

AIRY'S METHOD OF TREATING OUTLIERS—SEE OUTLIERS, AIRY'S METHOD OF TREATING

ALGEBRAIC DISTRIBUTION, JORDAN'S FIRST—SEE JORDAN'S ALGEBRAIC DISTRIBUTION, FIRST

ALGEBRAIC DISTRIBUTION, JORDAN'S SECOND—SEE JORDAN'S ALGEBRAIC DISTRIBUTION, SECOND

ALL ORDER STATISTICS OF SAMPLE—SEE ORDER STATISTICS OF SAMPLE, ALL

ALPHA—SEE ERROR RATE OF A TEST, TYPE I

AMOROSO'S DISTRIBUTION
Amoroso (1925), d'Addario (1932, 1934b, 1936)

ANALYSIS, CANONICAL—SEE CANONICAL ANALYSIS

ANALYSIS, DISCRIMINANT—SEE DISCRIMINANT ANALYSIS

ANALYSIS, HARMONIC—SEE HARMONIC ANALYSIS

ANALYSIS OF COVARIANCE
Snedecor (1937), Wilks (1943), Kendall (1946), Bartlett (1947b, 1948), Rao (1948)



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ANALYSIS OF VARIANCE (ANOVA OR MANOVA)—SEE ALSO K-SAMPLE TESTS
Student (1931), Gini (1932), Davies & Pearson (1934), Fisher (1935), Snedecor (1937), Roy (1942b), Scheffé (1943), Wilks (1943), Rodgers (1944), Anderson (1946), Cramér (1946), Greenwood (1946), Kendall (1946), Roy (1946a,b), Tukey (1946), Bartlett (1947b), Duncan (1947), Bartlett (1948), Geary (1948), Nair (1948a), Rao (1948), Tukey (1948c), Wilks (1948), Hartley (1949), Tukey (1949a)

ANTHROPOMETRY, APPLICATIONS TO
Quetelet (1835, 1846), Herschel (1850), Galton (1869, 1875, 1881), Edgeworth (1885, 1887e), Galton (1888), Pearson (1895), Sheppard (1899b), Bowley (1901), Yule (1911), Gini (1912), King (1912), Gini (1914a), Weinberg (1916), Gini (1917), Czuber (1921), Julin (1921), von Bortkiewicz (1922a,b), Rietz (1924), Whittaker & Robinson (1924), Irwin (1925b), Pietra (1925), Gini (1926), von Verschuer (1927), Cramér (1928), Lenz & von Verschuer (1928), Neyman & Pearson (1928), von Mises (1931), Pearson (1931), Castellano (1933b), Smith (1934), Savur (1937b), Snedecor (1937), Bruen (1938), Gini (1939), Pearson (1939), Pizzetti (1941a), Boldrini (1942), Kendall (1943), Cramér (1946), Bartlett (1947b)

ANY CONTINUOUS DISTRIBUTION—SEE CONTINUOUS DISTRIBUTION, ANY

ANY KNOWN DISTRIBUTION—SEE KNOWN DISTRIBUTION, ANY

APPLICATIONS—SEE (NAME OF FIELD), APPLICATIONS TO

APPROXIMATION, MINIMUM—SEE MINIMAX METHOD (MINIMUM APPROXIMATION)

APPROXIMATION, QUADRATIC—SEE QUADRATIC APPROXIMATION

AREA OF CONCENTRATION—SEE CONCENTRATION, AREA OF

ASCENDING MEANS—SEE MEANS, ASCENDING

ASTRONOMY, APPLICATIONS TO
Ptolemaeus (circa 150), Galilei (1632), Euler (1749), Mayer (1750), Simpson (1756, 1757), Short (1763), Lalande (1771), Laplace (1774, 1799), Legendre (1805), von Zach (1805), Gauss (1809), Laplace (1812), Delambre (1813), Quetelet (1846), Airy (1861), Chauvenet (1863), Stone (1868), Jordan (1869), von Andrae (1872), Glaisher (1872), Stone (1873b), Merriman (1877), Newcomb (1886), Edgeworth (1887e), Lehmann-Filhes (1887), Turner (1887), Saunder (1903), Mansion (1906), Goedseels (1909, 1911), Newcomb (1912), Mansion (1913), Contarino (1914), Stewart (1920a), Keynes (1921), Edgeworth (1923), Wilson (1923), Whittaker & Robinson (1924), Goedseels (1925), Irwin (1925b), Ogrodnikoff (1928), Smith (1929), Scarborough (1930), Pearson (1931), Luyten (1932), Eddington (1933), Rider (1933), Calichiopulo (1937), Jeffreys (1938), Hulme & Symms (1939), Jeffreys (1939), Kendall (1946)

ASYMMETRIC DISTRIBUTION, UNSPECIFIED
Galton (1896), Yule (1896a,b), Edgeworth (1898), Hall (1921), Goodrich (1927), Gumbel (1943a), Kendall (1943), Gumbel (1944), Gurney (1945), Baker (1946), Cramér (1946), Kendall (1946), Gumbel (1947), Tanenhaus (1947), Epstein (1948b), Higuchi, Leeper & Davis (1948), Rao (1948), Cole (1949), Lane & Lei (1949)

ASYMPTOTIC DISTRIBUTIONS
Dodd (1923), Lévy (1925), Fréchet (1927), Fisher & Tippett (1928), Lévy (1929), Gnedenko (1943), Carlton (1946), Cramér (1946), Kendall (1946), Gumbel (1947), Hastings, Mosteller, Tukey & Winsor (1947), von Mises (1947), Anderson (1948), Epstein & Brooks (1948), Feller (1948), Gartštein (1948), Hoeffding (1948), Nanda (1948b), Wilks (1948), Gumbel (1949), Juncosa (1949), Meĭzler (1949), Smirnov (1949a,b), Walsh (1949d)

AVERAGE (ABSOLUTE) DEVIATION (FROM MEDIAN)

Lalande (1771), Boudin (1865), Venn (1866), Helmert (1875a, 1876a), Bréger (1881), Edgeworth (1887f), Gore (1889), Bowley (1901), Watkins (1909), Edgeworth (1911), Yule (1911), Dettori (1912), Gini (1912), King (1912), Czuber (1914), Gini (1914a), Savorgnan (1915), Secrist (1917), Niceforo (1919), Dalton (1920), Daniell (1920), Fisher (1920), de Pietri-Tonelli (1920), Czuber (1921), Jones (1921), Julin (1921), Rietz (1924), Winkler (1924), Estienne (1926-27), Burgess (1927), Jordan (1927), Dufton (1928), Lenz & von Verschuer (1928), d'Addario (1930), March (1930), Scarborough (1930), von Bortkiewicz (1931), Galvani (1931), Horst (1931), Gini (1932), Pietra (1932a,b), Julin (1933), Rhodes (1933), d'Addario (1934b), Dominedò (1934), Mortara (1934b), Münzner (1934), Anderson (1935), Bond (1935), Castellano (1935), Pietra (1935a,b), Wold (1935), d'Addario (1936), Gini (1936), Nair (1936), Pitman (1937), Barral Souto (1938), Bruen (1938), Bresciani-Turroni (1939), Gini (1939), Gottschick (1939), Jeffreys (1939), Pitman (1939), Arley & Buch (1940), Fréchet (1940a,b, 1941), Pizzetti (1941c), Simon (1941), Boldrini (1942), Dudding & Jennett (1942), Fréchet (1942), Gumbel (1942c), Kendall (1943), Reiersöl (1944), Cramér (1946), Kendall (1946), Kimball (1946a), Mosteller (1946), Tukey (1946), Nair (1947), Eisenhart & Martin (1948), Godwin (1949b), Nair (1949a,b)

AVERAGE OF QUARTILES—SEE QUARTILE AVERAGE

AVERAGES (MEASURES OF CENTRAL TENDENCY)

Ptolemaeus (circa 150), Cotes (1722), Simpson (1756), Boscovich (1757), Simpson (1757), Lambert (1760), Short (1763), Lambert (1765a,b), Lalande (1771), Lagrange (1774), Laplace (1774), Bernoulli (1778), Euler (1778), Laplace (1781), Bernoulli (1785), Laplace (1786, 1799), Trembley (1804), von Zach (1805), Delambre (1806-10), Adrain (1808), Gauss (1809), Laplace (1810, 1811a,b, 1812), Delambre (1813), Mathieu (1813-14), Legendre (1814), van Beeck Calkoen (1816), Anonymous (1821), Cauchy (1824), Fourier (1824c), Ivory (1825), Muncke (1825), Fourier (1826a, 1829), Hauber (1830-32), Cauchy (1831), Encke (1832-34), Quetelet (1835), Dirichlet (1836), Cauchy (1837), Hagen (1837), Stampfer (1839), Cournot (1843), Gerling (1843), Ellis (1844), Quetelet (1846), Herschel (1850), Lloyd (1855), Petzval (1857), von Andrae (1860), Chauvenet (1863), De Morgan (1864), Boudin (1865), Todhunter (1865), Venn (1866), Galton (1869), Zachariae (1871), Glaisher (1873), Peirce (1873), Stone (1873a,b), Fechner (1874), Glaisher (1874), Jevons (1874), Stone (1874), Galton (1875), Laurent (1875), Bréger (1876), Merriman (1877), Wilson (1877), Herschel (1878), Lalanne (1879), Chaplin (1880), Bréger (1881), Galton (1881), Bruns (1882), Chaplin (1882), Schreiber (1882), Edgeworth (1883a,b), Doolittle (1884), Wright (1884), Edgeworth (1885, 1886a,b), Newcomb (1886), Edgeworth (1887a,c,d,e,f, 1887-90), Turner (1887), Bertrand (1888a,b), Delauney (1888), Edgeworth (1888), Faye (1888), Galton (1888), Jordan (1888), Bertrand (1889), Galton (1889), Gore (1889), Pizzetti (1889), Czuber (1890), Estienne (1890), Runge (1890), Czuber (1891a,b), Venn (1891), Holmes (1892), Pizzetti (1892), Edgeworth (1893), Vallier (1894), Mendeleev (1895), Pearson (1895), Cranz (1896), Galton (1896), Poincaré (1896), Yule (1896a,b), Blümcke (1897), Fechner (1897), Edgeworth (1898), Rodewald (1898), Czuber (1899), Galton (1899a,b), Sheppard (1899a,b), Wölffing (1899), Estienne (1900a,b), Láska (1900), Bowley (1901), Galton (1902), Hayford (1902), Pearson (1902), Czuber (1903), Heydenreich (1903), Kapteyn (1903), Mitscherlich (1903), Saunder (1903), Edgeworth (1905), Holmes (1905), Lorenz (1905), Watkins (1905), Charlier (1906), Mansion (1906), Slocum & Hancock (1906), Galton (1907a,b,c), Hooker (1907), Rohne (1907), Yule (1907), Mazzuoli (1908), Watkins (1908), Bohlmann (1909), Dunkel (1909), Goedseels (1909), Persons (1909), de la Vallée Poussin (1909), Watkins (1909), Wellisch (1909), Charlier (1910), Gini (1910), Goedseels (1910), Edgeworth (1911), Goedseels (1911), Keynes (1911), Mortara (1911), de la Vallée Poussin (1911), Yule (1911), Dettori (1912), Fisher (1912), Gini (1912), King (1912), Newcomb (1912), Tits (1912), Dodd (1913), Edgeworth (1913), Goodwin (1913), Mansion (1913), Wellisch (1913), Contarino (1914), Czuber (1914), Dodd (1914), Fuller (1914), Gini (1914a,b), Hazen (1914) Gini (1915a,b), Pietra (1915), Savorgnan (1915), Trachtenberg (1915), Gini (1916a,b), Ricci (1916), Spillman, Tolley & Reed (1916), Tolley (1916), Becker (1917), Brunt (1917), Gini (1917), Secrist (1917), Gini (1918), Niceforo (1919), Persons (1919), Campbell (1920), Dalton (1920), Daniell (1920), Fisher (1920), Pearson (1920), de Pietri-Tonelli (1920), Stewart

(1920a,b), Czuber (1921), Hall (1921), Jackson (1921), Jones (1921), Julin (1921), Keynes (1921), Savorgnan (1921), von Bortkiewicz (1922a,b), Dodd (1922), Fisher (1922), Crum (1923), Dodd (1923), Edgeworth (1923), Goldhizer (1923), Jackson (1923), von Mises (1923), Sloane (1923), Neyman (1923), Wilson (1923), Foster (1924), Haag (1924), Horton (1924), Jackson (1924), Rietz (1924), Whittaker & Robinson (1924), Winkler (1924), Bottema (1925), Coolidge (1925), Goedseels (1925), Irwin (1925a,b), Lévy (1925), Pearson (1925), Pietra (1925), Reilly, Rae & Wheeler (1925), Tippett (1925), Estienne (1926-27), Gini (1926), Jarvis (1926), Pearson (1926), Peirce (1926), Alliaume (1927a,b), Burgess (1927), Estienne (1927), Friedrich (1927), Galvani (1927), Goodrich (1927), Gumbel (1927), Jordan (1927), Student (1927), Birkeland (1928), Bowley (1928), Cramér (1928), Dufton (1928), Gumbel (1928), Köppen (1928), Neyman & Pearson (1928), Ogrodnikoff (1928), Pearson & Adyanthāya (1928), Sophister (1928), Gini & Galvani (1929), Lévy (1929), Pearson & Adyanthāya (1929), Rider (1929), Smith (1929), Wilson & Hilferty (1929), d'Addario (1930), Eells (1930), de Finetti & Paciello (1930), Fogelson (1930), Gibrat (1930), Gini (1930), Hazen (1930), Lidstone (1930), March (1930), de Montessus de Ballore (1930), Rhodes (1930), Ross (1930), Scarborough (1930), d'Addario (1931), von Bortkiewicz (1931), Edgett (1931), de Finetti (1931a,b), Galvani (1931), Gibrat (1931), Hojo (1931), Horst (1931), Kolmogoroff (1931), von Mises (1931), Pearson (1931), Pearson & Pearson (1931), Shewhart (1931), Student (1931), d'Addario (1932), Craig (1932a,b), Dodge (1932), de Finetti (1932), Galvani (1932), Garver (1932), Gibrat (1932a,b), Gini (1932), Gumbel (1932), Hotelling & Solomons (1932), Jeffreys (1932a,b), Luyten (1932), Pearson (1932), Pearson & Pearson (1932), Pietra (1932a,b), Bartlett (1933), Castellano (1933a,b), Craig (1933), Crowe (1933), Dodge (1933), Eddington (1933), Galvani (1933), Gini, Boldrini, Galvani & Venere (1933), Grassberger (1933), Griffin (1933), Gumbel (1933a,b,c,d,f,g), Hojo (1933), Jeffreys (1933), Julin (1933), Linders (1933), Rhodes (1933), Rider (1933), Romanovsky (1933), Scates (1933), Tricomi (1933), Yang (1933), Yntema (1933), Baidaff & Barral Souto (1934), Davies & Pearson (1934), Eyraud (1934a,b), Foster (1934), Gumbel (1934 a,c,d,e,f,g), Jeffreys (1934), Mortara (1934a), Münzner (1934), Pollard (1934), Rosin & Rammler (1934), Slade (1934), Smith (1934), Anderson (1935), Baidaff & Barral Souto (1935), Barral Souto (1935), Birkeland & Frogner (1935), Bond (1935), Castellano (1935), Fréchet (1935), Gumbel (1935a,b,c,d, 1935-36), McKay (1935), Olds (1935), Pearson (1935), Pearson & Haines (1935), Pietra (1935a,b), Sastry (1935), Smirnoff (1935), Thompson (1935), Wold (1935), d'Addario (1936), Crowe (1936), Darmois (1936), Foster (1936), Gini (1936), Grassberger (1936), Gumbel (1936a,b,c, 1936-37), Logan (1936), Mathews (1936), Pearson & Chandra Sekar (1936), Saville (1936), Slade (1936), Ville (1936), Bowley (1937), Bresciani-Turroni (1937), Calichiopulo (1937), Coutagne (1937), Friedrich (1937), Gumbel (1937a,b,c,d,e, 1937-38), Pitman (1937), Savur (1937a,b), Snedecor (1937), Thibaud & Ferber (1937), Wold (1937), Barral Souto (1938), Bose (1938), Brenet & Armand (1938), Bruen (1938), Cisbani (1938), Dodd (1938), von Eberhard (1938), Gini (1938), Gumbel (1938a,b,c), Jeffreys (1938), Kimball (1938), Savur (1938), d'Addario (1939), Coutagne (1939), Derevitsky (1939), Eisenhart (1939), Gini (1939), Gumbel (1939a,b,c,d,e,f), l'Hermite (1939), Hsu (1939), Hulme & Symms (1939), Jeffreys (1939), Pitman (1939), de Vergottini (1939), Wald (1939), Zappa (1939), Arley (1940), Arley & Buch (1940), Dodd (1940), Fréchet (1940a), Nair (1940a), Pizzetti (1940), de Vergottini (1940), Wald (1940), Wilson (1940), Zappa (1940), Brookner (1941), Daniels (1941), Fréchet (1941), Gumbel (1941a,b), Mood (1941), Pizzetti (1941a,b,c), Roller (1941), von Schelling (1941), Simon (1941), Smirnoff (1941), Aldanondo (1942), Beard (1942), Boldrini (1942), Dudding & Jennett (1942), Evans (1942), Fréchet (1942), Gaede (1942), Gumbel (1942a,b,c), Hartley (1942), Henry (1942), Kimball (1942), Livada (1942), Nair & Shrivastava (1942), Pearson & Hartley (1942), Roy (1942a,b), Barnard (1943), Barricelli (1943), Craig (1943), Curtiss (1943), Gumbel (1943a,c), Kendall (1943), Knudsen (1943), Nair & Banerjee (1943), Powell (1943), Rajalakshman (1943), Scheffé (1943), Wilks (1943), Geary (1944), Gumbel (1944), Hartley (1944), Robbins (1944b), Rodgers (1944), Wing, Price & Douglass (1944), Gumbel (1945a), Gurney (1945), Anderson (1946), Baker (1946), Brown & Tukey (1946), Carlton (1946), Cramér (1946), Daly (1946), Freudenthal (1946), Grant (1946), Greenwood (1946), Gumbel (1946b), Hoel (1946), Jones (1946), Kendall (1946), Kimball (1946a,b), Kontorova (1946), Mosteller (1946), Oding (1946), Roy (1946a,b), Tintner (1946), Tukey (1946), Walsh (1946a),

Winsten (1946), Bartlett (1947a,b), Bickerstaff (1947), Cohan (1947), Davidenkov, Shevandin & Wittmann (1947), Duncan (1947), Egudin (1947), Elfving (1947), Fisher & Hollomon (1947), Grubbs & Weaver (1947), Gumbel (1947), Gurney (1947), Gurney & Pearson (1947), Hastings, Mosteller, Tukey & Winsor (1947), Jacobson (1947), Kimball (1947), Lord (1947), Nair (1947), Purcell (1947), Ruggles & Brodie (1947), Tanenhaus (1947), Walsh (1947), Anderson (1948), Cox (1948), Eisenhart, Deming & Martin (1948b), Eisenhart & Martin (1948), Epstein (1948a,b), Epstein & Brooks (1948), Frankel (1948), Geary (1948), Gumbel (1948), Higuchi, Leeper & Davis (1948), Hill & Schmidt (1948), Hoeffding (1948), Housner & Brennan (1948), Jones (1948), Nair (1948a,b), Pillai (1948), Rao (1948), Stevens (1948), Thomas (1948), Tukey (1948b,c), Wilks (1948), Bartlett (1949), Benson (1949), Cole (1949), Cox (1949), Godwin (1949a,b), Gumbel (1949), Harris & Tukey (1949), Hartley (1949), Howell (1949), Kimball (1949), Kontorova & Timoshenko (1949), Lane & Lei (1949), Langbein (1949), Mather (1949), van Meer & Plantema (1949), Nair (1949a,b), Noether (1949), Paulson (1949), Peterson (1949a,b), Potter (1949a,b), Press (1949), Prot (1949a,b,c), Purcell (1949), Scheffé (1949), Shone (1949), Tukey (1949a,b,c,d,e,f), Walsh (1949a,b,c,d), Weibull (1949), Wolfowitz (1949), Yamanouchi (1949), Youden (1949)

AVERAGES, DISCARD—SEE DISCARD AVERAGES

AVERAGES, METHOD OF—SEE METHOD OF AVERAGES

AVERAGES, WALSH—SEE WALSH AVERAGES

AVERAGES, WEIGHTED—SEE WEIGHTED AVERAGES

BALLISTICS AND GUNNERY, APPLICATIONS TO—SEE ALSO MISS DISTANCE, APPLICATIONS TO

Cranz (1896), Estienne (1900b), Heydenreich (1903), Rohne (1907), Kozák (1908-10), Mazzuoli (1908), Estienne (1926-27), Scarborough (1930), Bond (1935), von Eberhard (1938), Arley (1940), Arley & Buch (1940), Simon (1941), Hartley (1942), Gumbel (1945a), Grant (1946), Hoel (1946)

BERNOULLI DISTRIBUTION—SEE BINOMIAL DISTRIBUTION

BERTRAND'S CRITERION FOR REJECTION OF OUTLIERS—SEE OUTLIERS, BERTRAND'S CRITERION FOR REJECTION

BEST TWO OUT OF THREE

Youden (1949)

BETA—SEE ERROR RATE OF A TEST, TYPE II

BETA DISTRIBUTION

Eyraud (1935), Pitman (1939), Thompson (1941), Scheffé (1943), Cramér (1946), Bickerstaff (1947), Eisenhart, Deming & Martin (1948a), Murphy (1948), Noether (1948), Wilks (1948), Noether (1949)

BIMODAL DISTRIBUTIONS

Baker (1946), Cramér (1946), Grant (1946), Kendall (1946), Tanenhaus (1947)

BINOMIAL DISTRIBUTION

Galton (1875), Lexis (1875), Merriman (1877), Pizzetti (1892), Rodewald (1898), Edgeworth (1905), Bohlmann (1909), Edgeworth (1911), Gini (1912), Fisher (1922), Lenz & von Verschuer (1928), Rider (1929), de Montessus de Ballore (1930), Weinberg (1933), Eyraud (1934b), Gumbel (1934e, 1935a), Snedecor (1937), Gini (1939), Gottschick (1939), Smirnov (1939c), Grant (1940), Curtiss (1943),

Scheffé (1943), Wilks (1943), Cramér (1946), Grant (1946), Kendall (1946), Feller (1948), Noether (1948), Tukey (1948b), Cox (1949), Noether (1949)

BIOMETRICS, APPLICATIONS TO
Pearson (1895), Kapteyn (1903), Charlier (1906), King (1912), Edgeworth (1913), Gini (1914b), Weinberg (1916), Niceforo (1919), Czuber (1921), Jones (1921), Julin (1921), Fisher (1922), Wilson (1923), Whittaker & Robinson (1924), Pearson (1925), Gini (1926), Burgess (1927), Goodrich (1927), Cramér (1928), Steffensen (1930), Gumbel (1933e,g), Julin (1933), Snedecor (1937), Fisher (1938), Fisher & Yates (1938), Thompson (1938), Eisenhart (1939), Pearson (1939), von Schelling (1941), Boldrini (1942), Kendall (1943), Cramér (1946), Greenwood (1946), Kendall (1946), Lord (1947), Bartlett (1948), Rao (1948), Bartlett (1949), Cole (1949), Mather (1949), Noether (1949), Tukey (1949b)

BIVARIATE DISTRIBUTIONS
Hayford (1902), Sloane (1923), Gini & Galvani (1929), Eells (1930), Ross (1930), Galvani (1933), Gini, Boldrini, Galvani & Venere (1933), Griffin (1933), Linders (1933), Scates (1933), Zappa (1939), Wald (1943), Wilks (1943), Cramér (1946), Kendall (1946), Mosteller (1946), Bickerstaff (1947), Tukey (1947), Wilks (1948)

BONFERRONI'S RATIO OF CONCENTRATION—SEE CONCENTRATION RATIO, BONFERRONI'S

BOUNDS, CONFIDENCE— SEE CONFIDENCE BOUNDS

BOWLEY'S QUARTILE MEASURE OF SKEWNESS—SEE SKEWNESS, BOWLEY'S QUARTILE MEASURE OF

BOWLEY'S QUARTILE MEASURE OF RELATIVE VARIABILITY—SEE RELATIVE VARIABILITY, BOWLEY'S QUARTILE MEASURE OF

BUNDLES OF FIBERS (OR THREADS), STRENGTH OF
Peirce (1926), Kontorova (1943), Daniels (1945), Epstein (1948a,b), Wilks (1948)

CALICHIOPULO'S MEASURE OF DISPERSION—SEE DISPERSION, CALICHIOPULO'S MEASURE OF

CANONICAL ANALYSIS
Hotelling (1936b), Girshick (1939), Tintner (1946), Bartlett (1947a,b)

CAUCHY DISTRIBUTION
Edgeworth (1898, 1911), Dodd (1922), Fisher (1922), Lévy (1925), Darmois (1936), Jeffreys (1938, 1939), Paulson (1940), Wilson (1940), Deming (1943), Brown & Tukey (1946), Cramér (1946), Kendall (1946), Eisenhart, Deming & Martin (1948b), Epstein (1948a), Huzurbazar (1948), Stevens (1948)

CAUCHY'S METHOD OF INTERPOLATION
Cauchy (1837), Goedseels (1909, 1925), Gibrat (1930), Gini (1932), Boldrini (1942), Deming (1943)

CENSORED SAMPLES, DOUBLY
Wald (1943)

CENSORED SAMPLES, FROM ABOVE
Gini (1914b), von Bortkiewicz (1915), Ruark & Devol (1936), Anderson (1948), Bartlett (1948)

CENSORED SAMPLES, FROM BELOW
Gini (1909, 1910), Furlan (1911), Mortara (1911), Porru (1912), Wellisch (1913), Czuber (1914), Savorgnan (1915), Gini (1939), Ruark (1939), Anderson (1948)

CENSORED SAMPLES, PROGRESSIVELY
Ruark & Devol (1936)

CENTILES—SEE PERCENTILES

CENTRAL TENDENCY, MEASURES OF—SEE AVERAGES (MEASURES OF CENTRAL TENDENCY)

CHANCE, GAMES OF, APPLICATIONS TO—SEE GAMES OF CHANCE, APPLICATIONS TO

CHARACTERISTIC FUNCTIONS
Anderson (1946), Brown & Tukey (1946), Daly (1946), Kendall (1946), Gumbel (1947), Smirnov (1949a)

CHARACTERISTIC LARGEST VALUE
Gibrat (1932b), Kimball (1946b), Gumbel (1948)

CHARACTERISTIC ROOTS OF MATRICES
Hotelling (1933), Thomson (1934), Girshick (1936), Hotelling (1936a,b), Aitken (1937), Bartlett (1938), Fisher (1939), Girshick (1939), Hsu (1939), Roy (1939, 1940a,b), Hsu (1941a,b), Roy (1942a,b), Wilks (1943), Anderson (1945), Roy (1945), Tintner (1945), Kendall (1946), Roy (1946a,b), Bartlett (1947b), Kincaid (1947), Anderson (1948)

CHARACTERISTIC SMALLEST VALUE
Gibrat (1932b), Kimball (1946b), Epstein (1948a), Gumbel (1948)

CHARLIER DISTRIBUTION, TYPE A
Charlier (1906, 1910), Crum (1923), Rietz (1924), Cramér (1928), Steffensen (1930), Luyten (1932), Kendall (1943), Wilks (1943), Cramér (1946), Kendall (1946), Benson (1949)

CHARLIER DISTRIBUTION, TYPE B
Charlier (1906, 1910), Dodd (1923), Rietz (1924), Cramér (1928), Steffensen (1930), Kendall (1943)

CHAUVENET'S CRITERION FOR REJECTION OF OUTLIERS—SEE OUTLIERS, CHAUVENET'S CRITERION FOR REJECTION

CHEMISTRY, APPLICATIONS TO
Coolidge (1925), Reilly, Rae & Wheeler (1925), Estienne (1926-27), Student (1927), Neyman & Pearson (1928), Scarborough (1930), Pearson (1935), Newman (1939), Pearson (1939), Grant (1946), Eisenhart & Martin (1948), Walsh (1949c)

CHI-SQUARE DISTRIBUTION
Pearson (1931), Pitman (1937), Snedecor (1937), Bartlett (1938), Lawley (1940), Chandra Sekar & Francis (1941), Cochran (1941), Finney (1941), Hsu (1941b), Dodd (1942), Gumbel (1942a), Pearson (1942), Kendall (1943), Scheffé (1943), Wilks (1943), Scheffé & Tukey (1944), Anderson (1945), Tintner (1945), Cramér (1946), Kendall (1946), Kimball (1946b), Bartlett (1947b), Duncan (1947), Eisenhart, Hastay & Wallis (1947), Eisenhart & Solomon (1947), Lehmann (1947), von Mises (1947), Anderson (1948), Bartlett (1948), Geary (1948), Murphy (1948), Nair (1948a), Rao (1948), Wilks (1948), Benson (1949), Hartley (1949), Mather (1949), Wolfowitz (1949)

CLOSEST ESTIMATES—SEE ESTIMATES, CLOSEST

COCHRAN'S RATIO (TEST STATISTIC)
Bresciani-Turroni (1939), Cochran (1941), Finney (1941), Kendall (1946), Eisenhart & Solomon (1947)

COCHRAN'S TEST—SEE COCHRAN'S RATIO (TEST STATISTIC)

COGRADUATE QUANTITIES
Gini (1914b, 1915a,b, 1916b, 1917), Pietra (1925), Gini (1926), Saibante (1926), d'Addario (1930), Watkins (1933)

COGRADUATION, CONSTANT OF
Dettori (1912), Gini (1917)

COGRADUATION, CURVE OF
Gini (1917)

COGRADUATION, EQUATION OF
Gini (1917)

COGRADUATION, INDEX OF
Saibante (1926, 1928), Watkins (1933)

COLLECTIVE JUDGMENTS, APPLICATIONS TO
Galton (1899a,b, 1907a,b,c), Hooker (1907), Yule (1907), Pearson (1939)

COMBINATION (COMBINATORIAL) MEANS — SEE MEANS, COMBINATION (COMBINATORIAL)

COMBINATION THEORY OF STRENGTH—SEE STRENGTH THEORY, COMBINATION

COMPETITION, APPLICATIONS TO—SEE PRIZES AND COMPETITION, APPLICATIONS TO

COMPOUND NORMAL (GAUSSIAN) DISTRIBUTION
Crum (1923)

COMPUTER APPLICATIONS
Mosteller (1946)

CONCENTRATION, AREA OF
Dalton (1920), Amoroso (1925), Gumbel (1927, 1928), Gini (1930), von Bortkiewicz (1931), de Finetti (1931a), Galvani (1931), d'Addario (1932), Galvani (1932), Pietra (1932a,b), Castellano (1933a,b), Yntema (1933), Dominedò (1934), Castellano (1935), Pietra (1935a), Wold (1935), d'Addario (1936), Gini (1936, 1939), Davis (1941), Kendall (1943)

CONCENTRATION CURVE—SEE ALSO LORENZ CURVE
Pietra (1915), Ricci (1916), Niceforo (1919), Amoroso (1925), Gumbel (1927, 1928), d'Addario (1930), Gini (1930), von Bortkiewicz (1931), de Finetti (1931a), Galvani (1931), d'Addario (1932), Galvani (1932), Pietra (1932a,b), Castellano (1933a,b), Yntema (1933), d'Addario (1934a), Castellano (1935), Pietra (1935a), Wold (1935), d'Addario (1936), Gini (1936), d'Addario (1939), Gini (1939), Davis (1941), Livada (1942), Kendall (1943)

CONCENTRATION CURVE, WATKINS' LOGARITHMIC
Watkins (1908), Persons (1909), Watkins (1909), Ricci (1916), von Bortkiewicz (1931)

CONCENTRATION, HOLMES' MEASURE OF
Holmes (1892, 1905), Watkins (1905), von Bortkiewicz (1931)

CONCENTRATION INDEX, GIBRAT'S
Gibrat (1930)

CONCENTRATION INDEX, GINI'S
Gini (1909, 1910), Furlan (1911), Mortara (1911), Porru (1912), Czuber (1914), Savorgnan (1915), Gini (1917, 1926), Saibante (1928), von Bortkiewicz (1931), Castellano (1933b), Yntema (1933), d'Addario (1934a,b), Castellano (1935), Pietra (1935b), Gini (1936), Bresciani-Turroni (1937), d'Addario (1939), Gini (1939), Pizzetti (1941b)

CONCENTRATION, MEASURES OF
Holmes (1892), Pareto (1897), Holmes (1905), Lorenz (1905), Watkins (1905, 1908), Gini (1909), Persons (1909), Watkins (1909), Gini (1910), Furlan (1911), Mortara (1911), King (1912), Porru (1912), Czuber (1914), Gini (1914a,b), Pietra (1915), Savorgnan (1915), Ricci (1916), Gini (1917), Vinci (1918), Niceforo (1919), Dalton (1920), Amoroso (1925), Gini (1926), Saibante (1926), Gumbel (1927, 1928), Saibante (1928), d'Addario (1930), de Finetti & Paciello (1930), Gibrat (1930), Gini (1930), von Bortkiewicz (1931), de Finetti (1931a), Galvani (1931), d'Addario (1932), Galvani (1932), Pietra (1932a,b), Castellano (1933a,b), Yntema (1933), d'Addario (1934a,b), Dominedò (1934), Mortara (1934a,b), Anderson (1935), Castellano (1935), Pietra (1935a,b), Wold (1935), d'Addario (1936), Gini (1936), Bresciani-Turroni (1937), d'Addario (1939), Bresciani-Turroni (1939), Gini (1939), de Vergottini (1939, 1940), Davis (1941), Pizzetti (1941b), Boldrini (1942), Livada (1942), Kendall (1943), Cramér (1946), Kendall (1946), Hoeffding (1948)

CONCENTRATION RATIO, BONFERRONI'S
de Vergottini (1940), Boldrini (1942)

CONCENTRATION RATIO, GINI'S
Gini (1914a), Pietra (1915), Savorgnan (1915), Ricci (1916), Vinci (1918), Niceforo (1919), Amoroso (1925), Gini (1926), Saibante (1926), Gumbel (1927, 1928), Saibante (1928), d'Addario (1930), de Finetti & Paciello (1930), Gini (1930), de Finetti (1931a), Galvani (1931), d'Addario (1932), Galvani (1932), Pietra (1932a,b), Castellano (1933a,b), Yntema (1933), d'Addario (1934b), Dominedò (1934), Anderson (1935), Castellano (1935), Pietra (1935a,b), Wold (1935), d'Addario (1936), Gini (1936, 1939), de Vergottini (1940), Davis (1941), Pizzetti (1941b), Boldrini (1942), Kendall (1943), Hoeffding (1948)

CONCORDANCE AND DISCORDANCE
Gini (1916a,b), Pietra (1925), Gini (1926), Kendall (1946)

CONFIDENCE BOUNDS
Thompson (1938), Wilks (1938), Wald & Wolfowitz (1939), Kullback & Frankel (1940), Nair (1940b), Kolmogoroff (1941), von Schelling (1941), Wald & Wolfowitz (1941), Gumbel (1943a), Pearson & Hartley (1943), Scheffé (1943), Wilks (1943), Scheffé & Tukey (1945), Cramér (1946), Kendall (1946), Kimball (1946a,b), Bickerstaff (1947), Eisenhart, Deming & Martin (1948b), Noether (1948), Wilks (1948), Kimball (1949), Tukey (1949b), Walsh (1949a), Wolfowitz (1949)

CONFIDENCE INTERVALS—SEE CONFIDENCE BOUNDS

CONFIDENCE PROCEDURES, ONE-SAMPLE—SEE ONE-SAMPLE TESTS (OR CONFIDENCE PROCEDURES)

CONFIDENCE PROCEDURES, TWO-SAMPLE—SEE TWO-SAMPLE TESTS (OR CONFIDENCE PROCEDURES)

CONFIDENCE REGIONS
Kimball (1946a), Wilks (1948), Wolfowitz (1949)

CONNECTION, MEASURES OF
Gini (1914b), Pietra (1925)

CONTAMINATED DISTRIBUTIONS
Tukey (1946), Harris & Tukey (1949), Tukey (1949c,d,e,f)

CONTINUOUS DISTRIBUTION, ANY
Galvani (1931), Wald & Wolfowitz (1941), Wilks (1941), Mathisen (1943), Scheffé (1943), Wald (1943), Bowker (1944), Scheffé & Tukey (1944), Smirnov (1944), Gumbel (1945a), Scheffé & Tukey (1945), Cramér (1946), Kendall (1946), Bickerstaff (1947), Kimball (1947), Eisenhart, Deming & Martin (1948a), Feller (1948), Gartštejn (1948), Gumbel (1948), Murphy (1948), Noether (1948), Schützenberger (1948), Smirnov (1948), Wilks (1948), Chung (1949), Doob (1949), Godwin (1949a,b), Juncosa (1949), Kac (1949), Maniya (1949), McMillan (1949), Meĭzler (1949), Noether (1949), Smirnov (1949a,b), Wolfowitz (1949)

CONTINUOUS ISOSCELES TRIANGULAR DISTRIBUTION—SEE ISOSCELES TRIANGULAR DISTRIBUTION, CONTINUOUS

CONTINUOUS RECTANGULAR DISTRIBUTION—SEE UNIFORM DISTRIBUTION, CONTINUOUS

CONTINUOUS SYMMETRIC DISTRIBUTIONS — SEE SYMMETRIC DISTRIBUTIONS, CONTINUOUS

CONTINUOUS UNIFORM DISTRIBUTION—SEE UNIFORM DISTRIBUTION, CONTINUOUS

CONTRAGRADUATE QUANTITIES
Gini (1915a,b, 1916b, 1917), Pietra (1925), Gini (1926), Saibante (1926), Reiersöl (1944)

CORRELATIONS
Edgeworth (1911), Dettori (1912), Walker (1914), Gini (1915a,b, 1916b), Weinberg (1916), Jones (1921), Rietz (1924), Bottema (1925), Pearson (1925), Gini (1926), Pearson (1926), Saibante (1926, 1928), d'Addario (1930), Pearson & Pearson (1931, 1932), Craig (1933), Hojo (1933), Hotelling (1933), Watkins (1933), Thomson (1934), Girshick (1936), Hotelling (1936a,b), Aitken (1937), Smirnoff (1937b), Pearson (1939), Lawley (1940), Mood (1941), Henry (1942), Roy (1942a), Kendall (1943), Wilks (1943), Anderson (1946), Baker (1946), Cramér (1946), Grant (1946), Kendall (1946), Mosteller (1946), Tintner (1946), Bartlett (1947a,b), Bickerstaff (1947), Egudin (1947), Bartlett (1948), Geary (1948), Hoeffding (1948), Nair (1948a), Nanda (1948a,b), Wilks (1948), Hartley (1949), Purcell (1949), Scheffé (1949), Wolfowitz (1949)

COSINE DISTRIBUTION
Lagrange (1774), Bernoulli (1785), Merriman (1877), Pizzetti (1892)

COVARIANCE, ANALYSIS OF—SEE ANALYSIS OF COVARIANCE

COVARIANCE MATRICES
Foster (1936), Girshick (1939), Hsu (1939), Kendall (1940), Roy (1940a,b), Mood (1941), Roy (1942a,b), Wilks (1943), Anderson (1945), Tintner (1945), Anderson (1946), Cramér (1946), Kendall (1946), Tintner (1946), Bartlett (1947b), Hastings, Mosteller, Tukey & Winsor (1947), Anderson (1948), Geary (1948), Hoeffding (1948), Nanda (1948a,b), Rao (1948), Kimball (1949)

COVERAGES
Bickerstaff (1947), Wilks (1948)

CRAMÉR-VON MISES STATISTIC (OR TEST)
Cramér (1928), von Mises (1931), Kolmogoroff (1933), Wold (1935), Smirnoff (1936), Neyman (1937a), Smirnoff (1937a), Gumbel (1942a), Pearson (1942), Smirnov (1944), Cramér (1946), Kendall (1946), von Mises (1947), Feller (1948), Smirnov (1949a)

CROSS-PRODUCT MOMENTS—SEE MOMENTS, CROSS-PRODUCT

CURVE FITTING
Galton (1896), Yule (1896a,b), Charlier (1906), Tolley (1916), Jones (1921), Hazen (1930), Scarborough (1930), d'Addario (1931), Gibrat (1931, 1932a,b), Grassberger (1932), Luyten (1932), Grassberger (1933), Slade (1934), Saville (1936), Slade (1936), Thibaud (1937), Deming (1943), Price (1943), Wilks (1943), Cramér (1946), Kendall (1946), Geary (1948)

CURVES, DURATION—SEE DURATION CURVES

CYCLICAL SERIES
Pietra (1925), Kendall (1946)

DALY'S G DISTRIBUTION
Daly (1946), Walsh (1947), Wilks (1948)

DECILES
Galton (1881), Edgeworth (1886b), Galton (1896), Yule (1896b), Bowley (1901), Persons (1909), King (1912), Edgeworth (1913), Secrist (1917), Vinci (1918), Niceforo (1919), Pearson (1920), Czuber (1921), Jones (1921), Julin (1921), Goldhizer (1923), Burgess (1927), Jordan (1927), March (1930), Galvani (1931), von Mises (1931), Craig (1932a), Yang (1933), Bresciani-Turroni (1937), Gini (1939), Arley & Buch (1940), Pizzetti (1940), Fréchet (1941), Boldrini (1942), Gumbel (1943a), Kendall (1943), Cramér (1946), Schützenberger (1948)

DEMOGRAPHY, APPLICATIONS TO
Quetelet (1835), Bowley (1901), Hayford (1902), Yule (1911), Gini (1912, 1914a,b), von Bortkiewicz (1915), Gini (1918), Niceforo (1919), Czuber (1921), von Bortkiewicz (1922b), Sloane (1923), Wilson (1923), Whittaker & Robinson (1924), Pietra (1925), Gini (1926), Goodrich (1927), Jordan (1927), Saibante (1928), Gini & Galvani (1929), Eells (1930), de Finetti & Paciello (1930), Gini (1930), de Gleria (1930), Ross (1930), de Finetti (1931b), Galvani (1931), Gibrat (1931), von Mises (1931), Gibrat (1932b), Galvani (1933), Gini, Boldrini, Galvani & Venere (1933), Griffin (1933), Linders (1933), Scates (1933), Watkins (1933), Gini (1939), Zappa (1939), Kendall (1943), Cramér (1946), Kendall (1946)

DE MORGAN'S ALTERNATIVE TO REJECTION OF OUTLIERS—SEE OUTLIERS, DE MORGAN'S ALTERNATIVE TO REJECTION

DEPENDENCE—SEE INDEPENDENCE AND DEPENDENCE

DEPENDENCE, INDICES OF—SEE INDEPENDENCE AND DEPENDENCE, INDICES OF

DESCENDING MEANS—SEE MEANS, DESCENDING

DESTRUCTIVE TESTS, APPLICATIONS TO
Jacobson (1947), Purcell (1949)

DETERMINANTAL EQUATIONS, ORDERED ROOTS OF—SEE ORDERED ROOTS OF DETERMINANTAL EQUATIONS

DEVIATE, EXTREME—SEE EXTREME DEVIATE

DEVIATE, STUDENTIZED EXTREME—SEE STUDENTIZED EXTREME DEVIATE

DEVIATION, DISCARD—SEE DISCARD DEVIATION

DEVIATION, LARGEST (ABSOLUTE)—SEE LARGEST (ABSOLUTE) DEVIATION

DEVIATION, QUARTILE—SEE QUARTILE DEVIATION

DEVICE LIFE, APPLICATIONS TO
Gumbel (1933e), Pearson (1935), Grant (1946), Kendall (1946), Purcell (1947), Epstein (1948b), Purcell (1949)

DIFFERENCES AT HALF RANGE
Bose (1938), Nair & Shrivastava (1942)

DIFFERENCES, SUCCESSIVE—SEE SUCCESSIVE DIFFERENCES

DISCARD AVERAGES—SEE ALSO TRIMMED MEANS
Anonymous (1821), Cournot (1843), Mendeleev (1895), Daniell (1920), Estienne (1926-27, 1927), Lévy (1929), Cramér (1946), Bartlett (1949), Harris & Tukey (1949), Tukey (1949d)

DISCARD DEVIATION
Daniell (1920), Harris & Tukey (1949), Tukey (1949d)

DISCORDANCE—SEE CONCORDANCE AND DISCORDANCE

DISCRETE ISOSCELES TRIANGULAR DISTRIBUTION—SEE ISOSCELES TRIANGULAR DISTRIBUTION, DISCRETE

DISCRETE RECTANGULAR DISTRIBUTION—SEE UNIFORM DISTRIBUTION, DISCRETE

DISCRETE UNIFORM DISTRIBUTION—SEE UNIFORM DISTRIBUTION, DISCRETE

DISCRIMINANT ANALYSIS
Fisher (1938), Tintner (1946)

DISCRIMINANT FUNCTIONS—SEE DISCRIMINANT ANALYSIS

DISPERSION, CALICHIOPULO'S MEASURE OF
Calichiopulo (1937)

DISPERSION, MEASURES OF
Ptolemaeus (circa 150), Lalande (1771), Mathieu (1813-14), Gauss (1816), Fourier (1824a, 1826a, 1829), Hauber (1830, 1830-32), Encke (1832-34), Quetelet (1835), Dirichlet (1836), Dove (1838), Stampfer (1839), Cournot (1843), Quetelet (1846), Herschel (1850), Lloyd (1855), De Morgan (1864), Boudin (1865), Venn (1866), von Andrae (1869), Galton (1869), Jordan (1869), von Andrae (1872), Helmert (1872), Jordan (1872), Peirce (1873), Galton (1875), Helmert (1875a,b), Mees (1875), Helmert (1876a,b), Mees (1876), Helmert (1877a,b), Jordan (1877a,b), Merriman (1877), Herschel (1878), Jordan (1879), Bréger (1881), Galton (1881), Doolittle (1884), Edgeworth (1885, 1886a,b, 1887f, 1887-90), Galton (1888, 1889), Gore (1889), Czuber (1891a), Holmes (1892), Pizzetti (1892), Vallier (1894), Cranz (1896), Galton (1896), Poincaré (1896), Yule (1896a), Blümcke (1897), Fechner (1897), Blümcke (1898), Edgeworth (1898), Rodewald (1898), Czuber (1899), Galton (1899a,b), Sheppard (1899a,b), Estienne (1900a), Blümcke (1901), Bowley (1901), Czuber (1903), Heydenreich (1903), Mitscherlich (1903), Edgeworth (1905), Holmes (1905), Lorenz (1905), Watkins (1905), Charlier (1906), Kozák (1907), Rohne (1907), Vogeler (1907), Kozák (1908-10), Mazzuoli (1908), Watkins (1908), Bohlmann (1909), Persons (1909), Watkins (1909), Wellisch (1909), Charlier (1910), Goedseels (1910), Edgeworth (1911), Goedseels (1911), Mortara (1911), de la Vallée Poussin (1911), Yule (1911), Dettori (1912), Fisher (1912), Gini (1912), King (1912), Dodd (1913), Edgeworth (1913), Goodwin (1913), Wellisch (1913), Czuber (1914), Dodd (1914), Gini (1914a, 1915a,b), Pietra (1915), Savorgnan (1915), Gini (1916a,b), Ricci (1916), Spillman, Tolley & Reed (1916), Tolley (1916), Weinberg (1916), Becker (1917), Brunt (1917), Gini (1917), Secrist (1917), Czuber (1918), Gini (1918), Dalton (1920), Daniell (1920), Fisher (1920), Pearson (1920), de Pietri-Tonelli (1920), Stewart (1920b), Czuber (1921), Hall (1921), Jackson (1921), Jones (1921), Julin (1921), Savorgnan (1921), von Bortkiewicz (1922a,b), Fisher (1922), Crum (1923), Dodd (1923), von Mises (1923), Neyman (1923), Foster (1924), Haag (1924), Horton (1924), Rietz (1924), Whittaker & Robinson (1924), Winkler (1924), Bottema (1925), Coolidge (1925), Goedseels (1925), Irwin (1925a,b), Lévy (1925), Pearson (1925), Pietra (1925), Reilly, Rae & Wheeler (1925), Tippett (1925), Estienne (1926-27), Gini (1926), Jarvis (1926), Pearson (1926), Peirce (1926), Burgess (1927), Estienne (1927), Goodrich (1927), Gumbel (1927), Jordan (1927), Student (1927), Tucker (1927), von Verschuer (1927), Bartels (1928), Bowley (1928), Cramér (1928), Dufton (1928), Gumbel (1928), Köppen (1928), Lenz & von Verschuer (1928), Neyman & Pearson (1928), Ogrodnikoff (1928), Pearson & Adyanthāya (1928), Sophister (1928), de Gleria (1929), Lévy (1929), Rider (1929), d'Addario (1930), de Finetti & Paciello (1930), Fogelson (1930), Gibrat (1930), Gini (1930), de Gleria (1930), Hazen (1930), Lidstone (1930), March (1930), de Montessus de Ballore (1930), Rhodes (1930), Scarborough (1930), d'Addario (1931), von Bortkiewicz (1931), Edgett (1931), de Finetti (1931a), Galvani (1931), Hojo (1931), Horst (1931), von Mises (1931), Pearson (1931), Pearson & Pearson (1931), Shewhart (1931), Student (1931), Craig (1932a,b), Dodge (1932), Galvani (1932), Garver (1932), Gibrat (1932a), Hotelling & Solomons (1932), Jeffreys (1932a,b), Luyten (1932), Pearson (1932), Pearson & Pearson (1932), Pietra (1932a,b), Bartlett (1933), Craig (1933), Crowe (1933), Dodge (1933), Eddington (1933), Gumbel (1933a,f), Hojo (1933), Hotelling (1933), Jeffreys (1933), Julin (1933), McKay & Pearson (1933), Rhodes (1933), Rider (1933), Weinberg (1933), Yang (1933), Yntema (1933), d'Addario (1934b), Davies & Pearson (1934), Dominedò (1934), Eyraud (1934a,b), Foster (1934), Gumbel (1934c,d,e,f), Jeffreys (1934), Mortara (1934a,b), Münzner (1934), Pollard (1934), Slade (1934), Anderson (1935), Birkeland & Frogner (1935), Bond (1935), Castellano (1935), Fréchet (1935), Gumbel (1935a,b,c,d, 1935-36), Olds (1935), Pearson (1935), Pearson & Haines (1935), Pietra (1935a,b), Sastry (1935), Smirnoff (1935), Thompson (1935), Wold (1935), d'Addario (1936), Crowe (1936), Foster (1936), Gini (1936), Girshick (1936), Grassberger (1936), Gumbel (1936a,b,c, 1936-37), Hotelling (1936a,b), Mathews (1936), Nair (1936), Neyman & Pearson (1936), Pearson & Chandra Sekar (1936), Saville (1936), Slade (1936), Bowley (1937), Bresciani-Turroni (1937), Calichiopulo (1937), Gumbel (1937b,c,d,e, 1937-38), Pitman (1937), Savur (1937b), Snedecor (1937), Thibaud & Ferber (1937), Wold (1937), Barral Souto (1938), Bose (1938), Brenet & Armand

(1938), Bruen (1938), von Eberhard (1938), Fisher & Yates (1938), Gumbel (1938a,b,c), Hertzman (1938), Jeffreys (1938), Kimball (1938), Bresciani-Turroni (1939), Derevitsky (1939), Eisenhart (1939), Gini (1939), Girshick (1939), Gottschick (1939), Gumbel (1939a,b,c,e,f), l'Hermite (1939), Hsu (1939), Hulme & Symms (1939), Jeffreys (1939), Pitman (1939), de Vergottini (1939), Wald (1939), Arley (1940), Arley & Buch (1940), Dodd (1940), Fréchet (1940a,b), Kendall (1940), Nair (1940a), Pizzetti (1940), de Vergottini (1940), Wald (1940), Wilson (1940), Brookner (1941), Cochran (1941), Daniels (1941), Fréchet (1941), Gumbel (1941a,b), Mood (1941), Pizzetti (1941a,b,c), Roller (1941), Simon (1941), Smirnoff (1941), Aldanondo (1942), Boldrini (1942), Dudding & Jennett (1942), Evans (1942), Fréchet (1942), Gaede (1942), Gumbel (1942a,b,c), Hartley (1942), Kimball (1942), Livada (1942), Nair & Shrivastava (1942), Pearson & Hartley (1942), Roy (1942a,b), Barricelli (1943), Gumbel (1943a,c), Kendall (1943), Knudsen (1943), Pearson & Hartley (1943), Powell (1943), Rajalakshman (1943), Wilks (1943), Barnard (1944), Geary (1944), Gumbel (1944), Hartley (1944), Reiersöl (1944), Rodgers (1944), Wing, Price & Douglass (1944), Gurney (1945), Anderson (1946), Baker (1946), Brown & Tukey (1946), Carlton (1946), Cramér (1946), Daly (1946), Freudenthal (1946), Grant (1946), Greenwood (1946), Gumbel (1946b), Hoel (1946), Jones (1946), Kendall (1946), Kimball (1946a), Kontorova (1946), Mosteller (1946), Oding (1946), Roy (1946a,b), Tintner (1946), Tukey (1946), Walsh (1946a), Winsten (1946), Bartlett (1947a,b), Bickerstaff (1947), Cohan (1947), Davidenkov, Shevandin & Wittmann (1947), Duncan (1947), Egudin (1947), Eisenhart, Hastay & Wallis (1947), Eisenhart & Solomon (1947), Elfving (1947), Fisher & Hollomon (1947), Grubbs & Weaver (1947), Gumbel (1947), Gurney (1947), Gurney & Pearson (1947), Hastings, Mosteller, Tukey & Winsor (1947), Jacobson (1947), Kimball (1947), Lord (1947), Nair (1947), Plackett (1947), Purcell (1947), Walsh (1947), Anderson (1948), Cox (1948), Eisenhart, Deming & Martin (1948b), Eisenhart & Martin (1948), Epstein (1948a,b), Epstein & Brooks (1948), Frankel (1948), Gartštein (1948), Geary (1948), Gumbel (1948), Higuchi, Leeper & Davis (1948), Hill & Schmidt (1948), Hoeffding (1948), Housner & Brennan (1948), Huzurbazar (1948), Jones (1948), Nair (1948a,b), Pillai (1948), Rao (1948), Stevens (1948), Tukey (1948b,c), Wilks (1948), Bartlett (1949), Benson (1949), Cole (1949), Cox (1949), Godwin (1949a,b), Gumbel (1949), Harris & Tukey (1949), Hartley (1949), Howell (1949), Kimball (1949), Lane & Lei (1949), Mather (1949), McMillan (1949), van Meer & Plantema (1949), Nair (1949a,b), Noether (1949), Paulson (1949), Peterson (1949a,b), Potter (1949a,b), Press (1949), Prot (1949a,b,c), Purcell (1949), Scheffé (1949), Shone (1949), Tukey (1949a,b,c,d,e,f), Walsh (1949a,b,c,d), Weibull (1949), Wolfowitz (1949), Yamanouchi (1949), Youden (1949)

DISSIMILARITY, INDEX OF
Gini (1914b, 1915b, 1926)

DISSIMILARITY, MEASURES OF
Gini (1915b), Pietra (1925), Gini (1926)

DISTRIBUTION-FREE METHODS
Hauber (1830), Bréger (1881), Estienne (1890), Pizzetti (1892), Estienne (1900a), Savorgnan (1915), Jackson (1921), Goldhizer (1923), Jackson (1923), Neyman (1923), de Finetti (1932), Garver (1932), Hotelling & Solomons (1932), Bartlett (1933), Cantelli (1933), de Finetti (1933), Glivenko (1933), Kolmogoroff (1933), Yang (1933), Fisher (1934), Jeffreys (1934), Pollard (1934), Smirnoff (1936), Thompson (1936), Ville (1936), Kozakiewicz (1937), Madow (1937), Neyman (1937a), Savur (1937a,b), Smirnoff (1937a,b), Snedecor (1937), Cisbani (1938), Hertzman (1938), Savur (1938), Thompson (1938), Waschakidse (1938), Gottschick (1939), Smirnov (1939a,b), Wald & Wolfowitz (1939), Nair (1940a,b), Kolmogoroff (1941), Evans (1942), Fréchet (1942), Livada (1942), Wald (1942), Wilks (1942), Mathisen (1943), Scheffé (1943), Wald (1943), Barnard (1944), Bowker (1944), Robbins (1944a,b), Smirnov (1944), Scheffé & Tukey (1945), Grant (1946), Kendall (1946), Winsten (1946), Bickerstaff (1947), Egudin (1947), Kimball (1947), Tukey (1947), Feller (1948), Murphy (1948),

Noether (1948), Schützenberger (1948), Smirnov (1948), Thomas (1948), Tukey (1948a), Wilks (1948), Birnbaum & Zuckerman (1949), Chung (1949), Doob (1949), Epstein (1949b), Godwin (1949a,b), Kac (1949), Maniya (1949), Noether (1949), Smirnov (1949a,b), Wolfowitz (1949)

DISTRIBUTION, NAMED—SEE (NAME) DISTRIBUTION

DISTRIBUTION THEORY

Edgeworth (1886b), Holmes (1892), Gini (1909), Watkins (1909), Furlan (1911), Mortara (1911), Czuber (1914), Gini (1914a, 1917), Wilson (1923), Amoroso (1925), Fréchet (1927), Goodrich (1927), Fisher & Tippett (1928), Pearson & Adyanthāya (1928), Sophister (1928), Fisher (1929), Pearson & Adyanthāya (1929), Rider (1929), Fogelson (1930), Gibrat (1930), Edgett (1931), Gibrat (1931), Hojo (1931), Pearson (1931), Shewhart (1931), Craig (1932a,b), de Finetti (1932), Grassberger (1932), Gumbel (1932), Jeffreys (1932b), Pearson (1932), Pearson & Pearson (1932), Bartlett (1933), Cantelli (1933), de Finetti (1933), Fisher (1933), Glivenko (1933), Gumbel (1933a,b,c,d,e,g), Hojo (1933), Hotelling (1933), Jeffreys (1933), Kolmogoroff (1933), McKay & Pearson (1933), Romanovsky (1933), Fisher (1934), Gumbel (1934b,c,d,e,f), Jeffreys (1934), Pollard (1934), Rosin & Rammler (1934), Slade (1934), Eyraud (1935), Gumbel (1935a,b,c,d, 1935-36), Olds (1935), Pearson (1935), Pearson & Haines (1935), Sastry (1935), Smirnoff (1935), Foster (1936), Girshick (1936), Grassberger (1936), Gumbel (1936a,b,c, 1936-37), Horton (1936), Hotelling (1936a,b), von Mises (1936), Ruark & Devol (1936), Aitken (1937), Camp (1937), Coutagne (1937), Gumbel (1937a,b,c,d,e), Kozakiewicz (1937), Madow (1937), Smirnoff (1937a,b), Sukhatme (1937), Thibaud (1937), Gumbel (1938a,b,c), Jeffreys (1938), Kimball (1938), Waschakidse (1938), Wilks (1938), d'Addario (1939), Coutagne (1939), Gini (1939), Girshick (1939), Gumbel (1939a,e,f), Hsu (1939), Newman (1939), Pearson (1939), Ruark (1939), Smirnov (1939a,b,c), Stevens (1939), Feller (1940), Fisher (1940), Garwood (1940), Gumbel (1940), Kendall (1940), Nair (1940a), Roy (1940a,b), Chandra Sekar & Francis (1941), Daniels (1941), Davis (1941), Finney (1941), Gnedenko (1941), Gumbel (1941b), Hsu (1941a,b), Kolmogoroff (1941), Mood (1941), Rajalakshman (1941), Roller (1941), Thompson (1941), Aldanondo (1942), Beard (1942), Fréchet (1942), Gumbel (1942a,b,c), Hartley (1942), Kimball (1942), Pearson & Hartley (1942), Roy (1942a,b), Curtiss (1943), Gnedenko (1943), Gumbel (1943c), Rajalakshman (1943), Wilks (1943), Gumbel (1944), Hartley (1944), Smirnov (1944), Anderson (1945), Baker (1946), Brown & Tukey (1946), Carlton (1946), Cramér (1946), Daly (1946), Gumbel (1946b), Hoel (1946), Kendall (1946), Kimball (1946a), Mosteller (1946), Roy (1946a,b), Walsh (1946b), Bartlett (1947a), Bickerstaff (1947), Domb (1947), Duncan (1947), Eisenhart & Solomon (1947), Elfving (1947), Gumbel (1947), Gurney (1947), Gurney & Pearson (1947), Jacobson (1947), Lord (1947), Moran (1947), Cox (1948), Eisenhart, Deming & Martin (1948a), Epstein (1948a), Feller (1948), Gartšteĭn (1948), Hoeffding (1948), Huzurbazar (1948), Nair (1948b), Nanda (1948a,b), Pillai (1948), Rao (1948), Tukey (1948c), Wilks (1948), Epstein (1949a,b), Juncosa (1949), Kac (1949), Kimball (1949), McMillan (1949), Meĭzler (1949), Smirnov (1949a,b), Tukey (1949a,b), Weibull (1949)

DISTRIBUTIONS, MIXTURE OF—SEE MIXTURE OF DISTRIBUTIONS

DOUBLE EXPONENTIAL DISTRIBUTION

Laplace (1774), Glaisher (1872), Todhunter (1873), Merriman (1877), Edgeworth (1887f, 1887-90, 1888), Gore (1889), Czuber (1891a,b), Pizzetti (1892), Czuber (1899), Keynes (1911), Brunt (1917), Fisher (1920), Keynes (1921), Dodd (1922), Fisher (1922), Wilson (1923), Estienne (1926-27), Wilson & Hilferty (1929), Luyten (1932), Eddington (1933), Rider (1933), Gumbel (1934e), Münzner (1934), Bond (1935), Fréchet (1935), Darmois (1936), Gumbel (1937c,d), Bruen (1938), Gumbel (1938b), Jeffreys (1939), Fréchet (1941), Gumbel (1943c), Cramér (1946), Eisenhart, Deming & Martin (1948b), Epstein (1948a,b), Walsh (1949b)

DOUBLE SPIKE (POLAR) DISTRIBUTION

von Bortkiewicz (1931), Smirnov (1949b)

DOUBLY CENSORED SAMPLES—SEE CENSORED SAMPLES, DOUBLY

DUFTON'S METHOD OF FITTING A STRAIGHT LINE
Dufton (1928), Bruen (1938)

DUNCAN'S RANKED DIFFERENCE TEST—SEE RANKED DIFFERENCE TEST, DUNCAN'S

DURATION CURVES
Foster (1924), Goodrich (1927), Foster (1934), Slade (1934), Foster (1936), Beard (1942), Lane & Lei (1949)

ECONOMETRICS, APPLICATIONS TO
Edgeworth (1887f, 1887-90), Holmes (1892), Pareto (1897), Bowley (1901), Holmes (1905), Lorenz (1905), Watkins (1905, 1908), Gini (1909), Persons (1909), Watkins (1909), Gini (1910), Furlan (1911), Mortara (1911), Dettori (1912), Gini (1912), King (1912), Porru (1912), Edgeworth (1913), Czuber (1914), Gini (1914a), Savorgnan (1915), Ricci (1916), Gini (1917), Secrist (1917), Czuber (1918), Vinci (1918), Niceforo (1919), Persons (1919), Dalton (1920), de Pietri-Tonelli (1920), Czuber (1921), Jones (1921), Julin (1921), Keynes (1921), Savorgnan (1921), Crum (1923), Sloane (1923), Wilson (1923), Rietz (1924), Winkler (1924), Amoroso (1925), Pietra (1925), Gini (1926), Saibante (1926), Burgess (1927), Goodrich (1927), Gumbel (1927), Jordan (1927), Bowley (1928), Gumbel (1928), Saibante (1928), d'Addario (1930), Gibrat (1930), Gini (1930), March (1930), de Montessus de Ballore (1930), Rhodes (1930), d'Addario (1931), von Bortkiewicz (1931), Galvani (1931), Gibrat (1931), Shewhart (1931), d'Addario (1932), Galvani (1932), Gibrat (1932b), Gini (1932), Pietra (1932a,b), Castellano (1933a,b), Julin (1933), Rhodes (1933), Yntema (1933), d'Addario (1934a,b), Dominedò (1934), Mortara (1934a,b), Castellano (1935), Pietra (1935a,b), Wold (1935), d'Addario (1936), Gini (1936), Bresciani-Turroni (1937), Neyman (1937b), Bruen (1938), d'Addario (1939), Bresciani-Turroni (1939), Gini (1939), Davis (1941), Pizzetti (1941b), Boldrini (1942), Evans (1942), Kendall (1943), Price (1943), Cramér (1946), Grant (1946), Kendall (1946), Tintner (1946), Anderson (1948), Geary (1948), Noether (1949)

EDGEWORTH SERIES DISTRIBUTION
Edgeworth (1905, 1911), Edgett (1931), Kendall (1943), Cramér (1946)

EDGEWORTH'S MODIFICATION OF STONE'S FIRST CRITERION—SEE OUTLIERS, EDGEWORTH'S MODIFICATION OF STONE'S FIRST CRITERION FOR REJECTION

EDUCATION, APPLICATIONS TO
Pearson (1920), Jones (1921), Burgess (1927), Gibrat (1930), Hotelling (1933, 1936a), Snedecor (1937), Lawley (1940), Kendall (1943, 1946), Bickerstaff (1947)

EFFICIENCY OF ESTIMATION—SEE ESTIMATION, EFFICIENCY OF

EFFICIENCY OF TESTS—SEE TESTS, EFFICIENCY OF

EIGENVALUES OF MATRICES—SEE CHARACTERISTIC ROOTS OF MATRICES

ELECTRONICS, APPLICATIONS TO
Grant (1946), Epstein (1948b), Epstein & Brooks (1948), Hill & Schmidt (1948), Epstein (1949a)

EQUATION OF COGRADUATION—SEE COGRADUATION, EQUATION OF

ERROR RATE OF A TEST, TYPE I

Fisher (1929), Pearson & Adyanthāya (1929), Student (1931), Neyman & Pearson (1936), Pearson & Chandra Sekar (1936), Savur (1937a), Snedecor (1937), Newman (1939), Wald (1939), Fisher (1940), Nair (1940a), Paulson (1940), Chandra Sekar & Francis (1941), Cochran (1941), Davis (1941), Finney (1941), Kolmogoroff (1941), Pearson (1942), Knudsen (1943), Mathisen (1943), Pearson & Hartley (1943), Scheffé (1943), Wilks (1943), Bowker (1944), Cramér (1946), Daly (1946), Kendall (1946), Tintner (1946), Walsh (1946a), Bartlett (1947b), Duncan (1947), Eisenhart, Hastay & Wallis (1947), Eisenhart & Solomon (1947), Jacobson (1947), Kimball (1947), Lord (1947), Walsh (1947), Bartlett (1948), Nair (1948a), Rao (1948), Tukey (1948b,c), Wilks (1948), Noether (1949), Tukey (1949a,b), Walsh (1949a,b,c,d)

ERROR RATE OF A TEST, TYPE II—SEE ALSO POWER OF A TEST

Pearson & Adyanthāya (1929), Neyman & Pearson (1936), Wald (1939), Nair (1940a), Paulson (1940), Knudsen (1943), Wilks (1943), Cramér (1946), Daly (1946), Kendall (1946), Walsh (1946a), Eisenhart, Hastay & Wallis (1947), Jacobson (1947)

ERRORS, THEORY OF

Ptolemaeus (circa 150), Galilei (1632), Cotes (1722), Euler (1749), Mayer (1750), Maire & Boscovich (1755), Simpson (1756), Boscovich (1757), Simpson (1757), Boscovich (1760), Lambert (1760), Short (1763), Lambert (1765a,b), Lalande (1771), Lagrange (1774), Laplace (1774), Bernoulli (1778), Euler (1778), Laplace (1781), Bernoulli (1785), Laplace (1786, 1793, 1799), Prony (1804), Trembley (1804), Legendre (1805), Puissant (1805), Svanberg (1805), von Zach (1805), Delambre (1806-10), von Lindenau (1806), Adrain (1808), Gauss (1809), Laplace (1810, 1811a,b, 1812), Delambre (1813), Mathieu (1813-14), Legendre (1814), van Beeck Calkoen (1816), Gauss (1816), Adrain (1818), Anonymous (1821), Fourier (1823), Gauss (1823), Cauchy (1824), Fourier (1824a,b,c), Ivory (1825), Muncke (1825), Fourier (1826a, 1829), Hauber (1830, 1830-32), von Riese (1830), Cauchy (1831), Fourier (1831), Encke (1832-34), Poncelet (1835), Quetelet (1835), Dirichlet (1836), Cauchy (1837), Hagen (1837), Bessel & Baeyer (1838), Stampfer (1839), Cournot (1843), Gerling (1843), Ellis (1844), Quetelet (1846), De Morgan (1847), Herschel (1850), Peirce (1852), Tchebychef (1854), Gould (1855), Airy (1856), Winlock (1856), Petzval (1857), von Andrae (1860), Airy (1861), Chauvenet (1863), De Morgan (1864), Boudin (1865), Todhunter (1865), Venn (1866), Stone (1868), von Andrae (1869), Galton (1869), Jordan (1869), Zachariae (1871), von Andrae (1872), Glaisher (1872), Helmert (1872), Jordan (1872), Glaisher (1873), Peirce (1873), Stone (1873a,b), Todhunter (1873), Fechner (1874), Glaisher (1874), Jevons (1874), Stone (1874), Galton (1875), Helmert (1875a,b), Laurent (1875), Lexis (1875), Mees (1875), Bréger (1876), Helmert (1876a,b), Mees (1876), Helmert (1877a,b), Jordan (1877a,b), Merriman (1877), Wilson (1877), Herschel (1878), Peirce (1878), Schott (1878), Jordan (1879), Galton (1881), Bruns (1882), Schreiber (1882), Edgeworth (1883a,b), Doolittle (1884), Merriman (1884), Wright (1884), Edgeworth (1885, 1886a,b), Newcomb (1886), Bertrand (1887a,b), Edgeworth (1887a,b,c,d,e), Lehmann-Filhes (1887), Turner (1887), Bertrand (1888a,b,c), Delauney (1888), Edgeworth (1888), Faye (1888), Galton (1888), Jordan (1888), Bertrand (1889), Galton (1889), Gore (1889), Pizzetti (1889), Czuber (1890), Estienne (1890), Jordan (1890), Runge (1890), Czuber (1891a,b), Venn (1891), Pizzetti (1892), Edgeworth (1893), Goedseels & Mansion (1893), Vallier (1894), Pearson (1895), Cranz (1896), Galton (1896), Poincaré (1896), Yule (1896a,b), Blümcke (1897), Fechner (1897), Blümcke (1898), Edgeworth (1898), Rodewald (1898), Czuber (1899), Galton (1899a,b), Sheppard (1899a,b), Wölffing (1899), Estienne (1900a,b), Láska (1900), Blümcke (1901), Bowley (1901), Pearson (1902), Czuber (1903), Heydenreich (1903), Kapteyn (1903), Kirchberger (1903), Mitscherlich (1903), Saunder (1903), Edgeworth (1905), Charlier (1906), Mansion (1906), Galton (1907a,b,c), Hooker (1907), Kozák (1907), Rohne (1907), Vogeler (1907), Yule (1907), Kozák (1908-10), Mazzuoli (1908), Dunkel (1909), Goedseels (1909), de la Vallée Poussin (1909), Wellisch (1909), Charlier (1910), Goedseels (1910), Edgeworth (1911), Goedseels (1911), Keynes (1911), de la Vallée Poussin (1911), Yule (1911), Fisher (1912), Gini (1912), King (1912), Newcomb (1912), Tits (1912), Dodd (1913), Edgeworth (1913), Goodwin (1913), Mansion (1913), Contarino (1914), Czuber (1914), Dodd (1914), Fuller (1914), Hazen

(1914), Walker (1914), Trachtenberg (1915), Tolley (1916), Becker (1917), Brunt (1917), Secrist (1917), Czuber (1918), Frank (1918), Gini (1918), Campbell (1920), Daniell (1920), Fisher (1920), Pearson (1920), Stewart (1920a,b), Czuber (1921), Jackson (1921), Jones (1921), Keynes (1921), von Bortkiewicz (1922a,b), Dodd (1922), Fisher (1922), Crum (1923), Edgeworth (1923), Goldhizer (1923), Jackson (1923), von Mises (1923), Neyman (1923), Wilson (1923), Haag (1924), Jackson (1924), Rietz (1924), Whittaker & Robinson (1924), Archibald (1925), Coolidge (1925), Goedseels (1925), Irwin (1925a,b), Lévy (1925), Pearson (1925), Reilly, Rae & Wheeler (1925), Tippett (1925), Estienne (1926-27), Alliaume (1927a,b), Burgess (1927), Estienne (1927), Friedrich (1927), Jordan (1927), Student (1927), Bartels (1928), Birkeland (1928), Bowley (1928), Cramér (1928), Dufton (1928), Köppen (1928), Neyman & Pearson (1928), Ogrodnikoff (1928), Pearson & Adyanthāya (1928), Sophister (1928), Lévy (1929), Pearson & Adyanthāya (1929), Rider (1929), Smith (1929), Wilson & Hilferty (1929), Gibrat (1930), Hazen (1930), Lidstone (1930), March (1930), de Montessus de Ballore (1930), Rhodes (1930), Scarborough (1930), d'Addario (1931), de Finetti (1931a,b), Hojo (1931), Horst (1931), Kolmogoroff (1931), von Mises (1931), Pearson (1931), Pearson & Pearson (1931), Shewhart (1931), Student (1931), Craig (1932a,b), Dodge (1932), de Finetti (1932), Garver (1932), Gibrat (1932a,b), Grassberger (1932), Jeffreys (1932a,b), Luyten (1932), Pearson (1932), Bartlett (1933), Crowe (1933), Dodge (1933), Eddington (1933), Fisher (1933), Jeffreys (1933), Rhodes (1933), Rider (1933), Baidaff & Barral Souto (1934), Davies & Pearson (1934), Eyraud (1934a,b), Fisher (1934), Jeffreys (1934), Münzner (1934), Pollard (1934), Smith (1934), Baidaff & Barral Souto (1935), Barral Souto (1935), Birkeland & Frogner (1935), Bond (1935), Fréchet (1935), Thompson (1935), Crowe (1936), Darmois (1936), Girshick (1936), Mathews (1936), Nair (1936), Pearson & Chandra Sekar (1936), Slade (1936), Ville (1936), Bowley (1937), Calichiopulo (1937), Friedrich (1937), Gumbel (1937c), Pitman (1937), Savur (1937b), Snedecor (1937), Thibaud & Ferber (1937), Wold (1937), Barral Souto (1938), Bose (1938), Brenet & Armand (1938), Bruen (1938), Cisbani (1938), Dodd (1938), von Eberhard (1938), Gini (1938), Gumbel (1938b,c), Hertzman (1938), Jeffreys (1938), Derevitsky (1939), Eisenhart (1939), Gini (1939), l'Hermite (1939), Hulme & Symms (1939), Jeffreys (1939), Pitman (1939), Zappa (1939), Arley (1940), Arley & Buch (1940), Dodd (1940), Fréchet (1940a,b), Pizzetti (1940), Singleton (1940), Wald (1940), Wilson (1940), Zappa (1940), Brookner (1941), Daniels (1941), Fréchet (1941), Pizzetti (1941a,c), Rajalakshman (1941), Smirnoff (1941), Boldrini (1942), Dudding & Jennett (1942), Evans (1942), Fréchet (1942), Nair & Shrivastava (1942), Deming (1943), Wilks (1943), Wing, Price & Douglass (1944), Cramér (1946), Grant (1946), Kendall (1946), Tintner (1946), Eisenhart, Deming & Martin (1948b), Nair (1948b), Bartlett (1949), Harris & Tukey (1949), Tukey (1949c,d,e,f), Youden (1949)

ESTIMATES, BEST LINEAR UNBIASED
Daniell (1920), Pearson (1920), Grubbs & Weaver (1947), Wilks (1948), Nair (1949a), Yamanouchi (1949)

ESTIMATES, CLOSEST
Pitman (1937, 1939), Craig (1943), Geary (1944), Cramér (1946), Kendall (1946)

ESTIMATION, EFFICIENCY OF
Fisher (1922), Geary (1944), Cramér (1946), Kendall (1946), Mosteller (1946), Eisenhart, Deming & Martin (1948b), Geary (1948), Huzurbazar (1948), Stevens (1948), Wilks (1948), Bartlett (1949), Benson (1949), Nair (1949b), Walsh (1949d), Yamanouchi (1949)

ESTIMATION, INTERVAL—SEE INTERVAL ESTIMATION

ESTIMATION, POINT—SEE POINT ESTIMATION

ESTIMATORS, BEST LINEAR UNBIASED—SEE ESTIMATES, BEST LINEAR UNBIASED

ESTIMATORS, LINEAR—SEE LINEAR ESTIMATORS

ESTIMATORS, MEDIAN—SEE MEDIAN ESTIMATORS

ESTIMATORS, UNBIASED—SEE UNBIASED ESTIMATORS

EXCEEDANCE INTERVAL
Horton (1936), Gumbel (1941a,b), Beard (1942), Gumbel (1942b,c), Kimball (1942), Gumbel (1943a,b,c), Powell (1943), Price (1943), Gumbel (1945a)

EXCESS, MEASURES OF—SEE ALSO KURTOSIS, MEASURES OF
Charlier (1906), Crum (1923), Cramér (1928), Saville (1936), Gumbel (1944), Baker (1946), Cramér (1946), Grubbs & Weaver (1947), Peterson (1949a)

EXPONENTIAL DISTRIBUTION
Whitworth (1867), Walker (1914), von Bortkiewicz (1915), Morant (1921), Gumbel (1928), Neyman & Pearson (1928), Fisher (1929), von Bortkiewicz (1931), Pearson & Pearson (1932), Castellano (1933a), Craig (1933), Jeffreys (1933), Gumbel (1935-36, 1936b,c), von Mises (1936), Nair (1936), Pitman (1936), Ruark & Devol (1936), Sukhatme (1936), Camp (1937), Coutagne (1937), Gumbel (1937a,b,c,d), Pitman (1937), Sukhatme (1937), Thibaud (1937), Thibaud & Ferber (1937), Brenet & Armand (1938), Gumbel (1938c, 1939a,b,f), Pitman (1939), Ruark (1939), Weibull (1939a,b), Feller (1940), Fisher (1940), Garwood (1940), Nair (1940b), Paulson (1940, 1941), Beard (1942), Gumbel (1942a,b, 1943a,c, 1945a), Cramér (1946), Greenwood (1946), Kendall (1946), Fisher & Hollomon (1947), Gumbel (1947), Lehmann (1947), Epstein & Brooks (1948), McAdam, Geil, Woodard & Jenkins (1948), McMillan (1949), Walsh (1949b)

EXPONENTIAL TYPE INITIAL DISTRIBUTION
Gumbel (1933c,d, 1937-38, 1938c, 1940, 1941a,b), Gnedenko (1943), Gumbel (1943c, 1946a), Jones (1946), Gumbel (1947), Cox (1948), Epstein (1948a,b), Wilks (1948), Gumbel (1949), Juncosa (1949), McMillan (1949), Meĭzler (1949), Press (1949)

EXTREME DEVIATE
Nair (1948b)

EXTREME DEVIATE, STUDENTIZED—SEE STUDENTIZED EXTREME DEVIATE

EXTREMES (LARGEST AND SMALLEST)—SEE ALSO LARGEST (SMALLEST) SAMPLE VALUE
Bernoulli (1709), Maire & Boscovich (1755), Lambert (1760), Short (1763), Lambert (1765a,b), Bernoulli (1785), Laplace (1786, 1793, 1799), Prony (1804), Legendre (1805), Svanberg (1805), von Zach (1805), Delambre (1806-10), Laplace (1812), Mathieu (1813-14), Legendre (1814), van Beeck Calkoen (1816), Anonymous (1821), Fourier (1823), Gauss (1823), Cauchy (1824), Fourier (1824b), Muncke (1825), Fourier (1826a,b, 1829), Cauchy (1831), Fourier (1831), Poncelet (1835), Quetelet (1835), Hagen (1837), Bessel & Baeyer (1838), Stampfer (1839), Cournot (1843), Peirce (1852), Tchebychef (1854), Gould (1855), Lloyd (1855), Airy (1856), Winlock (1856), Petzval (1857), Airy (1861), Chauvenet (1863), Stone (1868), Glaisher (1872), Helmert (1872), Glaisher (1873), Peirce (1873), Stone (1873b), Todhunter (1873), Glaisher (1874), Jevons (1874), Stone (1874), Lexis (1875), Merriman (1877), Peirce (1878), Schott (1878), Jordan (1879), Edgeworth (1883b), Doolittle (1884), Merriman (1884), Wright (1884), Newcomb (1886), Bertrand (1887a), Edgeworth (1887a,b), Lehmann-Filhes (1887), Bertrand (1888a,b), Faye (1888), Bertrand (1889), Czuber (1890), Jordan (1890), Czuber (1891a), Pizzetti (1892), Vallier (1894), Cranz (1896), Poincaré (1896), Fechner (1897), Rodewald (1898), Czuber (1899), Sheppard (1899b), Wölffing (1899), Heydenreich (1903), Kirchberger (1903), Mitscherlich (1903), Saunder (1903), Mansion (1906), Vogeler (1907), Mazzuoli (1908), Bohlmann (1909),

Dunkel (1909), Goedseels (1909), de la Vallée Poussin (1909), Wellisch (1909), Goedseels (1910), Edgeworth (1911), Goedseels (1911), de la Vallée Poussin (1911), Newcomb (1912), Tits (1912), Goodwin (1913), Mansion (1913), Contarino (1914), Fuller (1914), Hazen (1914), Walker (1914), Savorgnan (1915), Spillman, Tolley & Reed (1916), Tolley (1916), Brunt (1917), Frank (1918), Niceforo (1919), Persons (1919), de la Vallée Poussin (1919), Griffith (1920), Stewart (1920a), Hall (1921), Fisher (1922), Dodd (1923), Griffith (1924), Horton (1924), Whittaker & Robinson (1924), Archibald (1925), Coolidge (1925), Goedseels (1925), Irwin (1925a,b), Reilly, Rae & Wheeler (1925), Tippett (1925), Jarvis (1926), Alliaume (1927a,b), Burgess (1927), Fréchet (1927), Galvani (1927), Student (1927), Tucker (1927), Fisher & Tippett (1928), Ogrodnikoff (1928), Sophister (1928), Fisher (1929), Gini & Galvani (1929), Lévy (1929), Pearson & Adyanthāya (1929), Rider (1929), Gibrat (1930), Gini (1930), Lidstone (1930), Peterson (1930), Steffensen (1930), Galvani (1931), Insolera (1931), Logan & Grodsky (1931), Pearson (1931), Pearson & Pearson (1931), Reinkober (1931), Steffensen (1931), Student (1931), Craig (1932b), Dodge (1932), de Finetti (1932), Gibrat (1932b), Grassberger (1932), Gumbel (1932), Jeffreys (1932a), Alexandrov & Žurkov (1933), Faulhaber (1933), Faulhaber, Buchholtz & Schulz (1933), Grassberger (1933), Gumbel (1933a,b,c,d,e,f,g), Hotelling (1933), Insolera (1933), Rider (1933), Tricomi (1933), Baidaff & Barral Souto (1934), Eyraud (1934a), de Finetti (1934), Foster (1934), de Franchis (1934), Gumbel (1934a,b,c,d,e,f,g), Insolera (1934), Slade (1934), Smith (1934), Thomson (1934), Baticle (1935), Birkeland & Frogner (1935), Fisher (1935), Gumbel (1935a,b,c,d, 1935-36), Insolera (1935), McKay (1935), Olds (1935), Pearson (1935), Smirnoff (1935), Thompson (1935), Foster (1936), Girshick (1936), Grassberger (1936), Gumbel (1936a,b,c, 1936-37), Horton (1936), Hotelling (1936a,b), Logan (1936), Mathews (1936), von Mises (1936), Pearson & Chandra Sekar (1936), Pitman (1936), Saville (1936), Slade (1936), Aitken (1937), Calichiopulo (1937), Camp (1937), Coutagne (1937), Gumbel (1937a,b,d,e, 1937-38), Insolera (1937-38), Madow (1937), Neyman (1937b), Pitman (1937), Thibaud (1937), Barral Souto (1938), Bartlett (1938), Brenet & Armand (1938), Bruen (1938), Cisbani (1938), Dodd (1938), von Eberhard (1938), Fisher (1938), Gumbel (1938a,b,c), Jeffreys (1938), Kimball (1938), Wilks (1938), Wishart (1938), Coutagne (1939), Creager (1939), Derevitsky (1939), Fisher (1939), Gini (1939), Gumbel (1939c,e,f), l'Hermite (1939), Hulme & Symms (1939), Lévy (1939), Pitman (1939), Roy (1939), Stevens (1939), de Vergottini (1939), Wald (1939), Weibull (1939a,b), Afanassiev (1940), Arley (1940), Fisher (1940), Garwood (1940), Gillett (1940), Grant (1940), Gumbel (1940), Kontorova (1940), Kullback & Frankel (1940), Lawley (1940), Zappa (1940), Chandra Sekar & Francis (1941), Cochran (1941), Daniels (1941), Davis (1941), Finney (1941), Gnedenko (1941), Gumbel (1941a,b), Hsu (1941b), Kontorova & Frenkel (1941), Rajalakshman (1941), Simon (1941), Smirnoff (1941), Tucker (1941), Aldanondo (1942), Beard (1942), Dodd (1942), Gaede (1942), Gumbel (1942b,c), Hartley (1942), Henry (1942), Kimball (1942), Moore & Morkovin (1942-44), Nair & Shrivastava (1942), Pearson & Hartley (1942), Roy (1942a,b), Wald (1942), Wilks (1942), Barricelli (1943), Craig (1943), Deming (1943), Gnedenko (1943), Gumbel (1943a,b,c), Kendall (1943), Knudsen (1943), Kontorova (1943), Pearson & Hartley (1943), Powell (1943), Price (1943), Rajalakshman (1943), Scheffé (1943), Wilks (1943), Geary (1944), Gumbel (1944), Hartley (1944), Robbins (1944b), Wing, Price & Douglass (1944), Daniels (1945), Fowler (1945), Gumbel (1945a,b), Gurney (1945), Nordquist (1945), Roy (1945), Tintner (1945), Tucker (1945a,b), Carlton (1946), Cramér (1946), Freudenthal (1946), Gumbel (1946a,b), Jones (1946), Kendall (1946), Kimball (1946a,b), Kontorova (1946), Mosteller (1946), Oding (1946), Roy (1946a,b), Tintner (1946), Bartlett (1947b), Bickerstaff (1947), Davidenkov, Shevandin & Wittmann (1947), Egudin (1947), Eisenhart, Hastay & Wallis (1947), Eisenhart & Solomon (1947), Elfving (1947), Fisher & Hollomon (1947), Gensamer, Saibel & Lowrie (1947), Gensamer, Saibel & Ransom (1947), Gumbel (1947), Gurney (1947), Gurney & Pearson (1947), Kincaid (1947), Lehmann (1947), Lord (1947), Purcell (1947), Ruggles & Brodie (1947), Tanenhaus (1947), Walsh (1947), Anderson (1948), Bartlett (1948), Cox (1948), Eisenhart & Martin (1948), Epstein (1948a,b), Epstein & Brooks (1948), Frankel (1948), Gartštein (1948), Geary (1948), Gumbel (1948), Higuchi, Leeper & Davis (1948), Hill & Schmidt (1948), Huzurbazar (1948), McAdam, Geil, Woodard & Jenkins (1948), Nair (1948a,b), Nanda (1948a,b), Pillai (1948), Schützenberger (1948), Thomas (1948), Tukey (1948b), Wilks (1948), Birnbaum & Zuckerman (1949), Cole (1949), Epstein (1949a,b), Godwin (1949b), Gumbel (1949),

Hartley (1949), Howell (1949), Juncosa (1949), Kimball (1949), Kontorova & Timoshenko (1949), Langbein (1949), Lévi (1949), Mather (1949), McMillan (1949), van Meer & Plantema (1949), Meĭzler (1949), Paulson (1949), Peterson (1949a,b), Potter (1949a,b), Press (1949), Prot (1949a,b,c), Purcell (1949), Scheffé (1949), Shone (1949), Smirnov (1949b), Tukey (1949b), Walsh (1949a,b,c), Weibull (1949)

EXTREME—VALUE DISTRIBUTION, TYPE I LARGEST
Gumbel (1933c,d, 1934a,b,c,e,f, 1935a, 1935-36), Smirnoff (1935), Gumbel (1936a,b,c, 1936-37), von Mises (1936), Coutagne (1937), Gumbel (1937b,c,d,e, 1937-38), Thibaud (1937), Brenet & Armand (1938), Gumbel (1938a,b,c), Coutagne (1939), Gumbel (1939c,e,f, 1940), Gnedenko (1941), Gumbel (1941a,b), Beard (1942), Gumbel (1942b,c), Kimball (1942), Barricelli (1943), Gnedenko (1943), Gumbel (1943a,b,c), Powell (1943), Price (1943), Gumbel (1945a,b), Nordquist (1945), Gumbel (1946b), Kimball (1946a,b), Gumbel (1947), Cox (1948), Gartštein (1948), Gumbel (1948), Wilks (1948), Kimball (1949), Lane & Lei (1949), Langbein (1949), Mather (1949), Meĭzler (1949), Potter (1949a,b), Press (1949), Smirnov (1949b)

EXTREME-VALUE DISTRIBUTION, TYPE I SMALLEST
Gumbel (1934a,b,c,e,f, 1935a, 1940, 1942b), Barricelli (1943), Gnedenko (1943), Gumbel (1943c, 1947), Cox (1948), Gartštein (1948), Wilks (1948), Juncosa (1949), Mather (1949), Smirnov (1949b)

EXTREME-VALUE DISTRIBUTION, TYPE II LARGEST
von Mises (1936), Gnedenko (1941, 1943), Gumbel (1947), Gartštein (1948), Wilks (1948), Meĭzler (1949), Smirnov (1949b)

EXTREME-VALUE DISTRIBUTION, TYPE II SMALLEST
Gnedenko (1943), Gumbel (1947), Gartštein (1948), Wilks (1948), Juncosa (1949), Smirnov (1949b)

EXTREME-VALUE DISTRIBUTION, TYPE III LARGEST
von Mises (1936), Kimball (1942), Gnedenko (1943), Gumbel (1946b, 1947), Gartštein (1948), Wilks (1948), Meĭzler (1949), Smirnov (1949b)

EXTREME-VALUE DISTRIBUTION, TYPE III SMALLEST—SEE ALSO WEIBULL DISTRIBUTION
Gnedenko (1941), Roller (1941), Gnedenko (1943), Gumbel (1947), Gartštein (1948), Wilks (1948), Juncosa (1949), Smirnov (1949b)

EXTREME-VALUE PROBABILITY PAPER—SEE PROBABILITY PAPER, EXTREME-VALUE

EXTREME-VALUE THEORY
Bernoulli (1709), Dove (1838), Chaplin (1880, 1882), Blümcke (1897, 1898, 1901), Slocum & Hancock (1906), Bohlmann (1909), Fuller (1914), Hazen (1914), Spillman, Tolley & Reed (1916), Tolley (1916), Frank (1918), Dodd (1923), Horton (1924), Tippett (1925), Jarvis (1926), Peirce (1926), Fréchet (1927), Tucker (1927), Birkeland (1928), Fisher & Tippett (1928), Köppen (1928), Lévy (1929), Lidstone (1930), Steffensen (1930), Insolera (1931), Logan & Grodsky (1931), Steffensen (1931), de Finetti (1932), Gibrat (1932b), Grassberger (1932), Gumbel (1932), Grassberger (1933), Gumbel (1933 a,b,c,d,e,f,g), Insolera (1933), Tricomi (1933), Eyraud (1934a), de Finetti (1934), Foster (1934), de Franchis (1934), Gumbel (1934a,b,c,d,e,f,g), Insolera (1934), Slade (1934), Gumbel (1935a,b,c,d, 1935-36), Insolera (1935), Smirnoff (1935), Foster (1936), Gumbel (1936a,b,c, 1936-37), Horton (1936), Logan (1936), von Mises (1936), Saville (1936), Slade (1936), Coutagne (1937), Gumbel (1937a,b,d,e, 1937-38), Insolera (1937-38), Thibaud (1937), Brenet & Armand (1938), Gumbel (1938a,b,c), Kimball (1938), Coutagne (1939), Creager (1939), Gumbel ( 1939b,c,e,f), Weibull (1939a,b), Afanassiev (1940), Gillett (1940), Grant (1940), Gumbel (1940), Kontorova (1940), Daniels (1941), Gnedenko (1941), Gumbel (1941a,b), Kontorova & Frenkel (1941), Rajalakshman (1941), Roller (1941), Tucker (1941),

Aldanondo (1942), Beard (1942), Gaede (1942), Gumbel (1942b,c), Kimball (1942), Barricelli (1943), Gnedenko (1943), Gumbel (1943a,b,c), Kontorova (1943), Powell (1943), Price (1943), Wilks (1943), Geary (1944), Gumbel (1944), Daniels (1945), Fowler (1945), Gumbel (1945a,b), Gurney (1945), Nordquist (1945), Tucker (1945a,b), Cramér (1946), Freudenthal (1946), Gumbel (1946b), Kendall (1946), Kimball (1946a), Kontorova (1946), Mosteller (1946), Oding (1946), Fisher & Hollomon (1947), Gensamer, Saibel & Lowrie (1947), Gensamer, Saibel & Ransom (1947), Gurney (1947), Gurney & Pearson (1947), Cox (1948), Epstein (1948a,b), Epstein & Brooks (1948), Gartštein (1948), Gumbel (1948), Higuchi, Leeper & Davis (1948), Hill & Schmidt (1948), McAdam, Geil, Woodard & Jenkins (1948), Thomas (1948), Wilks (1948), Epstein (1949a,b), Juncosa (1949), Kimball (1949), Kontorova & Timoshenko (1949), Langbein (1949), Lévi (1949), Mather (1949), McMillan (1949), van Meer & Plantema (1949), Meĭzler (1949), Peterson (1949b), Potter (1949a,b), Press (1949), Prot (1949a,c), Smirnov (1949b), Weibull (1949)

F DISTRIBUTION—SEE FISHER-SNEDECOR F DISTRIBUTION

FACTOR ANALYSIS, MULTIPLE—SEE MULTIPLE FACTOR ANALYSIS

FIBERS, STRENGTH OF BUNDLES OF—SEE BUNDLES OF FIBERS (OR THREADS), STRENGTH OF

FIDUCIAL INTERVALS
Fisher (1933), Wilks (1938), Pitman (1939), Paulson (1940), Kendall (1946)

FISHER-SNEDECOR F DISTRIBUTION
Snedecor (1937), Sukhatme (1937), Wishart (1938), Fisher (1939), Pearson (1939), Finney (1941), Thompson (1941), Curtiss (1943), Wald (1943), Wilks (1943), Brown & Tukey (1946), Cramér (1946), Tintner (1946), Duncan (1947), Eisenhart & Solomon (1947), Eisenhart, Deming & Martin (1948a), Nair (1948a), Tukey (1948c), Bartlett (1949), Hartley (1949), Tukey (1949a)

FISHER'S MULTIPLE COMPARISON TEST—SEE MULTIPLE COMPARISON TEST, FISHER'S

FISHER'S Z DISTRIBUTION
Fisher (1935), Snedecor (1937), Fisher & Yates (1938), Curtiss (1943), Scheffé (1943), Hartley (1944), Cramér (1946), Kendall (1946), Eisenhart, Deming & Martin (1948a), Tukey (1948c, 1949a)

FLOODS AND HYDROLOGY, APPLICATIONS TO
Herschel (1878), Fuller (1914), Hazen (1914), Hall (1921), Foster (1924), Horton (1924), Jarvis (1926), Goodrich (1927), Hazen (1930), Gibrat (1932a,b), Grassberger (1932, 1933), Foster (1934), Slade (1934), Foster (1936), Grassberger (1936), Gumbel (1936a), Horton (1936), Saville (1936), Slade (1936), Coutagne (1937), Gumbel (1937d, 1937-38), Fisher & Yates (1938), Gumbel (1938a,b,c), Kimball (1938), Coutagne (1939), Creager (1939), Gumbel (1939c,f), Grant (1940), Gumbel (1941a,b), Beard (1942), Gumbel (1942c), Kimball (1942), Gumbel (1943b,c), Powell (1943), Price (1943), Gumbel (1945a,b), Cramér (1946), Kendall (1946), Kimball (1946a,b), Epstein (1948b), Epstein & Brooks (1948), Gumbel (1948), Thomas (1948), Kimball (1949), Lane & Lei (1949), Langbein (1949), Potter (1949a,b)

FOODS AND NUTRITION, APPLICATIONS TO
Snedecor (1937), Grant (1946), Kendall (1946), Tintner (1946)

FUNCTIONS, CHARACTERISTIC—SEE CHARACTERISTIC FUNCTIONS

FUNCTIONS, DISCRIMINANT—SEE DISCRIMINANT ANALYSIS

FUNCTIONS, MOMENT-GENERATING—SEE MOMENT-GENERATING FUNCTIONS

FUNCTIONS OF MOMENTS—SEE MOMENTS AND FUNCTIONS OF MOMENTS

G DISTRIBUTION, DALY'S—SEE DALY'S G DISTRIBUTION

GALTON DISTRIBUTION—SEE LOGNORMAL (GALTON) DISTRIBUTION

GAMES OF CHANCE, APPLICATIONS TO
von Bortkiewicz (1922a,b), Whittaker & Robinson (1924), Goodrich (1927), Smith (1929), von Mises (1931), de Finetti (1932), Fisher & Yates (1938), Wald (1939), Kendall (1943, 1946)

GAMMA DISTRIBUTION
Burgess (1927), Pitman (1937, 1939), Curtiss (1943), Cramér (1946), Kendall (1946), Cox (1949), Epstein (1949a)

GAMMA DISTRIBUTION, GENERALIZED—SEE GENERALIZED GAMMA DISTRIBUTION

GAPS
Galton (1902), Pearson (1902), von Bortkiewicz (1915), Morant (1921), Irwin (1925a,b), Pearson (1926), Pearson & Pearson (1931, 1932), Rajalakshman (1943), Greenwood (1946), Bickerstaff (1947), Domb (1947), Duncan (1947), Kimball (1947), Moran (1947), Ruggles & Brodie (1947), Nair (1948b), Tukey (1948c, 1949a)

GAP-STRAGGLER PROCEDURE, TUKEY'S
Tukey (1948c, 1949a)

GAP, STUDENTIZED LARGEST—SEE STUDENTIZED LARGEST GAP

GAUGING
Stevens (1948)

GAUSSIAN DISTRIBUTION—SEE NORMAL (GAUSSIAN) DISTRIBUTION

GENERALIZED GAMMA DISTRIBUTION
Pitman (1939)

GENERALIZED LOGISTIC DISTRIBUTION
Gumbel (1944)

GENERALIZED NORMAL (GAUSSIAN) DISTRIBUTION
Münzner (1934), Slade (1934)

GENETICS, APPLICATIONS TO
Edgeworth (1913), Weinberg (1916), Pearson (1925), von Verschuer (1927), Lenz & von Verschuer (1928), von Mises (1931), Weinberg (1933), Savur (1937b), Snedecor (1937), Fisher & Yates (1938), Gottschick (1939), von Schelling (1941), Kendall (1943), Cramér (1946), Kendall (1946)

GEODESY, APPLICATIONS TO
Maire & Boscovich (1755), Boscovich (1757, 1760), Lambert (1765b), Laplace (1786, 1793), Puissant (1805), Svanberg (1805), von Zach (1805), Delambre (1806-10), von Lindenau (1806), Adrain (1808), Laplace (1812), Mathieu (1813-14), Adrain (1818), Fourier (1824c), Bessel & Baeyer (1838), Helmert

(1872), Peirce (1873), Todhunter (1873), Helmert (1877a), Jordan (1877a,b), Merriman (1877), Jordan (1879), Bruns (1882), Schreiber (1882), Wright (1884), Jordan (1888), Gore (1889), Runge (1890), Goedseels (1909, 1910), Contarino (1914), Brunt (1917), Lévy (1925), Friedrich (1927), Scarborough (1930), von Mises (1931), Eddington (1933), Calichiopulo (1937), Friedrich (1937), Bruen (1938), Jeffreys (1938), Arley & Buch (1940), Deming (1943), Kendall (1946)

GEOGRAPHY, APPLICATIONS TO
Hayford (1902), Goedseels (1909), Sloane (1923), Eells (1930), Ross (1930), Galvani (1933), Gini, Boldrini, Galvani & Venere (1933), Griffin (1933), Linders (1933), Scates (1933), Watkins (1933), Zappa (1939)

GEOMETRIC QUASI-MIDRANGES
Láska (1900)

GEOPHYSICS, APPLICATIONS TO
Madow (1937)

GIBRAT'S INDEX OF CONCENTRATION—SEE CONCENTRATION INDEX, GIBRAT'S

GINI'S INDEX OF CONCENTRATION—SEE CONCENTRATION INDEX, GINI'S

GINI'S MEAN DIFFERENCE (SIMPLE OR QUADRATIC)
von Andrae (1869), Jordan (1869), von Andrae (1872), Jordan (1872), Helmert (1876a), Gore (1889), Bowley (1901), Czuber (1903), Wellisch (1909), Dettori (1912), Gini (1912), Czuber (1914), Gini (1914a), Pietra (1915), Savorgnan (1915), Gini (1916b), Ricci (1916), Weinberg (1916), Gini (1918), Niceforo (1919), Dalton (1920), de Pietri-Tonelli (1920), Julín (1921), Savorgnan (1921), Pietra (1925), Gini (1926), Tucker (1927), von Verschuer (1927), Gumbel (1928), Lenz & von Verschuer (1928), de Gleria (1929), de Finetti & Paciello (1930), Gini (1930), de Gleria (1930), March (1930), von Bortkiewicz (1931), de Finetti (1931a), Galvani (1931, 1932), Gini (1932), Pietra (1932a,b), Yntema (1933), Dominedò (1934), Mortara (1934a,b), Anderson (1935), Castellano (1935), Pietra (1935a,b), Wold (1935), Gini (1936), Nair (1936), Bowley (1937), Wold (1937), Hertzman (1938), Gini (1939), Gottschick (1939), Pizzetti (1940, 1941a,c), Boldrini (1942), Kendall (1943, 1946), Hoeffding (1948), Nair (1949b)

GINI'S RATIO OF CONCENTRATION—SEE CONCENTRATION RATIO, GINI'S

GLAISHER'S ALTERNATIVE TO REJECTION OF OUTLIERS—SEE OUTLIERS, GLAISHER'S ALTERNATIVE TO REJECTION

GOMPERTZ DISTRIBUTION
Gumbel (1935d, 1937d,e)

GOODNESS-OF-FIT TESTS
Cramér (1928), Edgett (1931), von Mises (1931), Cantelli (1933), de Finetti (1933), Glivenko (1933), Kolmogoroff (1933), Smirnoff (1936), Kozakiewicz (1937), Neyman (1937a), Smirnoff (1937a), Waschakidse (1938), Smirnov (1939a,b,c), Pearson (1942), Scheffé (1943), Smirnov (1944), Cramér (1946), Greenwood (1946), Kendall (1946), Kimball (1946b), Bickerstaff (1947), Kimball (1947), von Mises (1947), Smirnov (1948), Chung (1949), Doob (1949), Maniya (1949), Noether (1949), Smirnov (1949a)

GOODWIN'S CRITERION FOR REJECTION OF OUTLIERS—SEE OUTLIERS, GOODWIN'S CRITERION FOR REJECTION

GRAIN SIZE, APPLICATIONS TO
Rosin & Rammler (1934), Roller (1941)

GROUP AVERAGES, METHOD OF—SEE METHOD OF GROUP AVERAGES

GUMBEL DISTRIBUTION—SEE EXTREME-VALUE DISTRIBUTION, TYPE I (LARGEST OR SMALLEST)

GUMBEL'S MTH POINT STATISTIC
Gumbel (1942a)

GUNNERY, APPLICATIONS TO—SEE BALLISTICS AND GUNNERY, APPLICATIONS TO

GUST LOADS, APPLICATIONS TO
Press (1949)

HALF-NORMAL DISTRIBUTION
Bertrand (1888c), Gumbel (1928), McKay (1935), Kendall (1946), Smirnov (1949b)

HALF RANGE, DIFFERENCES AT—SEE DIFFERENCES AT HALF RANGE

HARMONIC ANALYSIS
Fisher (1929, 1939, 1940), Davis (1941), Greenwood (1946), Kendall (1946), Nair (1948a), Hartley (1949)

HEYDENREICH'S CRITERION FOR REJECTION OF OUTLIERS—SEE OUTLIERS, HEYDENREICH'S CRITERION FOR REJECTION

HIGHER MOMENTS—SEE MOMENTS, HIGHER

HOLMES' MEASURE OF CONCENTRATION—SEE CONCENTRATION, HOLMES' MEASURE OF

HOMOPHILY, BETWEEN DEVIATIONS, INDEX OF
Gini (1915a,b, 1916a,b, 1926)

HOMOPHILY, BETWEEN INTENSITIES, INDEX OF
Gini (1915a,b, 1916a,b), Weinberg (1916), Gini (1926)

HOMOPHILY, BETWEEN VARIATIONS , INDEX OF
Gini (1915a,b, 1916a,b, 1926)

HOMOPHILY, MEASURES OF
Gini (1915a,b, 1916a,b), Weinberg (1916), Pietra (1925), Gini (1926)

HOTELLING'S T-SQUARE DISTRIBUTION
Wilks (1943), Cramér (1946), Kendall (1946), Tintner (1946), Lehmann (1947), Rao (1948), Wolfowitz (1949)

HOUSNER-BRENNAN ESTIMATE OF REGRESSION
Housner & Brennan (1948)

HULME-SYMMS ALTERNATIVE TO REJECTION OF OUTLIERS—SEE OUTLIERS, HULME-SYMMS ALTERNATIVE TO REJECTION

HYDRAULIC PROBABILITY PAPER—SEE PROBABILITY PAPER, HYDRAULIC

HYDROLOGY, APPLICATIONS TO—SEE FLOODS AND HYDROLOGY, APPLICATIONS TO

HYPERBOLA, INDICATING—SEE INDICATING HYPERBOLA

INDEPENDENCE AND DEPENDENCE
Gini (1910, 1926), Gumbel (1946a), Kendall (1946), Bickerstaff (1947)

INDEPENDENCE AND DEPENDENCE, INDICES OF
Gini (1910)

INDEX NUMBERS, APPLICATIONS TO
Edgeworth (1887-90), Yule (1911), Persons (1919), Keynes (1921), Crum (1923), Rietz (1924), Pietra (1925), Jordan (1927), Bowley (1928), Kendall (1946), Tintner (1946)

INDICATING HYPERBOLA
Goedseels (1925)

INDICES OF DISSIMILARITY, CONNECTION AND CONCORDANCE (HOMOPHILY), QUADRATIC
Gini (1915b, 1916b), Pietra (1925), Gini (1926)

INDICES OF DISSIMILARITY, CONNECTION AND CONCORDANCE (HOMOPHILY), SIMPLE
Gini (1915a,b, 1916b), Pietra (1925), Gini (1926)

INITIAL DISTRIBUTION, EXPONENTIAL TYPE—SEE EXPONENTIAL TYPE INITIAL DISTRIBUTION

INSPECTION BY VARIABLES—SEE VARIABLES INSPECTION

INTERDECILE RANGE
Galton (1881), Kendall (1943)

INTER- (INTRA-) FRACTIONAL MEAN DIFFERENCE
von Bortkiewicz (1931)

INTERMEDIANT
March (1930), von Bortkiewicz (1931)

INTERPOLATION, CAUCHY'S METHOD OF—SEE CAUCHY'S METHOD OF INTERPOLATION

INTERQUARTILE RANGE
Quetelet (1835, 1846), Herschel (1850), Merriman (1877), Galton (1881), Gore (1889), Pizzetti (1892), Edgeworth (1898), Lévy (1925), von Bortkiewicz (1931), Gumbel (1936a), Fréchet (1942), Bickerstaff (1947), Benson (1949), Godwin (1949b)

INTERVAL ESTIMATION
Lexis (1875), Fisher (1933), Thompson (1936), Neyman (1937b), Savur (1937a), Snedecor (1937),

Thompson (1938), Wilks (1938), Eisenhart (1939), Pitman (1939), Wald (1939), Wald & Wolfowitz (1939), Kullback & Frankel (1940), Nair (1940b), Paulson (1940), Wald (1940), Kolmogoroff (1941), von Schelling (1941), Wald & Wolfowitz (1941), Wilks (1941), Gumbel (1943c), Pearson & Hartley (1943), Wilks (1943), Scheffé & Tukey (1945), Cramér (1946), Grant (1946), Kendall (1946), Kimball (1946a), Bickerstaff (1947), Eisenhart, Deming & Martin (1948b), Murphy (1948), Noether (1948), Schützenberger (1948), Wilks (1948), Birnbaum & Zuckerman (1949), Howell (1949), Kac (1949), Kimball (1949), Noether (1949), Press (1949), Tukey (1949b), Walsh (1949a), Wolfowitz (1949)

INTERVAL, EXCEEDANCE—SEE EXCEEDANCE INTERVAL

INTERVAL, OCCURRENCE—SEE OCCURRENCE INTERVAL

INTERVAL, RECURRENCE—SEE RECURRENCE INTERVAL

INTERVALS, BOTH ONE-SIDED AND TWO-SIDED—SEE TESTS (OR INTERVALS), BOTH ONE-SIDED AND TWO-SIDED

INTERVALS, CONFIDENCE—SEE CONFIDENCE BOUNDS

INTERVALS, FIDUCIAL—SEE FIDUCIAL INTERVALS

INTERVALS, ONE-SIDED—SEE TESTS (OR INTERVALS), ONE-SIDED

INTERVALS, TWO-SIDED—SEE TESTS (OR INTERVALS), TWO-SIDED

INTRINSIC ACCURACY
Fisher (1922), Pitman (1936), Kendall (1946), Huzurbazar (1948)

IRWIN'S CRITERION FOR REJECTION OF OUTLIERS—SEE OUTLIERS, IRWIN'S CRITERION FOR REJECTION

ISOSCELES TRIANGULAR DISTRIBUTION, CONTINUOUS
Simpson (1757), Gauss (1823), Bréger (1876), Merriman (1877), Pizzetti (1892), Gini (1932), Cramér (1946)

ISOSCELES TRIANGULAR DISTRIBUTION, DISCRETE
Simpson (1756, 1757), Lagrange (1774), Trembley (1804), Merriman (1877), Pizzetti (1892)

JEFFREYS' ALTERNATIVE TO REJECTION OF OUTLIERS—SEE OUTLIERS, JEFFREYS' ALTERNATIVE TO REJECTION

JORDAN'S ALGEBRAIC DISTRIBUTION, FIRST
Helmert (1877b), Jordan (1877a,b), Gore (1889), Vogeler (1907)

JORDAN'S ALGEBRAIC DISTRIBUTION, SECOND
Jordan (1877b), Gore (1889), Jordan (1890), Blümcke (1897, 1898, 1901), Vogeler (1907)

KAPTEYN'S SYSTEM OF SKEW FREQUENCY CURVES
Kapteyn (1903), Goodrich (1927), Gibrat (1930), Foster (1936)

KNOWN DISTRIBUTION, ANY
Pearson (1902, 1920), Gumbel (1939d), Daniels (1945), Epstein (1949b)

KOLMOGOROV-SMIRNOV STATISTIC (OR TEST)
Glivenko (1933), Kolmogoroff (1933), Smirnoff (1937a), Waschakidse (1938), Smirnov (1939a,b,c), Kolmogoroff (1941), Wald & Wolfowitz (1941), Smirnov (1944), Feller (1948), Smirnov (1948), Wilks (1948), Chung (1949), Doob (1949), Kac (1949), Maniya (1949), Noether (1949), Wolfowitz (1949)

K-SAMPLE TESTS (K=3 OR MORE)—SEE ALSO ANALYSIS OF VARIANCE
Fisher & Yates (1938), Wishart (1938), Fisher (1939), Newman (1939), Pearson (1939), Chandra Sekar & Francis (1941), Cochran (1941), Finney (1941), Roy (1942b), Cramér (1946), Kendall (1946), Roy (1946a,b), Bartlett (1947b), Eisenhart & Solomon (1947), Anderson (1948), Geary (1948), Nair (1948a), Rao (1948), Tukey (1948c), Wilks (1948), Paulson (1949), Tukey (1949a), Walsh (1949c)

KURTOSIS, MEASURES OF—SEE ALSO EXCESS, MEASURES OF
Yule (1896a), Niceforo (1919), Pearson (1926), Pearson & Adyanthāya (1928), Lidstone (1930), Pearson (1931), Shewhart (1931), Eddington (1933), Sastry (1935), Slade (1936), Gumbel (1942c), Hartley (1942), Cramér (1946), Greenwood (1946), Kendall (1946), Elfving (1947), Grubbs & Weaver (1947), Gumbel (1947), Gurney (1947), Gurney & Pearson (1947), Kimball (1947), Cox (1948), Tukey (1948b), Benson (1949), Shone (1949)

LAPLACE'S DISTRIBUTION, FIRST—SEE DOUBLE EXPONENTIAL DISTRIBUTION

LAPLACE'S DISTRIBUTION, SECOND—SEE NORMAL (GAUSSIAN) DISTRIBUTION

LARGEST (ABSOLUTE) DEVIATION
Dove (1838), Rohne (1907), Haar (1918), de la Vallée Poussin (1919), McKay (1935), Prot (1949b)

LARGEST SAMPLE VALUE
Bernoulli (1709), Svanberg (1805), Mathieu (1813-14), Fourier (1823, 1824a,b, 1826a,b, 1829, 1831), Poncelet (1835), Cournot (1843), Tchebychef (1854), Halphen (1873), Lemoine (1873), Lexis (1875), Bréger (1876), Helmert (1877b), Jordan (1877a,b), Merriman (1877), Herschel (1878), Jordan (1879), Lalanne (1879), van Pesch (1882, 1884), Bertrand (1887b), Edgeworth (1887a), Delauney (1888), Bertrand (1889), Czuber (1891a), Pizzetti (1892), Vallier (1894), Cranz (1896), Blümcke (1897), Fechner (1897), Blümcke (1898), Czuber (1899), Wölffing (1899), Láska (1900), Blümcke (1901), Kirchberger (1903), Kozák (1907), Vogeler (1907), Kozák (1908-10), Bohlmann (1909), Dunkel (1909), Goedseels (1909, 1910, 1911), de la Vallée Poussin (1911), Tits (1912), Mansion (1913), Fuller (1914), Hazen (1914), Walker (1914), Mantel (1915), Frank (1918), Griffith (1920), Stewart (1920a), Hall (1921), von Bortkiewicz (1922a,b), Dodd (1923), von Mises (1923), Neyman (1923), Wilson (1923), Griffith (1924), Horton (1924), Jackson (1924), Whittaker & Robinson (1924), Bottema (1925), Goedseels (1925), Irwin (1925a,b), Tippett (1925), Pearson (1926), Burgess (1927), Fréchet (1927), Galvani (1927), Bartels (1928), Birkeland (1928), Fisher & Tippett (1928), Köppen (1928), Ogrodnikoff (1928), Sophister (1928), Fisher (1929), Gini & Galvani (1929), Lévy (1929), Pearson & Adyanthāya (1929), Rider (1929), Gini (1930), Lidstone (1930), Peterson (1930), Steffensen (1930), Galvani (1931), Insolera (1931), Logan & Grodsky (1931), Pearson (1931), Reinkober (1931), Steffensen (1931), Student (1931), Dodge (1932), de Finetti (1932), Gibrat (1932b), Grassberger (1932), Gumbel (1932), Jeffreys (1932a), Alexandrov & Žurkov (1933), Faulhaber (1933), Faulhaber, Buchholtz & Schulz (1933), Glivenko (1933), Grassberger (1933), Gumbel (1933a,b,c,d,e,f,g), Hotelling (1933), Insolera (1933), Kolmogoroff (1933), Rider (1933), Tricomi (1933), Baidaff & Barral Souto (1934), Eyraud (1934a), de Finetti (1934), Foster (1934), de Franchis (1934), Gumbel (1934a,b,c,d,e,f,g), Insolera (1934), Smith (1934), Thomson (1934), Baticle (1935), Birkeland & Frogner (1935), Fisher (1935), Gumbel (1935a,b,c,d, 1935-36), Insolera (1935), McKay (1935), Olds (1935), Pearson (1935), Smirnoff (1935),

Thompson (1935), Foster (1936), Girshick (1936), Grassberger (1936), Gumbel (1936a,b,c, 1936-37), Horton (1936), Hotelling (1936a,b), Logan (1936), Mathews (1936), von Mises (1936), Pearson & Chandra Sekar (1936), Pitman (1936), Slade (1936), Aitken (1937), Camp (1937), Coutagne (1937), Gumbel (1937a,b,d,e, 1937-38), Insolera (1937-38), Kozakiewicz (1937), Neyman (1937b), Pitman (1937), Thibaud (1937), Barral Souto (1938), Bartlett (1938), Brenet & Armand (1938), Bruen (1938), Cisbani (1938), Dodd (1938), von Eberhard (1938), Fisher (1938), Gumbel (1938a,b,c), Kimball (1938), Waschakidse (1938), Wilks (1938), Wishart (1938), Coutagne (1939), Creager (1939), Derevitsky (1939), Fisher (1939), Gini (1939), Girshick (1939), Gumbel (1939b,c,e,f), l'Hermite (1939), Hsu (1939), Lévy (1939), Pitman (1939), Roy (1939), Wald (1939), Arley (1940), Fisher (1940), Garwood (1940), Gillett (1940), Grant (1940), Gumbel (1940), Kontorova (1940), Kullback & Frankel (1940), Lawley (1940), Roy (1940a,b), Zappa (1940), Chandra Sekar & Francis (1941), Cochran (1941), Daniels (1941), Davis (1941), Finney (1941), Gnedenko (1941), Gumbel (1941a,b), Rajalakshman (1941), Simon (1941), Smirnoff (1941), Tucker (1941), Aldanondo (1942), Beard (1942), Dodd (1942), Gumbel (1942b,c), Hartley (1942), Henry (1942), Kimball (1942), Moore & Morkovin (1942-44), Nair & Shrivastava (1942), Pearson & Hartley (1942), Roy (1942a,b), Wald (1942), Wilks (1942), Barricelli (1943), Craig (1943), Gnedenko (1943), Gumbel (1943a,b,c), Kendall (1943), Knudsen (1943), Kontorova (1943), Pearson & Hartley (1943), Powell (1943), Price (1943), Rajalakshman (1943), Scheffé (1943), Wilks (1943), Geary (1944), Gumbel (1944), Hartley (1944), Robbins (1944b), Gumbel (1945a,b), Nordquist (1945), Roy (1945), Carlton (1946), Cramér (1946), Gumbel (1946b), Kendall (1946), Kimball (1946a,b), Kontorova (1946), Mosteller (1946), Roy (1946a,b), Tintner (1946), Bartlett (1947b), Bickerstaff (1947), Davidenkov, Shevandin & Wittmann (1947), Egudin (1947), Eisenhart, Hastay & Wallis (1947), Eisenhart & Solomon (1947), Elfving (1947), Fisher & Hollomon (1947), Gensamer, Saibel & Lowrie (1947), Gensamer, Saibel & Ransom (1947), Gumbel (1947), Kincaid (1947), Lehmann (1947), Lord (1947), Purcell (1947), Ruggles & Brodie (1947), Walsh (1947), Anderson (1948), Cox (1948), Eisenhart & Martin (1948), Epstein (1948b), Epstein & Brooks (1948), Gartštein (1948), Geary (1948), Gumbel (1948), Hill & Schmidt (1948), Huzurbazar (1948), McAdam, Geil, Woodard & Jenkins (1948), Nair (1948a,b), Nanda (1948a,b), Pillai (1948), Schützenberger (1948), Thomas (1948), Tukey (1948b,c), Wilks (1948), Birnbaum & Zuckerman (1949), Cole (1949), Epstein (1949a,b), Gumbel (1949), Hartley (1949), Howell (1949), Kimball (1949), Kontorova & Timoshenko (1949), Langbein (1949), Mather (1949), Meĭzler (1949), Paulson (1949), Potter (1949a,b), Press (1949), Purcell (1949), Smirnov (1949b), Tukey (1949a,b), Walsh (1949a,b,c)

LEAST ABSOLUTE VALUES, METHOD OF

Boscovich (1757, 1760), Lalande (1771), Bernoulli (1785), Laplace (1793, 1799), Prony (1804), Puissant (1805), Svanberg (1805), von Lindenau (1806), Gauss (1809), Laplace (1812), Delambre (1813), Mathieu (1813-14), van Beeck Calkoen (1816), Adrain (1818), Fourier (1823), Gauss (1823), Fourier (1824b,c), von Riese (1830), Fourier (1831), Dirichlet (1836), Hagen (1837), De Morgan (1847), von Andrae (1860), Boudin (1865), Todhunter (1865), Glaisher (1872), Todhunter (1873), Fechner (1874), Merriman (1877), Bruns (1882), Schreiber (1882), Edgeworth (1886a, 1887b,c,d,f, 1887-90), Turner (1887), Edgeworth (1888), Jordan (1888), Gore (1889), Estienne (1890), Runge (1890), Czuber (1891b), Pizzetti (1892), Edgeworth (1898), Czuber (1899), Estienne (1900b), Edgeworth (1911), Keynes (1911), Mansion (1913), Fisher (1920), Jackson (1921), Keynes (1921), Edgeworth (1923), Jackson (1924), Whittaker & Robinson (1924), Estienne (1926-27), Friedrich (1927), Bowley (1928), Rhodes (1930), Smith (1934), Darmois (1936), Friedrich (1937), Barral Souto (1938), Bruen (1938), Eisenhart (1939), Jeffreys (1939), Pitman (1939), Singleton (1940), Boldrini (1942)

LEAST MTH POWERS, METHOD OF

Jackson (1924), Bruen (1938)

LEAST NUMBER, METHOD OF—SEE LEAST ZERO POWERS, METHOD OF

LEAST SIGNIFICANT DIFFERENCE (LSD) TEST—SEE MULTIPLE T (LSD) TEST

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LEAST SQUARES, METHOD OF
Legendre (1805), Puissant (1805), Delambre (1806-10), von Lindenau (1806), Adrain (1808), Gauss (1809), Laplace (1810, 1811a,b, 1812), Delambre (1813), Mathieu (1813-14), Legendre (1814), Adrain (1818), Anonymous (1821), Gauss (1823), Fourier (1824c), Ivory (1825), Muncke (1825), Hauber (1830-32), von Riese (1830), Encke (1832-34), Dirichlet (1836), Hagen (1837), Bessel & Baeyer (1838), Cournot (1843), Gerling (1843), Ellis (1844), De Morgan (1847), Herschel (1850), Petzval (1857), Airy (1861), Chauvenet (1863), Boudin (1865), Todhunter (1865), Zachariae (1871), Glaisher (1872, 1873), Peirce (1873), Stone (1873b), Todhunter (1873), Fechner (1874), Glaisher (1874), Jevons (1874), Stone (1874), Laurent (1875), Merriman (1877), Schreiber (1882), Edgeworth (1883b), Doolittle (1884), Merriman (1884), Wright (1884), Edgeworth (1886a, 1887b,c,d,f, 1887-90), Turner (1887), Edgeworth (1888), Gore (1889), Estienne (1890), Czuber (1891a,b), Pizzetti (1892), Goedseels & Mansion (1893), Poincaré (1896), Edgeworth (1898), Czuber (1899), Wölffing (1899), Estienne (1900b), Bowley (1901), Czuber (1903), Saunder (1903), Mansion (1906), Goedseels (1909), Charlier (1910), Goedseels (1910), Edgeworth (1911), Goedseels (1911), Keynes (1911), de la Vallée Poussin (1911), Fisher (1912), Goodwin (1913), Mansion (1913), Tolley (1916), Becker (1917), Campbell (1920), Fisher (1920), Jackson (1921), Jones (1921), Keynes (1921), Edgeworth (1923), Jackson (1924), Rietz (1924), Whittaker & Robinson (1924), Goedseels (1925), Lévy (1925), Reilly, Rae & Wheeler (1925), Estienne (1926-27), Friedrich (1927), Bowley (1928), Dufton (1928), Smith (1929), Gibrat (1930), Rhodes (1930), Scarborough (1930), von Bortkiewicz (1931), von Mises (1931), Jeffreys (1932b), Eddington (1933), Smith (1934), Bond (1935), Darmois (1936), Slade (1936), Friedrich (1937), Barral Souto (1938), Bose (1938), Bruen (1938), Fisher (1938), Gumbel (1938c), Eisenhart (1939), Jeffreys (1939), Pitman (1939), Arley & Buch (1940), Wald (1940), Boldrini (1942), Nair & Shrivastava (1942), Deming (1943), Nair & Banerjee (1943), Wilks (1943), Tintner (1945), Cramér (1946), Kendall (1946), Tintner (1946), Housner & Brennan (1948), Bartlett (1949), Potter (1949a,b), Yamanouchi (1949)

LEAST ZERO POWERS, METHOD OF
Bruen (1938)

LIFE INSURANCE, APPLICATIONS TO—SEE MORTALITY AND LIFE INSURANCE, APPLICATIONS TO

LIFE LENGTH, HUMAN, APPLICATIONS TO—SEE MORTALITY AND LIFE INSURANCE, APPLICATIONS TO

LIFE LENGTH OF DEVICES, APPLICATIONS TO—SEE DEVICE LIFE, APPLICATIONS TO

LIFE LENGTH OF ORGANISMS (NONHUMAN), APPLICATIONS TO
Gumbel (1933e), von Schelling (1941)

LIKELIHOOD-RATIO TESTS
Anderson (1946), Bartlett (1947b), Lehmann (1947), Rao (1948), Wilks (1948)

LIMITING AGE
Bohlmann (1909), Dodd (1923), Lidstone (1930), Steffensen (1930), Insolera (1931), Steffensen (1931), Gumbel (1932, 1933e,f,g), Insolera (1933), de Finetti (1934), de Franchis (1934), Gumbel (1934d,f,g), Insolera (1934), Gumbel (1935c,d), Insolera (1935), Gumbel (1936c, 1936-37), Coutagne (1937), Gumbel (1937a,e), Insolera (1937-38), Gumbel (1938a,c, 1939f), Kendall (1946), Epstein (1948b)

LIMITS, CONFIDENCE—SEE CONFIDENCE BOUNDS

LIMITS, TOLERANCE—SEE TOLERANCE LIMITS

LINEAR ESTIMATORS
Craig (1943), Mosteller (1946), Wilks (1948), Yamanouchi (1949)

LINEAR HYPOTHESES, TESTS OF
Anderson (1945), Roy (1945), Anderson (1946), Kendall (1946), Lehmann (1947)

LINEAR REGRESSION—SEE REGRESSION, LINEAR

LINEAR UNBIASED ESTIMATES, BEST—SEE ESTIMATES, BEST LINEAR UNBIASED

LOGISTIC DISTRIBUTION
Gumbel (1944, 1945a)

LOGISTIC DISTRIBUTION, GENERALIZED—SEE GENERALIZED LOGISTIC DISTRIBUTION

LOGNORMAL (GALTON) DISTRIBUTION
Edgeworth (1887-90), Galton (1889), Yule (1907), Keynes (1911), Contarino (1914), Fuller (1914), Hazen (1914), Keynes (1921), Gibrat (1930), Hazen (1930), d'Addario (1931), Gibrat (1931, 1932a,b), Grassberger (1932, 1933), d' Addario (1934a), Castellano (1935), Grassberger (1936), Gumbel (1936a), Coutagne (1937), Gumbel (1937c,d, 1937-38, 1938a,c), Kimball (1938), Coutagne (1939), Gumbel (1939f), Roller (1941), Beard (1942), Kimball (1942), Gumbel (1943c), Tanenhaus (1947), Epstein (1948b), Lane & Lei (1949), Lévi (1949)

LOGNORMAL PROBABILITY PAPER—SEE PROBABILITY PAPER, LOGNORMAL

LONG-TAILED DISTRIBUTIONS
Edgeworth (1886a, 1887d), Brown & Tukey (1946), Hastings, Mosteller, Tukey & Winsor (1947)

LORD'S U DISTRIBUTION
Lord (1947)

LORENZ CURVE—SEE ALSO CONCENTRATION CURVE
Holmes (1905), Lorenz (1905), Watkins (1905, 1908), Persons (1909), Watkins (1909), King (1912), Gini (1914a), Ricci (1916), Dalton (1920), Amoroso (1925), von Bortkiewicz (1931), de Finetti (1931a), Galvani (1931), d'Addario (1932), Castellano (1933a,b), Yntema (1933), Dominedò (1934), Pietra (1935a), d'Addario (1936), Gini (1936), Bresciani-Turroni (1939), Gini (1939), Davis (1941)

LOWER CONFIDENCE BOUND—SEE CONFIDENCE BOUNDS

LOWER SEMIRANGE—SEE SEMIRANGE, LOWER

LOWER TOLERANCE LIMIT—SEE TOLERANCE LIMITS

MAGNETISM, APPLICATIONS TO
Lloyd (1855), Grant (1946), Kendall (1946)

MATERIAL STRENGTH—SEE ALSO BUNDLES OF FIBERS (OR THREADS), STRENGTH OF
Pearson (1931), Dodge (1932), Davies & Pearson (1934), Pearson (1935), Derevitsky (1939), Dudding & Jennett (1942), Gaede (1942), Grant (1946), Kendall (1946), Eisenhart & Solomon (1947), Tanenhaus (1947), Epstein (1948a), Benson (1949), Epstein (1949a), Howell (1949), van Meer & Plantema (1949), Prot (1949b), Weibull (1949)

MATERIAL STRENGTH, SIZE EFFECT ON—SEE SIZE EFFECT ON MATERIAL STRENGTH

MATRICES, CHARACTERISTIC ROOTS OF—SEE CHARACTERISTIC ROOTS OF MATRICES

MATRICES, COVARIANCE—SEE COVARIANCE MATRICES

MATRICES, EIGENVALUES OF—SEE CHARACTERISTIC ROOTS OF MATRICES

MAXIMUM LIKELIHOOD, METHOD OF (AND/OR MAXIMUM PROBABILITY, METHOD OF)
Lambert (1760), Lagrange (1774), Bernoulli (1778), Laplace (1781), Bernoulli (1785), Gauss (1809), Laplace (1810), Todhunter (1865), Merriman (1877), Gore (1889), Czuber (1890, 1891a), Pizzetti (1892), Czuber (1899), Fisher (1912, 1922, 1934), Sukhatme (1937), Lawley (1940), Nair & Shrivastava (1942), Kendall (1943), Wilks (1943), Gumbel (1945a), Tintner (1945), Cramér (1946), Kendall (1946), Kimball (1946a), Eisenhart & Martin (1948), Geary (1948), Huzurbazar (1948), Kimball (1949), Mather (1949), Nair (1949b)

MAXIMUM SUM OF FOURTH POWERS OF ERROR P.D.F., METHOD OF
Euler (1778), von Zach (1805), Todhunter (1865), Merriman (1877), Pizzetti (1892)

MAXWELL DISTRIBUTION
Gumbel (1927, 1928)

MAZZUOLI'S CRITERION FOR REJECTION OF OUTLIERS—SEE OUTLIERS, MAZZUOLI'S CRITERION FOR REJECTION

MCKAY'S CRITERION FOR REJECTION OF OUTLIERS—SEE OUTLIERS, MCKAY'S CRITERION FOR REJECTION

MEAN DEVIATION (FROM MEDIAN)—SEE AVERAGE (ABSOLUTE) DEVIATION (FROM MEDIAN)

MEAN DIFFERENCE, GINI'S—SEE GINI'S MEAN DIFFERENCE

MEAN DIFFERENCE, INTER- (INTRA-) FRACTIONAL—SEE INTER- (INTRA-) FRACTIONAL MEAN DIFFERENCE

MEAN OSCILLATION
Pizzetti (1941c)

MEAN QUOTIENT
Hojo (1933), Pizzetti (1941c)

MEANS—SEE AVERAGES (MEASURES OF CENTRAL TENDENCY)

MEANS, ASCENDING
Pizzetti (1940)

MEANS, COMBINATION (COMBINATORIAL)
Fechner (1874), Merriman (1877), Gore (1889), Pizzetti (1892), Dunkel (1909), Gini (1938), Zappa (1940)

MEANS, DESCENDING
Pizzetti (1940)

MEANS, POWER

Fechner (1874), Merriman (1877), Gore (1889), Pizzetti (1892), Rietz (1924), de Finetti (1931b), Galvani (1931), Barral Souto (1938), Bruen (1938), Gini (1938), Zappa (1940), Boldrini (1942), Tukey (1949c)

MEANS, TRIMMED—SEE TRIMMED MEANS

MEANS, WINSORIZED—SEE WINSORIZED MEANS

MEDIAL

Winkler (1924), March (1930), von Bortkiewicz (1931), Julin (1933)

MEDIAN

Lalande (1771), Laplace (1774), D. Bernoulli (1778), J. Bernoulli (1785), Gauss (1809), Laplace (1812), Delambre (1813), Anonymous (1821), Fourier (1824c), Quetelet (1835), Dirichlet (1836), Cournot (1843), Herschel (1850), von Andrae (1860), Boudin (1865), Todhunter (1865), Venn (1866), Galton (1869), Zachariae (1871), Glaisher (1872), Todhunter (1873), Fechner (1874), Galton (1875), Merriman (1877), Wilson (1877), Galton (1881), Wright (1884), Edgeworth (1885, 1886a,b, 1887a,b,c,d,e,f, 1887-90), Turner (1887), Edgeworth (1888), Galton (1888, 1889), Gore (1889), Pizzetti (1889), Estienne (1890), Czuber (1891a,b), Venn (1891), Holmes (1892), Pizzetti (1892), Edgeworth (1893), Mendeleev (1895), Pearson (1895), Galton (1896), Yule (1896a,b), Fechner (1897), Edgeworth (1898), Czuber (1899), Galton (1899a,b), Sheppard (1899a), Estienne (1900a,b), Bowley (1901), Hayford (1902), Kapteyn (1903), Edgeworth (1905), Holmes (1905), Lorenz (1905), Watkins (1905); Charlier (1906), Galton (1907a,b,c), Hooker (1907), Yule (1907), Persons (1909), Watkins (1909), Edgeworth (1911), Keynes (1911), Yule (1911), Dettori (1912), Gini (1912), King (1912), Dodd (1913), Edgeworth (1913), Czuber (1914), Dodd (1914), Gini (1914a), Pietra (1915), Trachtenberg (1915), Ricci (1916), Tolley (1916), Becker (1917), Brunt (1917), Secrist (1917), Vinci (1918), Niceforo (1919), Persons (1919), Dalton (1920), Daniell (1920), Fisher (1920), Pearson (1920); de Pietri-Tonelli (1920), Czuber (1921), Jackson (1921), Jones (1921), Julin (1921), Keynes (1921), Dodd (1922), Fisher (1922), Crum (1923), Dodd (1923), Edgeworth (1923), Goldhizer (1923), Jackson (1923), Sloane (1923), Wilson (1923), Foster (1924), Haag (1924), Rietz (1924), Whittaker & Robinson (1924), Winkler (1924), Coolidge (1925), Pietra (1925), Estienne (1926-27), Gini (1926), Peirce (1926), Burgess (1927), Estienne (1927), Jordan (1927), Bowley (1928), Dufton (1928), Pearson & Adyanthāya (1928), Gini & Galvani (1929), Lévy (1929), Rider (1929), Wilson & Hilferty (1929), d'Addario (1930), Eells (1930), Fogelson (1930), Gibrat (1930), Gini (1930), March (1930), de Montessus de Ballore (1930), Rhodes (1930), Ross (1930), d'Addario (1931), von Bortkiewicz (1931), Edgett (1931), de Finetti (1931b), Galvani (1931), Gibrat (1931), Hojo (1931), Horst (1931), Kolmogoroff (1931), von Mises (1931), Pearson (1931), Pearson & Pearson (1931), Shewhart (1931), Craig (1932a), de Finetti (1932), Garver (1932), Gibrat (1932a,b), Gini (1932), Hotelling & Solomons (1932), Luyten (1932), Pearson & Pearson (1932), Pietra (1932a,b), Craig (1933), Crowe (1933), Eddington (1933), Galvani (1933), Gini, Boldrini, Galvani & Venere (1933), Grassberger (1933), Griffin (1933), Hojo (1933), Julin (1933), Linders (1933), Rhodes (1933), Rider (1933), Scates (1933), Yang (1933), Eyraud (1934a), Fisher (1934), Foster (1934), Gumbel (1934e,g), Münzner (1934) Pollard (1934), Rosin & Rammler (1934), Anderson (1935), Birkeland & Frogner (1935), Bond (1935), Castellano (1935), Fréchet (1935), Gumbel (1935a), Pietra (1935a), Wold (1935), Crowe (1936), Darmois (1936), Gini (1936), Gumbel (1936a,c), Mathews (1936), Ville (1936), Bowley (1937), Bresciani-Turroni (1937), Coutagne (1937), Gumbel (1937c,d, 1937-38), Pitman (1937), Savur (1937a,b), Snedecor (1937), Barral Souto (1938), Brenet & Armand (1938), Bruen (1938), Cisbani (1938), Dodd (1938), Gini (1938), Gumbel (1938a,b,c), Jeffreys (1938), Savur (1938), d'Addario (1939), Coutagne (1939), Eisenhart (1939), Gini (1939), Gumbel (1939a,d,e), l'Hermite (1939), Jeffreys (1939), Pitman (1939), Zappa (1939), Arley & Buch (1940), Dodd (1940), Fréchet (1940a), Nair (1940a,b), Paulson (1940), Pizzetti (1940), Wilson (1940), Fréchet (1941), Mood (1941), Pizzetti (1941a), Roller (1941), von Schelling (1941), Smirnoff (1941), Aldanondo (1942), Beard (1942), Bol-

drini (1942), Dudding & Jennett (1942), Evans (1942), Fréchet (1942), Gaede (1942), Gumbel (1942a,c), Livada (1942), Barnard (1943), Curtiss (1943), Gumbel (1943a,c), Kendall (1943), Mathisen (1943), Scheffé (1943), Wilks (1943), Bowker (1944), Geary (1944), Gumbel (1945a), Brown & Tukey (1946), Carlton (1946), Cramér (1946), Grant (1946), Kendall (1946), Kimball (1946a,b), Mosteller (1946), Tintner (1946), Bickerstaff (1947), Gumbel (1947), Jacobson (1947), Nair (1947), Purcell (1947), Tanenhaus (1947), Eisenhart, Deming & Martin (1948a,b), Noether (1948), Pillai (1948), Stevens (1948), Wilks (1948), Benson (1949), Cox (1949), Godwin (1949a,b), Lane & Lei (1949), Nair (1949a,b), Noether (1949), Prot (1949a,b), Purcell (1949), Tukey (1949b), Walsh (1949a,c), Wolfowitz (1949), Youden (1949)

MEDIAN ESTIMATORS
Galton (1899a,b), Hayford (1902), Galton (1907a,b,c), Hooker (1907), Yule (1907), Gumbel (1945a), Eisenhart & Martin (1948)

MEDIAN, NEIGHBORS OF—SEE NEIGHBORS OF MEDIAN

MEDIAN OF ABSOLUTE DEVIATIONS
Gauss (1816), Hauber (1830), Encke (1832-34), Gore (1889), Pizzetti (1892), Rohne (1907), Kozák (1908-10), Whittaker & Robinson (1924)

MEDICINE, APPLICATIONS TO—SEE PHYSIOLOGY AND MEDICINE, APPLICATIONS TO

MERRIMAN'S CRITERION FOR REJECTION OF OUTLIERS—SEE OUTLIERS, MERRIMAN'S CRITERION FOR REJECTION

METALLURGY, APPLICATIONS TO
Grant (1946), Epstein (1948b), Murphy (1948)

METEOROLOGY, APPLICATIONS TO
von Zach (1805), Dove (1838), Lloyd (1855), Fechner (1874, 1897), Edgeworth (1898), Yule (1911), Fuller (1914), Hazen (1914), Walker (1914), Spillman, Tolley & Reed (1916), Tolley (1916), Becker (1917), Czuber (1921), Hall (1921), Jones (1921), von Bortkiewicz (1922b), Foster (1924), Horton (1924), Whittaker & Robinson (1924), Pietra (1925), Jarvis (1926), Goodrich (1927), Bartels (1928), Birkeland (1928), Cramér (1928), Köppen (1928), Gini & Galvani (1929), Hazen (1930), March (1930), de Montessus de Ballore (1930), Gibrat (1932a,b), Grassberger (1932), Crowe (1933), Foster (1934), Slade (1934), Birkeland & Frogner (1935), Crowe (1936), Foster (1936), Horton (1936), Mathews (1936), Slade (1936), Coutagne (1937), Gumbel (1937-38), Barral Souto (1938), Fisher & Yates (1938), Kimball (1938), Coutagne (1939), Creager (1939), Gumbel (1939e), Grant (1940), Gumbel (1940, 1941a), Beard (1942), Gumbel (1942b,c), Kimball (1942), Barricelli (1943), Gumbel (1943a,b,c), Kendall (1943), Gumbel (1945a), Grant (1946), Kendall (1946), Hartley (1949), Potter (1949b)

METHOD OF AVERAGES
Euler (1749), Mayer (1750), Lalande (1771), Puissant (1805), Hauber (1830-32), Merriman (1877), Gore (1889), Pizzetti (1892), Goedseels (1909), Campbell (1920), Rietz (1924), Whittaker & Robinson (1924), Scarborough (1930), Bruen (1938), Wald (1940), Nair & Shrivastava (1942), Deming (1943), Nair & Banerjee (1943), Tintner (1946), Housner & Brennan (1948), Bartlett (1949)

METHOD OF GROUP AVERAGES
Eddington (1933), Nair & Shrivastava (1942), Nair & Banerjee (1943), Bartlett (1949)

METHOD OF INTERPOLATION, CAUCHY'S—SEE CAUCHY'S METHOD OF INTERPOLATION

METHOD OF MOMENTS
Tolley (1916), Jones (1921), Fisher (1922), Rietz (1924), Gibrat (1930), Nair & Shrivastava (1942), Gumbel (1943c), Wilks (1943), Cramér (1946), Gumbel (1946b), Kendall (1946)

METHOD OF REPEATING CIRCLES—SEE REPEATING CIRCLES, METHOD OF

METHOD OF SELECTED POINTS
Scarborough (1930), Bose (1938), Nair & Shrivastava (1942), Deming (1943)

METHOD OF SITUATION—SEE LEAST ABSOLUTE VALUES, METHOD OF

METHOD USING THE RANGE
Bose (1938), Nair & Shrivastava (1942)

METROLOGY, APPLICATIONS TO—SEE STANDARDS OF LENGTH, APPLICATIONS TO

MIDRANGE
Laplace (1774), D. Bernoulli (1778), J. Bernoulli (1785), Prony (1804), von Zach (1805), Gauss (1809), Anonymous (1821), Quetelet (1835), Boudin (1865), Todhunter (1865, 1873), Merriman (1877), Faye (1888), Gore (1889), Venn (1891), Pizzetti (1892), Fechner (1897), Láska (1900), Mitscherlich (1903), Goedseels (1909, 1910, 1911), de la Vallée Poussin (1911), Tits (1912), Mansion (1913), Haar (1918), Jackson (1921), Dodd (1922), Haag (1924), Winkler (1924), Goedseels (1925), Alliaume (1927a,b), Neyman & Pearson (1928), Pearson & Adyanthāya (1928, 1929), Rider (1929), Hojo (1931), Pearson (1931), Shewhart (1931), Craig (1932b), Smith (1934), Barral Souto (1938), Bruen (1938), Gini (1938), Jeffreys (1938), Arley & Buch (1940), Nair (1940a), Brookner (1941), Dudding & Jennett (1942), Craig (1943), Scheffé (1943), Geary (1944), Gumbel (1944), Carlton (1946), Cramér (1946), Kendall (1946), Gumbel (1947), Eisenhart, Deming & Martin (1948b), Tukey (1948b), Wilks (1948), Cole (1949), Gumbel (1949), van Meer & Plantema (1949), Tukey (1949b), Walsh (1949a,b,c)

MIDRANGE, R-PLY—SEE R-PLY RANGE, SEMIRANGE AND/OR MIDRANGE

MINIMAX METHOD (MINIMUM APPROXIMATION)
Euler (1749), Lambert (1765a), Laplace (1786, 1793, 1799), Prony (1804), Legendre (1805), Puissant (1805), Svanberg (1805), Delambre (1806-10), Gauss (1809), Laplace (1812), Mathieu (1813-14), Fourier (1823), Gauss (1823), Cauchy (1824), Fourier (1824b), Cauchy (1831), Fourier (1831), Poncelet (1835), Tchebychef (1854), Petzval (1857), Boudin (1865), Todhunter (1865, 1873), Merriman (1877), Edgeworth (1887b), Gore (1889), Pizzetti (1892), Wölffing (1899), Kirchberger (1903), Mansion (1906), Goedseels (1909), de la Vallée Poussin (1909), Goedseels (1910, 1911), de la Vallée Poussin (1911), Tits (1912), Mansion (1913), Haar (1918), de la Vallée Poussin (1919), Jackson (1921), Dodd (1922), Jackson (1924), Whittaker & Robinson (1924), Goedseels (1925), Alliaume (1927a,b), Smith (1934), Barral Souto (1938), Bruen (1938), Boldrini (1942)

MINIMUM APPROXIMATION—SEE MINIMAX METHOD (MINIMUM APPROXIMATION)

MISS DISTANCE, APPLICATIONS TO
Herschel (1850), Bréger (1876), Delauney (1888), Estienne (1890), Vallier (1894), Cranz (1896), von Eberhard (1938)

MIXTURE OF DISTRIBUTIONS
Jeffreys (1932a), Pollard (1934)

MOMENT-GENERATING FUNCTIONS
Kimball (1946a, 1947)

MOMENTS AND FUNCTIONS OF MOMENTS
Yule (1896a), Edgeworth (1898), Fisher (1912), Jones (1921), Julin (1921), Rietz (1924), Bottema (1925), Tippett (1925), Pearson & Pearson (1931), Gumbel (1934b,c), Münzner (1934), Gumbel (1935a), Sastry (1935), Slade (1936), Rajalakshman (1941), Kendall (1943), Rajalakshman (1943), Wilks (1943), Gumbel (1944), Anderson (1945), Cramér (1946), Kendall (1946), Domb (1947), Moran (1947), Hoeffding (1948), Jones (1948), Godwin (1949b)

MOMENTS, CROSS-PRODUCT
Bottema (1925), Pearson (1926), Cramér (1946), Bickerstaff (1947), Hastings, Mosteller, Tukey & Winsor (1947), Kimball (1947), Hoeffding (1948), Jones (1948), Godwin (1949a,b), Nair (1949a)

MOMENTS, HIGHER
Charlier (1906), Niceforo (1919), Fisher (1920), Julin (1921), Cramér (1928), Pearson & Pearson (1932), Gumbel (1942c), Brown & Tukey (1946), Carlton (1946), Cramér (1946), Gumbel (1946b), Jones (1946), Kendall (1946), Bartlett (1947a), Bickerstaff (1947), Egudin (1947)

MOMENTS, METHOD OF—SEE METHOD OF MOMENTS

MORTALITY AND LIFE INSURANCE, APPLICATIONS TO—SEE ALSO LIMITING AGE
Bernoulli (1709), Bohlmann (1909), Yule (1911), Edgeworth (1913), von Bortkiewicz (1915), Secrist (1917), Gini (1918), Niceforo (1919), Czuber (1921), Jones (1921), Dodd (1923), Whittaker & Robinson (1924), Burgess (1927), Goodrich (1927), Bowley (1928), Gini & Galvani (1929), Gini (1930), Lidstone (1930), Steffensen (1930), Insolera (1931), von Mises (1931), Steffensen (1931), Gumbel (1932, 1933e,f,g), Insolera (1933), de Finetti (1934), de Franchis (1934), Gumbel (1934d,f,g), Insolera (1934), Smith (1934), Gumbel (1935c,d), Insolera (1935), Gumbel (1936c, 1936-37), Coutagne (1937), Gumbel (1937a,e), Insolera (1937-38), Gumbel (1938a,c), Gini (1939), Gumbel (1939f), Pearson (1939), Boldrini (1942), Kendall (1943), Cramér (1946), Grant (1946), Kendall (1946), Epstein & Brooks (1948)

MOST ADVANTAGEOUS METHOD
Laplace (1774, 1781, 1810, 1811a, 1812), Ivory (1825), Todhunter (1865), Glaisher (1872), Merriman (1877), Edgeworth (1883b, 1886a), Gore (1889), Czuber (1891a), Pizzetti (1892), Estienne (1900b)

MOST APPROXIMATIVE METHOD
Goedseels (1909, 1910, 1911), Tits (1912), Goedseels (1925), Alliaume (1927a,b)

MOVING RANGE
Grant (1946), Hoel (1946)

MTH LARGEST VALUE
Gumbel (1933a,b,c,d, 1934a,c,e, 1935a,b), Smirnoff (1935), Gumbel (1937b), Madow (1937), Fisher (1938), Gumbel (1938c, 1939b,c,f, 1940), Roy (1940a,b), Finney (1941), Thompson (1941), Wilks (1941), Gumbel (1943a,b,c), Kendall (1943), Wald (1943), Gumbel (1944, 1946a), Kendall (1946), Kimball (1946b), Mosteller (1946), Walsh (1946a), Bickerstaff (1947), Gumbel (1947), Eisenhart, Deming & Martin (1948a), Murphy (1948), Nair (1948b), Nanda (1948a,b), Noether (1948), Thomas (1948), Wilks (1948), Benson (1949), Epstein (1949a,b), Godwin (1949a), Press (1949), Smirnov (1949b), Tukey (1949b), Walsh (1949b,c)

MTH POINT STATISTIC, GUMBEL'S—SEE GUMBEL'S MTH POINT STATISTIC

MTH SMALLEST VALUE
Gumbel (1933b,c,d, 1934a,c,e, 1935a,b), Smirnoff (1935), Madow (1937), Gumbel (1939b,c,f, 1940), Hsu (1941b), Thompson (1941), Wilks (1941), Gumbel (1943a,b,c), Kendall (1943), Wald (1943), Gumbel (1945a,b), Nordquist (1945), Tintner (1945), Cramér (1946), Gumbel (1946a), Kendall (1946), Kimball (1946b), Mosteller (1946), Tintner (1946), Walsh (1946a), Bickerstaff (1947), Gumbel (1947), Eisenhart, Deming & Martin (1948a), Murphy (1948), Nair (1948b), Nanda (1948a,b), Noether (1948), Wilks (1948), Benson (1949), Epstein (1949a,b), Godwin (1949a), Press (1949), Smirnov (1949b), Tukey (1949b), Walsh (1949a,b,c)

MULTINOMIAL DISTRIBUTION
Wilks (1943), Cramér (1946), Kendall (1946), Stevens (1948)

MULTIPLE COMPARISON TEST, FISHER'S
Fisher (1935), Duncan (1947)

MULTIPLE COMPARISON TEST, NEWMAN-KEULS
Newman (1939), Tukey (1948c, 1949a)

MULTIPLE COMPARISON TESTS
Student (1931), Fisher (1935), Wishart (1938), Newman (1939), Pearson (1939), Duncan (1947), Tukey (1948c, 1949a)

MULTIPLE DECISION PROCEDURE, PAULSON'S
Paulson (1949)

MULTIPLE FACTOR ANALYSIS
Hotelling (1933), Thomson (1934), Girshick (1936), Hotelling (1936a), Aitken (1937), Madow (1937), Girshick (1939), Lawley (1940), Kendall (1946), Tintner (1946), Bartlett (1947b, 1948)

MULTIPLE T (LSD) TEST
Duncan (1947)

MULTIVARIATE NORMAL DISTRIBUTION
Hotelling (1933), Girshick (1936), Hotelling (1936a,b), Bartlett (1938), Fisher (1938, 1939), Girshick (1939), Hsu (1939), Roy (1939, 1940a,b), Fréchet (1941), Hsu (1941a,b), Mood (1941), Roy (1942a,b), Wald (1942), Kendall (1943), Wilks (1943), Anderson (1945), Roy (1945), Tintner (1945), Anderson (1946), Cramér (1946), Kendall (1946), Mosteller (1946), Roy (1946a,b), Tintner (1946), Bartlett (1947a,b), Duncan (1947), Tukey (1947), Anderson (1948), Geary (1948), Nanda (1948a,b), Rao (1948), Wilks (1948)

MUSIC, APPLICATIONS TO
Kendall (1946)

NEGATIVE BINOMIAL DISTRIBUTION
Wilks (1943), Kendall (1946)

NEIGHBORS OF MEDIAN
Gumbel (1934e), Pillai (1948), Youden (1949)

NEWCOMB'S METHOD OF TREATING OUTLIERS—SEE OUTLIERS, NEWCOMB'S METHOD OF TREATING

NEWMAN-KEULS TEST—SEE MULTIPLE COMPARISON TEST, NEWMAN-KEULS

NONLINEAR REGRESSION—SEE REGRESSION, NONLINEAR

NORMAL (GAUSSIAN) DISTRIBUTION

Adrain (1808), Gauss (1809), Laplace (1810, 1811a,b, 1812), Delambre (1813), Gauss (1816, 1823), Fourier (1824a), Ivory (1825), Fourier (1826a, 1829), Hauber (1830, 1830-32), Encke (1832-34), Hagen (1837), Cournot (1843), Gerling (1843), Ellis (1844), De Morgan (1847), Herschel (1850), Peirce (1852), Gould (1855), von Andrae (1860), Airy (1861), Chauvenet (1863), De Morgan (1864), Boudin (1865), Venn (1866), Stone (1868), von Andrae (1869), Galton (1869), Jordan (1869), Zachariae (1871), von Andrae (1872), Glaisher (1872), Helmert (1872), Jordan (1872), Glaisher (1873), Peirce (1873), Stone (1873a,b), Fechner (1874), Glaisher (1874), Jevons (1874), Stone (1874), Galton (1875), Helmert (1875a,b), Laurent (1875), Lexis (1875), Mees (1875), Bréger (1876), Helmert (1876a,b), Mees (1876), Helmert (1877a,b), Jordan (1877a,b), Merriman (1877), Peirce (1878), Schott (1878), Jordan (1879), Chaplin (1880), Bréger (1881), Galton (1881), Chaplin (1882), Edgeworth (1883a,b), Doolittle (1884), Merriman (1884), Wright (1884), Edgeworth (1885, 1886a,b), Newcomb (1886), Edgeworth (1887 a,b,c,d,e,f, 1887-90), Lehmann-Filhes (1887), Bertrand (1888a,b,c), Delauney (1888), Edgeworth (1888), Faye (1888), Bertrand (1889), Galton (1889), Gore (1889), Pizzetti (1889), Estienne (1890), Jordan (1890), Czuber (1891a,b), Venn (1891), Pizzetti (1892), Edgeworth (1893), Vallier (1894), Mendeleev (1895), Cranz (1896), Galton (1896), Poincaré (1896), Yule (1896a,b), Blümcke (1897), Fechner (1897), Blümcke (1898), Edgeworth (1898), Rodewald (1898), Czuber (1899), Galton (1899a,b), Sheppard (1899a,b), Wölffing (1899), Estienne (1900a), Blümcke (1901), Bowley (1901), Galton (1902), Pearson (1902), Czuber (1903), Heydenreich (1903), Kapteyn (1903), Mitscherlich (1903), Saunder (1903), Edgeworth (1905), Charlier (1906), Mansion (1906), Slocum & Hancock (1906), Galton (1907b), Kozák (1907), Rohne (1907), Vogeler (1907), Yule (1907), Kozák (1908-10), Mazzuoli (1908), Bohlmann (1909), Wellisch (1909), Charlier (1910), Goedseels (1910), Edgeworth (1911), Keynes (1911), Yule (1911), Fisher (1912), Gini (1912), King (1912), Newcomb (1912), Dodd (1913), Edgeworth (1913), Goodwin (1913), Mansion (1913), Contarino (1914), Czuber (1914), Dodd (1914), Fuller (1914), Hazen (1914), Walker (1914), Spillman, Tolley & Reed (1916), Tolley (1916), Becker (1917), Brunt (1917), Secrist (1917), Czuber (1918), Frank (1918), Niceforo (1919), Campbell (1920), Daniell (1920), Fisher (1920), Pearson (1920), Stewart (1920a,b), Czuber (1921), Jones (1921), Julin (1921), Keynes (1921), von Bortkiewicz (1922a,b), Dodd (1922), Fisher (1922), Crum (1923), Dodd (1923), Edgeworth (1923), von Mises (1923), Neyman (1923), Wilson (1923), Foster (1924), Haag (1924), Rietz (1924), Whittaker & Robinson (1924), Winkler (1924), Archibald (1925), Coolidge (1925), Irwin (1925a,b), Lévy (1925), Pearson (1925), Reilly, Rae & Wheeler (1925), Tippett (1925), Estienne (1926-27), Gini (1926), Jarvis (1926), Pearson (1926), Peirce (1926), Burgess (1927), Estienne (1927), Fréchet (1927), Goodrich (1927), Jordan (1927), Student (1927), Bartels (1928), Bowley (1928), Cramér (1928), Fisher & Tippett (1928), Gumbel (1928), Neyman & Pearson (1928), Ogrodnikoff (1928), Pearson & Adyanthāya (1928), Sophister (1928), Lévy (1929), Pearson & Adyanthāya (1929), Rider (1929), Smith (1929), Wilson & Hilferty (1929), Fogelson (1930), Gibrat (1930), Hazen (1930), Lidstone (1930), Scarborough (1930), d'Addario (1931), von Bortkiewicz (1931), Gibrat (1931), Hojo (1931), Kolmogoroff (1931), von Mises (1931), Pearson (1931), Pearson & Pearson (1931), Shewhart (1931), Student (1931), Dodge (1932), de Finetti (1932), Gibrat (1932b), Gini (1932), Grassberger (1932), Gumbel (1932), Jeffreys (1932a,b), Luyten (1932), Pearson (1932), Bartlett (1933), Castellano (1933a), Dodge (1933), Eddington (1933), Fisher (1933), Gumbel (1933a,b,e,f,g), Hojo (1933), Insolera (1933), Jeffreys (1933), McKay & Pearson (1933), Rider (1933), Romanovsky (1933), Tricomi (1933), Davies & Pearson (1934), Eyraud (1934a,b), Gumbel (1934d,e,f,g), Jeffreys (1934), Münzner (1934), Rosin & Rammler (1934), Slade (1934), Thomson (1934), Birkeland & Frogner (1935), Bond (1935), Fisher (1935), Fréchet (1935), Gumbel (1935a,b,c, 1935-36), McKay (1935), Pearson (1935), Pearson & Haines (1935), Pietra (1935b), Thompson (1935), Darmois (1936), Foster (1936), Grassberger (1936), Gumbel (1936a), Horton (1936), von Mises (1936), Nair (1936), Neyman & Pearson (1936), Pearson & Chandra Sekar (1936), Pitman (1936), Saville (1936), Slade (1936), Sukhatme (1936), Bowley (1937),

Calichiopulo (1937), Coutagne (1937), Gumbel (1937a,c,d, 1937-38), Neyman (1937b), Pitman (1937), Savur (1937a,b), Snedecor (1937), Sukhatme (1937), Wold (1937), Bruen (1938), Fisher (1938), Fisher & Yates (1938), Gumbel (1938b), Jeffreys (1938), Kimball (1938), Savur (1938), Wilks (1938), Wishart (1938), Coutagne (1939), Derevitsky (1939), Eisenhart (1939), Gini (1939), Gumbel (1939b,f), l'Hermite (1939), Hulme & Symms (1939), Jeffreys (1939), Newman (1939), Pearson (1939), Pitman (1939), Wald (1939), Arley (1940), Arley & Buch (1940), Fisher (1940), Fréchet (1940a), Kendall (1940), Kontorova (1940), Lawley (1940), Nair (1940a), Wald (1940), Chandra Sekar & Francis (1941), Cochran (1941), Daniels (1941), Davis (1941), Finney (1941), Fréchet (1941), Kontorova & Frenkel (1941), von Schelling (1941), Simon (1941), Smirnoff (1941), Tucker (1941), Wilks (1941), Beard (1942), Boldrini (1942), Dudding & Jennett (1942), Evans (1942), Gaede (1942), Gumbel (1942a,b,c), Hartley (1942), Nair & Shrivastava (1942), Pearson (1942), Pearson & Hartley (1942), Wald (1942), Barnard (1943), Barricelli (1943), Curtiss (1943), Deming (1943), Gumbel (1943a,b,c), Kendall (1943), Knudsen (1943), Kontorova (1943), Nair & Banerjee (1943), Pearson & Hartley (1943), Wilks (1943), Geary (1944), Gumbel (1944), Hartley (1944), Rodgers (1944), Daniels (1945), Gumbel (1945a), Gurney (1945), Tintner (1945), Tucker (1945a,b), Baker (1946), Brown & Tukey (1946), Cramér (1946), Daly (1946), Freudenthal (1946), Grant (1946), Hoel (1946), Jones (1946), Kendall (1946), Kimball (1946a), Kontorova (1946), Mosteller (1946), Tukey (1946), Walsh (1946a,b), Bartlett (1947a,b), Bickerstaff (1947), Cohan (1947), Davidenkov, Shevandin & Wittmann (1947), Duncan (1947), Egudin (1947), Eisenhart, Hastay & Wallis (1947), Eisenhart & Solomon (1947), Elfving (1947), Fisher & Hollomon (1947), Gensamer, Saibel & Lowrie (1947), Grubbs & Weaver (1947), Gumbel (1947), Gurney (1947), Gurney & Pearson (1947), Hastings, Mosteller, Tukey & Winsor (1947), Jacobson (1947), Lehmann (1947), Lord (1947), von Mises (1947), Moran (1947), Nair (1947), Plackett (1947), Purcell (1947), Walsh (1947), Anderson (1948), Cox (1948), Eisenhart, Deming & Martin (1948b), Eisenhart & Martin (1948), Epstein (1948a,b), Frankel (1948), Geary (1948), Gumbel (1948), Higuchi, Leeper & Davis (1948), Hoeffding (1948), Housner & Brennan (1948), Jones (1948), Nair (1948a,b), Nanda (1948a,b), Pillai (1948), Rao (1948), Stevens (1948), Tukey (1948b,c), Wilks (1948), Bartlett (1949), Benson (1949), Cole (1949), Cox (1949), Godwin (1949a,b), Gumbel (1949), Harris & Tukey (1949), Hartley (1949), Howell (1949), Kac (1949), Kimball (1949), Kontorova & Timoshenko (1949), Lane & Lei (1949), Lévi (1949), McMillan (1949), Nair (1949a,b), Noether (1949), Paulson (1949), Peterson (1949a,b), Prot (1949a,b,c), Purcell (1949), Scheffé (1949), Shone (1949), Smirnov (1949b), Tukey (1949a,b,c,d,e,f), Walsh (1949a,c,d), Weibull (1949), Yamanouchi (1949), Youden (1949)

NORMAL (GAUSSIAN) DISTRIBUTION, GENERALIZED—SEE GENERALIZED NORMAL (GAUSSIAN) DISTRIBUTION

NORMAL (GAUSSIAN) DISTRIBUTION, MULTIVARIATE—SEE MULTIVARIATE NORMAL DISTRIBUTION

NORMAL PROBABILITY PAPER—SEE PROBABILITY PAPER, NORMAL

NUMBERS, INDEX, APPLICATIONS TO—SEE INDEX NUMBERS, APPLICATIONS TO

NUTRITION, APPLICATIONS TO—SEE FOODS AND NUTRITION, APPLICATIONS TO

OCCURRENCE INTERVAL
Gumbel (1945a)

OCTILES
Edgeworth (1886b)

OGRODNIKOFF'S METHOD OF TREATING OUTLIERS—SEE OUTLIERS, OGRODNIKOFF'S METHOD OF TREATING

ONE-SAMPLE TESTS (OR CONFIDENCE PROCEDURES)
Fisher (1929), Savur (1937a,b), Smirnoff (1937a), Snedecor (1937), Sukhatme (1937), Thibaud & Ferber (1937), Bartlett (1938), Savur (1938), Smirnov (1939a,b,c), Wald (1939), Nair (1940a), Paulson (1940), Davis (1941), Kolmogoroff (1941), Paulson (1941), Dodd (1942), Gumbel (1942a), Pearson (1942), Knudsen (1943), Pearson & Hartley (1943), Scheffé (1943), Wilks (1943), Smirnov (1944), Scheffé & Tukey (1945), Tintner (1945), Cramér (1946), Daly (1946), Greenwood (1946), Kendall (1946), Kimball (1946a), Mosteller (1946), Tintner (1946), Walsh (1946a), Bickerstaff (1947), Eisenhart, Hastay & Wallis (1947), Jacobson (1947), Kimball (1947), Lehmann (1947), Lord (1947), von Mises (1947), Bartlett (1948), Eisenhart, Deming & Martin (1948b), Feller (1948), Geary (1948), Murphy (1948), Nair (1948b), Noether (1948), Rao (1948), Smirnov (1948), Tukey (1948a), Wilks (1948), Birnbaum & Zuckerman (1949), Chung (1949), Cox (1949), Doob (1949), Hartley (1949), Kac (1949), Kimball (1949), Maniya (1949), Noether (1949), Smirnov (1949a), Tukey (1949b), Walsh (1949a,b,c,d), Wolfowitz (1949)

ONE-SIDED TESTS (OR INTERVALS)—SEE TESTS (OR INTERVALS), ONE-SIDED

OPTICS, APPLICATIONS TO
Lambert (1760), Petzval (1857)

ORDERED ROOTS OF DETERMINANTAL EQUATIONS
Fisher (1939), Girshick (1939), Hsu (1939), Roy (1939, 1940a,b), Hsu (1941a,b), Roy (1942a,b), Wilks (1943), Anderson (1945, 1946), Kendall (1946), Roy (1946a,b), Tintner (1946), Bartlett (1947a,b), Anderson (1948), Bartlett (1948), Geary (1948), Nanda (1948a,b), Rao (1948)

ORDER STATISTIC, FIRST (SMALLEST)—SEE SMALLEST SAMPLE VALUE

ORDER STATISTIC, MTH LARGEST (SMALLEST)—SEE MTH LARGEST (SMALLEST) SAMPLE VALUE

ORDER STATISTIC, NTH (LARGEST)—SEE LARGEST SAMPLE VALUE

ORDER STATISTICS OF SAMPLE, ALL
Boscovich (1760), Lalande (1771), Laplace (1774), D. Bernoulli (1778), J. Bernoulli (1785), Laplace (1793, 1799), Prony (1804), von Lindenau (1806), Laplace (1812), van Beeck Calkoen (1816), Adrain (1818), Anonymous (1821), Hagen (1837), Boudin (1865), Todhunter (1865), Clifford (1866), Whitworth (1867), von Andrae (1872), Todhunter (1873), Helmert (1875a, 1876a, 1877a), Merriman (1877), Herschel (1878), Lalanne (1879), Galton (1881), Bruns (1882), Schreiber (1882), Bertrand (1888c), Faye (1888), Galton (1888), Jordan (1888), Bertrand (1889), Pizzetti (1889), Estienne (1890), Runge (1890), van den Berg (1891), Czuber (1891b), Pizzetti (1892), Wölffing (1899), Láska (1900), Galton (1902), Pearson (1902), Czuber (1903), Holmes (1905), Lorenz (1905), Watkins (1905), Kozák (1908-10), Watkins (1908), Gini (1909), Persons (1909), Watkins (1909), Gini (1910), Furlan (1911), Mortara (1911), Gini (1912), Wellisch (1913), Czuber (1914), Gini (1914b, 1915a,b), Mantel (1915), Savorgnan (1915), Trachtenberg (1915), Gini (1916b, 1917), Campbell (1920), Daniell (1920), de Pietri-Tonelli (1920), Hall (1921), Julin (1921), von Bortkiewicz (1922a,b), Foster (1924), Bottema (1925), Irwin (1925a,b), Lévy (1925), Pearson (1925), Chadwick (1926), Gini (1926), Jarvis (1926), Pearson (1926), Alliaume (1927a), Friedrich (1927), Goodrich (1927), Tucker (1927), Cramér (1928), de Gleria (1929), d'Addario (1930), de Finetti & Paciello (1930), de Gleria (1930), Hazen (1930), March (1930), Rhodes (1930), Scarborough (1930), von Bortkiewicz (1931), de Finetti (1931a), Hojo (1931), von Mises (1931), Pearson & Pearson (1931), Shewhart (1931), Jeffreys (1932b), Luyten (1932), Pearson & Pearson (1932), Bartlett (1933), Baticle (1933a,b), Cantelli (1933), Castellano (1933a), Eddington (1933), de Finetti (1933), Fisher (1933), Glivenko (1933), Hotelling (1933), Kolmogoroff (1933), Rider (1933), Watkins (1933), Yang (1933), Eyraud (1934b), Fisher (1934), Foster (1934), Jeffreys (1934), Mortara (1934a), Slade

(1934), Anderson (1935), Eyraud (1935), Gumbel (1935a), Pietra (1935a), Wold (1935), Foster (1936), Girshick (1936), Grassberger (1936), Horton (1936), Hotelling (1936a,b), Nair (1936), Pearson & Chandra Sekar (1936), Ruark & Devol (1936), Saville (1936), Smirnoff (1936), Sukhatme (1936), Thompson (1936), Aitken (1937), Bowley (1937), Friedrich (1937), Kozakiewicz (1937), Neyman (1937a), Pitman (1937), Savur (1937a), Smirnoff (1937a,b), Sukhatme (1937), Thibaud & Ferber (1937), Wold (1937), Bartlett (1938), Bose (1938),Cisbani (1938), Dodd (1938), Fisher (1938), Fisher & Yates (1938), Hertzman (1938), Kimball (1938), Thompson (1938), Waschakidse (1938), d'Addario (1939), Coutagne (1939), Fisher (1939), Gini (1939), Girshick (1939), Gumbel (1939a,b,d), Hsu (1939), Pitman (1939), Roy (1939), Ruark (1939), Smirnov (1939a,b,c), Stevens (1939), de Vergottini (1939), Wald & Wolfowitz (1939), Dodd (1940), Feller (1940), Garwood (1949), Kendall (1940), Roy (1940a,b), Singleton (1940), de Vergottini (1940), Wald (1940), Gumbel (1941a,b), Hsu (1941a), Kolmogoroff (1941), Paulson (1941), Pizzetti (1941a,b), Rajalakshman (1941), von Schelling (1941), Thompson (1941), Wald & Wolfowitz (1941), Wilks (1941), Beard (1942), Fréchet (1942), Gumbel (1942a,b,c), Hartley (1942), Livada (1942), Nair & Shrivastava (1942), Pearson (1942), Roy (1942a,b), Wald (1942), Wilks (1942), Craig (1943), Gumbel (1943a,b,c), Kendall (1943), Nair & Banerjee (1943), Powell (1943), Price (1943), Rajalakshman (1943), Scheffé (1943), Wald (1943), Wilks (1943), Bowker (1944), Reiersöl (1944), Robbins (1944a), Smirnov (1944), Anderson (1945), Daniels (1945), Gumbel (1945a,b), Nordquist (1945), Roy (1945), Scheffé & Tukey (1945), Anderson (1946), Cramér (1946), Greenwood (1946), Kimball (1946b), Mosteller (1946), Roy (1946a,b), Walsh (1946b), Bartlett (1947b), Bickerstaff (1947), Domb (1947), Duncan (1947), Egudin (1947), Hastings, Mosteller, Tukey & Winsor (1947), Kimball (1947), Kincaid (1947), Lehmann (1947), Lord (1947), von Mises (1947), Moran (1947), Nair (1947), Tukey (1947), Anderson (1948), Feller (1948), Gumbel (1948), Hoeffding (1948), Housner & Brennan (1948), Jones (1948), Murphy (1948), Nair (1948a,b), Nanda (1948a,b), Pillai (1948), Rao (1948), Smirnov (1948), Thomas (1948), Tukey (1948a,c), Wilks (1948), Bartlett (1949), Benson (1949), Chung (1949), Doob (1949), Godwin (1949a,b), Kac (1949), Lane & Lei (1949), Langbein (1949), Maniya (1949), Nair (1949b), Noether (1949), Paulson (1949), Potter (1949a,b), Purcell (1949), Smirnov (1949a,b), Tukey (1949a,b), Walsh (1949a,c,d), Weibull (1949), Wolfowitz (1949), Yamanouchi (1949), Youden (1949)

ORDER STATISTICS OF SAMPLE, SELECTED
Smirnoff (1937b), Kendall (1940), Nair (1940b), Wilks (1943), Scheffé & Tukey (1944), Tintner (1945), Mosteller (1946), Bickerstaff (1947), Tukey (1949b), Walsh (1949b,c,d), Wolfowitz (1949), Yamanouchi (1949)

OSCILLATION, MEAN—SEE MEAN OSCILLATION

OUTLIERS, AIRY'S METHOD OF TREATING
Airy (1856), Winlock (1856), Edgeworth (1887a), Wellisch (1909)

OUTLIERS, BERTRAND'S CRITERION FOR REJECTION
Bertrand (1888b), Gore (1889), Pizzetti (1892), Czuber (1899), Wölffing (1899), Wellisch (1909), Coolidge (1925)

OUTLIERS, CHAUVENET'S CRITERION FOR REJECTION
Chauvenet (1863), Stone (1868), Glaisher (1873), Stone (1873b), Jevons (1874), Merriman (1877, 1884), Wright (1884), Edgeworth (1887a), Lehmann-Filhes (1887), Pizzetti (1892), Vallier (1894), Cranz (1896), Czuber (1899), Sheppard (1899b), Wölffing (1899), Heydenreich (1903), Kozák (1907, 1908-10), Wellisch (1909), Goodwin (1913), Brunt (1917), Stewart (1920a), Coolidge (1925), Irwin (1925b), Ogrodnikoff (1928), Pearson (1931), Jeffreys (1932a), Rider (1933), Bond (1935), Derevitsky (1939), Jeffreys (1939), Arley (1940), Smirnoff (1941)

OUTLIERS, DE MORGAN'S ALTERNATIVE TO REJECTION
De Morgan (1847), Edgeworth (1887a), Gore (1889), Pizzetti (1892), Wellisch (1909), Ogrodnikoff (1928)

OUTLIERS, EDGEWORTH'S MODIFICATION OF STONE'S FIRST CRITERION FOR REJECTION
Edgeworth (1887a), Pizzetti (1892), Coolidge (1925), Rider (1933)

OUTLIERS, GLAISHER'S ALTERNATIVE TO REJECTION
Glaisher (1872, 1873), Stone (1873b), Glaisher (1874), Stone (1874), Merriman (1877), Doolittle (1884), Edgeworth (1886a, 1887a), Gore (1889), Pizzetti (1892), Wellisch (1909), Coolidge (1925), Ogrodnikoff (1928), Rider (1933)

OUTLIERS, GOODWIN'S CRITERION FOR REJECTION
Goodwin (1913)

OUTLIERS, HEYDENREICH'S CRITERION FOR REJECTION
Cranz (1896) [Cranz & Becker (1910)], Heydenreich (1903), Arley (1940)

OUTLIERS, HULME-SYMMS ALTERNATIVE TO REJECTION
Hulme & Symms (1939)

OUTLIERS, IRWIN'S CRITERION FOR REJECTION
Irwin (1925b), Pearson (1931), Rider (1933), McKay (1935), Thompson (1935), Kendall (1946)

OUTLIERS, JEFFREYS' ALTERNATIVE TO REJECTION
Jeffreys (1932a), Rider (1933), Hulme & Symms (1939), Jeffreys (1939)

OUTLIERS, MAZZUOLI'S CRITERION FOR REJECTION
Cranz (1896) [Cranz & Becker (1910)], Mazzuoli (1908), Arley (1940)

OUTLIERS, MCKAY'S CRITERION FOR REJECTION
McKay (1935)

OUTLIERS, MERRIMAN'S CRITERION FOR REJECTION
Merriman (1884), Gore (1889), Pizzetti (1892), Wölffing (1899), Bruen (1938)

OUTLIERS, NEWCOMB'S METHOD OF TREATING
Newcomb (1886), Edgeworth (1887a), Gore (1889), Pizzetti (1892), Wellisch (1909), Newcomb (1912), Ogrodnikoff (1928), Rider (1933), Hulme & Symms (1939)

OUTLIERS, OGRODNIKOFF'S METHOD OF TREATING
Ogrodnikoff (1928)

OUTLIERS, PEIRCE'S CRITERION FOR REJECTION
Peirce (1852), Gould (1855), Airy (1856), Winlock (1856), Chauvenet (1863), Stone (1868), Glaisher (1872, 1873), Peirce (1873), Jevons (1874), Merriman (1877), Peirce (1878), Schott (1878), Edgeworth (1883b), Doolittle (1884), Merriman (1884), Wright (1884), Edgeworth (1886a), Newcomb (1886), Edgeworth (1887a), Lehmann-Filhes (1887), Gore (1889), Pizzetti (1892), Cranz (1896), Czuber (1899), Wölffing (1899), Saunder (1903), Kozák (1907), Wellisch (1909), Goodwin (1913), Contarino (1914), Brunt (1917), Stewart (1920a), Archibald (1925), Coolidge (1925), Irwin (1925b), Ogrodnikoff (1928), Jeffreys (1932a), Rider (1933), Jeffreys (1939), Lawley (1940)

OUTLIERS, ROHNE'S CRITERION FOR REJECTION
Cranz (1896) [Cranz & Becker (1910)], Rohne (1907), Arley (1940)

OUTLIERS, SHEPPARD'S MODIFICATION OF CHAUVENET'S CRITERION FOR REJECTION
Sheppard (1899b)

OUTLIERS, STEWART'S METHOD OF TREATING
Stewart (1920b), Ogrodnikoff (1928), Rider (1933)

OUTLIERS, STONE'S FIRST CRITERION FOR REJECTION
Stone (1868), Glaisher (1873), Stone (1873b), Glaisher (1874), Stone (1874), Merriman (1877), Edgeworth (1883b), Doolittle (1884), Wright (1884), Edgeworth (1887a), Pizzetti (1892), Vallier (1894), Cranz (1896), Czuber (1899), Kozák (1907, 1908-10), Coolidge (1925), Ogrodnikoff (1928), Rider (1933)

OUTLIERS, STONE'S SECOND CRITERION FOR REJECTION
Stone (1873b), Edgeworth (1887a), Pizzetti (1892), Wellisch (1909), Rider (1933)

OUTLIERS, STUDENT'S CRITERION FOR REJECTION
Student (1927)

OUTLIERS, THOMPSON'S CRITERION FOR REJECTION
Thompson (1935), Arley (1940), Smirnoff (1941), Kendall (1946)

OUTLIERS, TIPPETT'S CRITERION FOR REJECTION
Tippett (1925), Rider (1933), Pearson & Chandra Sekar (1936)

OUTLIERS, TØPSOE-JENSEN CRITERION FOR REJECTION
Arley (1940)

OUTLIERS, TREATMENT OF
Maire & Boscovich (1755), Boscovich (1757), Lambert (1760), Short (1763), Lambert (1765b), D. Bernoulli (1778), J. Bernoulli (1785), Legendre (1805), von Zach (1805), Delambre (1806-10), Legendre (1814), van Beeck Calkoen (1816), Anonymous (1821), Muncke (1825), Hagen (1837), Bessel & Baeyer (1838), Stampfer (1839), Gerling (1843), De Morgan (1847), Peirce (1852), Gould (1855), Airy (1856), Winlock (1856), Airy (1861), Chauvenet (1863), Todhunter (1865), Stone (1868), Glaisher (1872), Helmert (1872), Glaisher (1873), Peirce (1873), Stone (1873b), Glaisher (1874), Jevons (1874), Stone (1874), Merriman (1877), Peirce (1878), Schott (1878), Edgeworth (1883b), Doolittle (1884), Merriman (1884), Wright (1884), Edgeworth (1886a), Newcomb (1886), Edgeworth (1887a,b,c,d), Lehmann-Filhes (1887), Bertrand (1888a,b), Faye (1888), Bertrand (1889), Gore (1889), Jordan (1890), Czuber (1891a), Pizzetti (1892), Edgeworth (1893), Vallier (1894), Mendeleev (1895), Cranz (1896), Poincaré (1896), Rodewald (1898), Czuber (1899), Sheppard (1899b), Wölffing (1899), Pearson (1902), Heydenreich (1903), Saunder (1903), Kozák (1907), Rohne (1907), Vogeler (1907), Kozák (1908-10), Mazzuoli (1908), Wellisch (1909), Edgeworth (1911), Goodwin (1913), Wellisch (1913), Contarino (1914), Brunt (1917), Daniell (1920), Stewart (1920a,b), Keynes (1921), Fisher (1922), Archibald (1925), Coolidge (1925), Irwin (1925b), Reilly, Rae & Wheeler (1925), Tippett (1925), Alliaume (1927a), Estienne (1927), Student (1927), Dufton (1928), Ogrodnikoff (1928), Scarborough (1930), Pearson (1931), Jeffreys (1932a), Rider (1933), Bond (1935), McKay (1935), Thompson (1935), Pearson & Chandra Sekar (1936), Calichiopulo (1937), Jeffreys (1938), Derevitsky (1939), Hulme & Symms (1939), Jeffreys (1939), Arley (1940), Arley & Buch (1940), Smirnoff (1941), Deming (1943), Wing, Price & Douglass (1944), Kendall (1946), Nair (1948b), Benson (1949), Youden (1949)

OUTLIERS, VALLIER'S CRITERION FOR REJECTION
Vallier (1894), Cranz (1896), Heydenreich (1903), Kozák (1908-10), Arley (1940)

OUTLIERS, WELLISCH'S CRITERION FOR REJECTION
Wellisch (1909, 1913)

OUTLIERS, WRIGHT-HAYFORD CRITERION FOR REJECTION
Wright (1884) [Wright & Hayford (1906)], Brunt (1917), Reilly, Rae & Wheeler (1925), Rider (1933)

OUTLIERS, WRIGHT'S CRITERION FOR REJECTION
Wright (1884), Gore (1889) [1903], Pizzetti (1892)

PARABOLIC DISTRIBUTION
Lagrange (1774), Bernoulli (1785), Merriman (1877), Pizzetti (1892), Blümcke (1898), Fisher (1922), Walsh (1949b)

PARETO DISTRIBUTION
Pareto (1897), Lorenz (1905), Watkins (1908), Gini (1909), Persons (1909), Watkins (1909), Gini (1910), Edgeworth (1911), Furlan (1911), Mortara (1911), Porru (1912), Savorgnan (1915), Ricci (1916), Gini (1917), Niceforo (1919), Dalton (1920), Winkler (1924), Amoroso (1925), Gumbel (1927), Saibante (1928), d'Addario (1930), Gibrat (1930), d'Addario (1931), von Bortkiewicz (1931), d'Addario (1932), Castellano (1933a), Yntema (1933), d'Addario (1934a,b), Dominedò (1934), Castellano (1935), Pietra (1935b), d'Addario (1936), Gini (1936), von Mises (1936), Bresciani-Turroni (1937), d'Addario (1939), Bresciani-Turroni (1939), Gini (1939), Davis (1941), Pizzetti (1941b), Gumbel (1943c, 1945a), Cramér (1946), Gumbel (1947)

PAULSON'S MULTIPLE DECISION PROCEDURE—SEE MULTIPLE DECISION PROCEDURE, PAULSON'S

PAUPERISM, APPLICATIONS TO
Galton (1896), Yule (1896a,b)

PEARSON'S MEASURE OF SKEWNESS—SEE SKEWNESS, PEARSON'S MEASURE OF

PEARSON SYSTEM OF DISTRIBUTIONS
Pearson (1895), Galton (1896), Yule (1896a,b), Kapteyn (1903), Charlier (1910), Edgeworth (1911, 1913), Tolley (1916), Daniell (1920), Jones (1921), Julin (1921), Fisher (1922), Dodd (1923), Foster (1924), Rietz (1924), Irwin (1925a), Burgess (1927), Goodrich (1927), Sophister (1928), Pearson & Adyanthāya (1929), Rider (1929), Gibrat (1930), Lidstone (1930), Edgett (1931), Pearson (1931), Pearson & Pearson (1931, 1932), McKay & Pearson (1933), Foster (1934), Rosin & Rammler (1934), Slade (1934), Gumbel (1935b), Foster (1936), Saville (1936), d'Addario (1939), Creager (1939), Gumbel (1942b), Hartley (1942), Kendall (1943), Wilks (1943), Cramér (1946), Kendall (1946), Gurney (1947), Gurney & Pearson (1947), Cox (1949)

PEARSON TYPE I DISTRIBUTION
Pearson (1895), Foster (1924), Sophister (1928), Hazen (1930), Steffensen (1930), Edgett (1931), Pearson & Pearson (1931), Foster (1934), Slade (1934), Sastry (1935), Foster (1936), Saville (1936), Kimball (1938), Wilks (1943), Cramér (1946), Greenwood (1946), Kendall (1946), Bickerstaff (1947)

PEARSON TYPE II DISTRIBUTION
Pearson (1895), Pearson & Adyanthāya (1928), Steffensen (1930), Pearson & Pearson (1931), Foster (1936), Kendall (1946), Bickerstaff (1947), Benson (1949)

PEARSON TYPE III DISTRIBUTION
Pearson (1895), Czuber (1921), Fisher (1922), Foster (1924), Amoroso (1925), Burgess (1927), Sophister (1928), Gibrat (1930), Hazen (1930), Edgett (1931), d'Addario (1932), Foster (1934), Slade (1934), Gumbel (1935-36), d'Addario (1936), Foster (1936), Saville (1936), Slade (1936), Gumbel (1937d), Kimball (1938), d'Addario (1939), Roller (1941), Pearson (1942), Wilks (1943), Cramér (1946), Kendall (1946), Stevens (1948), Cox (1949), Press (1949)

PEARSON TYPE IV DISTRIBUTION
Pearson (1895), Edgett (1931), Luyten (1932), Kendall (1946)

PEARSON TYPE V DISTRIBUTION
Pearson (1895), Amoroso (1925), Gibrat (1930), Edgett (1931), d'Addario (1932, 1936)

PEARSON TYPE VI DISTRIBUTION
Pearson (1895), Sophister (1928), Gibrat (1930), Edgett (1931), Saville (1936), Cramér (1946)

PEARSON TYPE VII DISTRIBUTION
Pearson (1895), Pearson & Adyanthāya (1928), Steffensen (1930), Foster (1936), Jeffreys (1939), Wilks (1943), Cramér (1946), Benson (1949)

PEARSON TYPE IX DISTRIBUTION
Pearson (1895), Pearson & Pearson (1931), Rajalakshman (1941), Bickerstaff (1947)

PEIRCE'S CRITERION FOR REJECTION OF OUTLIERS—SEE OUTLIERS, PEIRCE'S CRITERION FOR REJECTION

PERCENTAGE POINTS
Vinci (1918), Pearson (1932), McKay & Pearson (1933), Pollard (1934), McKay (1935), Pearson (1935), Chandra Sekar & Francis (1941), Cochran (1941), Finney (1941), Kolmogoroff (1941), Simon (1941), Thompson (1941), Pearson & Hartley (1942), Curtiss (1943), Knudsen (1943), Mathisen (1943), Pearson & Hartley (1943), Scheffé (1943), Scheffé & Tukey (1944), Brown & Tukey (1946), Cramér (1946), Daly (1946), Kendall (1946), Kimball (1946a), Eisenhart, Hastay & Wallis (1947), Eisenhart & Solomon (1947), Elfving (1947), Grubbs & Weaver (1947), Gumbel (1947), Jacobson (1947), Lord (1947), Eisenhart, Deming & Martin (1948a), Murphy (1948), Nair (1948a), Rao (1948), Schützenberger (1948), Tukey (1948b,c), Wilks (1948), Godwin (1949b), Gumbel (1949), Tukey (1949a,b), Walsh (1949b,d)

PERCENTILES
Galton (1889), Gore (1889), Galton (1896), Yule (1896b), Edgeworth (1898), Galton (1899a,b), Bowley (1901), Edgeworth (1905), King (1912), Vinci (1918), Czuber (1921), Goldhizer (1923), Jackson (1923), Burgess (1927), Jordan (1927), March (1930), von Mises (1931), Yang (1933), Bresciani-Turroni (1937), Gini (1939), Arley & Buch (1940), Fréchet (1941), Boldrini (1942), Evans (1942), Schützenberger (1948), Tukey (1949e), Wolfowitz (1949)

PERIODICITIES
Walker (1914), Whittaker & Robinson (1924), Fisher (1929, 1940), Kendall (1946), Hartley (1949)

PERIOD, RETURN—SEE RETURN PERIOD

PHOTOGRAPHY, APPLICATIONS TO
Grant (1946)

PHYSICS, APPLICATIONS TO
Edgeworth (1913), Goodwin (1913), Frank (1918), Rietz (1924), Reilly, Rae & Wheeler (1925), Gumbel (1927, 1928), Scarborough (1930), von Mises (1931), Eddington (1933), Jeffreys (1933), Smith (1934), Gumbel (1936b), Ruark & Devol (1936), Coutagne (1937), Gumbel (1937b,d), Neyman (1937b), Thibaud (1937), Thibaud & Ferber (1937), Brenet & Armand (1938), Bruen (1938), Fisher & Yates (1938), Gumbel (1938a,c), Jeffreys (1938), Gumbel (1939b,f), Jeffreys (1939), Ruark (1939), Weibull (1939b), Arley & Buch (1940), Kendall (1943, 1946), Eisenhart & Martin (1948), Doob (1949), Epstein (1949a)

PHYSIOLOGY AND MEDICINE, APPLICATIONS TO
Gini (1914a), Weinberg (1916), Czuber (1921), Jones (1921), von Bortkiewicz (1922b), Wilson (1923), Gini (1926), von Verschuer (1927), Neyman & Pearson (1928), Weinberg (1933), Savur (1937b), Snedecor (1937), Bruen (1938), Gini (1939), Pearson (1939), von Schelling (1941), Boldrini (1942), Henry (1942), Kendall (1943), Greenwood (1946), Kendall (1946), Lord (1947), Moran (1947)

PLOTTING POSITIONS (ON PROBABILITY PAPER)
Fuller (1914), Hazen (1914), Hall (1921), Goodrich (1927), Hazen (1930), Foster (1934, 1936), Grassberger (1936), Horton (1936), Saville (1936), Kimball (1938), Gumbel (1939a,d), Davis (1941), Gumbel (1941a,b), Beard (1942), Gumbel (1942a,c), Pearson (1942), Gumbel (1943a,b), Powell (1943), Price (1943), Gumbel (1945a,b), Nordquist (1945), Kimball (1946b), Gumbel (1947, 1948), Thomas (1948), Langbein (1949), Peterson (1949a), Potter (1949a)

POINT ESTIMATION
Galton (1899a,b), Edgeworth (1911), Fisher (1912), Daniell (1920), Pearson (1920), Fisher (1922), Gibrat (1930), von Bortkiewicz (1931), Pearson & Pearson (1931), Fisher (1934), Pitman (1936), Slade (1936), Gumbel (1937-38), Neyman (1937b), Pitman (1937), Snedecor (1937), Sukhatme (1937), Wold (1937), Fisher & Yates (1938), Gumbel (1938b,c), Pitman (1939), Wald (1939), Grant (1940), Lawley (1940), Craig (1943), Geary (1944), Cramér (1946), Grant (1946), Kendall (1946), Kimball (1946a), Mosteller (1946), Tintner (1946), Egudin (1947), Grubbs & Weaver (1947), Gurney (1947), Gurney & Pearson (1947), Ruggles & Brodie (1947), Eisenhart & Martin (1948), Huzurbazar (1948), Pillai (1948), Stevens (1948), Wilks (1948), Benson (1949), Cole (1949), Godwin (1949b), Howell (1949), Kimball (1949), Mather (1949), Nair (1949b), Peterson (1949a), Press (1949), Prot (1949b), Shone (1949), Walsh (1949d), Wolfowitz (1949), Yamanouchi (1949)

POISSON DISTRIBUTION
Whitworth (1867), Charlier (1906), Edgeworth (1911), von Bortkiewicz (1915), Morant (1921), Fisher (1922), Dodd (1923), Gumbel (1933a,b, 1934e,f, 1935b), Ruark & Devol (1936), Sukhatme (1936), Snedecor (1937), Thibaud (1937), Ruark (1939), Feller (1940), Garwood (1940), Gumbel (1942b), Curtiss (1943), Kendall (1943), Wilks (1943), Cramér (1946), Grant (1946), Greenwood (1946), Kendall (1946), Domb (1947), Feller (1948), Cox (1949), Epstein (1949a), Kac (1949)

POLAR DISTRIBUTION—SEE DOUBLE SPIKE (POLAR) DISTRIBUTION

POPULATION SIZE, ESTIMATION OF, APPLICATIONS TO
Geary (1944), Kendall (1946), Ruggles & Brodie (1947)

POWER EFFICIENCY OF TESTS—SEE TESTS, EFFICIENCY OF

POWER LAW DISTRIBUTION
Camp (1937), Craig (1943)

POWER MEANS—SEE MEANS, POWER

POWER OF A TEST—SEE ALSO ERROR RATE OF A TEST, TYPE II
Neyman & Pearson (1936), Wilks (1943), Bowker (1944), Cramér (1946), Kendall (1946), Walsh (1946a), Jacobson (1947), Walsh (1947), Tukey (1948b), Wilks (1948), Tukey (1949b), Walsh (1949a,c,d)

PRIZES AND COMPETITION, APPLICATIONS TO
Galton (1902), Pearson (1902), Irwin (1925a), Snedecor (1937), Davis (1941), Henry (1942), Wilks (1948)

PROBABILITY INTEGRAL
Gumbel (1942a), Hartley (1942), Pearson (1942), Pearson & Hartley (1942), Wilks (1943), Hartley (1944), Cramér (1946), Grant (1946), Kendall (1946), Walsh (1946b), Bickerstaff (1947), Egudin (1947), Eisenhart, Hastay & Wallis (1947), Eisenhart & Solomon (1947), Elfving (1947), Gumbel (1947), Jacobson (1947), Nair (1947, 1948a), Pillai (1948), Smirnov (1948), Wilks (1948), Gumbel (1949), Paulson (1949)

PROBABILITY-INTEGRAL TRANSFORMATION—SEE TRANSFORMATION, PROBABILITY-INTEGRAL

PROBABILITY PAPER, EXTREME-VALUE
Beard (1942), Gumbel (1943b), Price (1943), Gumbel (1945b), Nordquist (1945), Gumbel (1948), Langbein (1949), Mather (1949), Potter (1949a,b)

PROBABILITY PAPER, HYDRAULIC
Hall (1921)

PROBABILITY PAPER, LOGNORMAL
Fuller (1914), Hazen (1914), Goodrich (1927), Hazen (1930), Slade (1934), Foster (1936), Grassberger (1936), Saville (1936), Kimball (1938), Beard (1942), Gumbel (1942c), Kimball (1942), Tanenhaus (1947), Lane & Lei (1949), Langbein (1949)

PROBABILITY PAPER, NORMAL
Galton (1899b), Fuller (1914), Hazen (1914), Wilson (1923), Foster (1924), Jarvis (1926), Goodrich (1927), Hazen (1930), Luyten (1932), Slade (1934), Foster (1936), Grassberger (1936), Saville (1936), Kimball (1938), Beard (1942), Gumbel (1943a,b, 1945a), Grant (1946), Jacobson (1947), Lane & Lei (1949), Peterson (1949a,b), Scheffé (1949)

PROBABILITY PAPER, PLOTTING POSITIONS ON—SEE PLOTTING POSITIONS (ON PROBABILITY PAPER)

PROBABILITY PAPER, SKEW
Goodrich (1927), Saville (1936), Lane & Lei (1949)

PROPORTIONS
Lexis (1875), Cramér (1946), Grant (1946), Kendall (1946), Noether (1949)

P-STATISTICS
Roy (1939, 1940a,b, 1942a,b, 1945), Anderson (1946), Kendall (1946), Roy (1946a,b), Bartlett (1947b), Rao (1948)

PSYCHOLOGY, APPLICATIONS TO
Neyman & Pearson (1928), Wilson & Hilferty (1929), Hotelling (1933), Fréchet (1935), Hotelling (1936a), Madow (1937), Hertzman (1938), Bartlett (1948), Schützenberger (1948)

QUADRATIC APPROXIMATION
Goedseels (1925)

QUADRATIC INDICES OF DISSIMILARITY, CONNECTION AND CONCORDANCE (HOMOPHILY)—SEE INDICES OF DISSIMILARITY, CONNECTION AND CONCORDANCE (HOMOPHILY), QUADRATIC

QUADRATIC QUASI-MIDRANGES
Láska (1900)

QUALITY CONTROL, APPLICATIONS TO
Pearson (1931), Shewhart (1931), Dodge (1932), Pearson (1932), Dodge (1933), Pearson (1935), Pearson & Haines (1935), Neyman & Pearson (1936), Eisenhart (1939), Simon (1941), Dudding & Jennett (1942), Hartley (1942), Pearson & Hartley (1942), Barnard (1943), Knudsen (1943), Pearson & Hartley (1943), Barnard (1944), Gumbel (1944), Rodgers (1944), Gumbel (1945a), Grant (1946), Jones (1946), Kendall (1946), Mosteller (1946), Walsh (1946a), Winsten (1946), Bickerstaff (1947), Cohan (1947), Egudin (1947), Eisenhart, Hastay & Wallis (1947), Eisenhart & Solomon (1947), Grubbs & Weaver (1947), Jacobson (1947), Lord (1947), Purcell (1947), Eisenhart & Martin (1948), Murphy (1948), Nair (1948a), Stevens (1948), Cole (1949), Howell (1949), Purcell (1949), Scheffé (1949), Walsh (1949c)

QUANTALS
March (1930)

QUANTILES
von Andrae (1860), Whitworth (1867), Galton (1869), Whitworth (1897), Sheppard (1899a), Yule (1911), Vinci (1918), Niceforo (1919), Daniell (1920), Pearson (1920), Julin (1921), Goldhizer (1923), Jackson (1923), Lévy (1925), Burgess (1927), Jordan (1927), de Montessus de Ballore (1930), Galvani (1931), Yang (1933), Mathews (1936), Savur (1937b), Dodd (1938), Gini (1939), Kendall (1940), Boldrini (1942), Evans (1942), Curtiss (1943), Kendall (1943), Scheffé (1943), Scheffé & Tukey (1945), Cramér (1946), Kendall (1946), Noether (1948), Wilks (1948), Benson (1949), Cox (1949), Godwin (1949b), Noether (1949)

QUARTALS
von Bortkiewicz (1931)

QUARTILE AVERAGE
Bowley (1901), Daniell (1920), Wold (1935)

QUARTILE DEVIATION (SEMI-INTERQUARTILE RANGE)
Quetelet (1835), Venn (1866), Zachariae (1871), Galton (1875), Merriman (1877), Galton (1881), Edgeworth (1885, 1886a,b, 1887-90), Galton (1888, 1889), Gore (1889), Pizzetti (1892), Yule (1896a), Bowley (1901), Galton (1907b), Kozák (1907, 1908-10), Yule (1911), Gini (1912), King (1912), Secrist (1917), Niceforo (1919), Daniell (1920), Pearson (1920), Czuber (1921), Jones (1921), Julin (1921), Rietz (1924), Winkler (1924), Lévy (1925), Burgess (1927), Goodrich (1927), Jordan (1927), March (1930), von Bortkiewicz (1931), Hojo (1931), Pearson & Pearson (1931), Luyten (1932), Pearson & Pearson (1932), Crowe (1933), Hojo (1933), Rhodes (1933), Bond (1935), Wold (1935), Gini (1939), Fréchet (1940a,b, 1942), Kendall (1943), Reiersöl (1944), Cramér (1946), Mosteller (1946)

QUARTILE MEASURE OF SKEWNESS—SEE SKEWNESS, BOWLEY'S QUARTILE MEASURE OF

QUARTILE RATIO
Goodrich (1927)

QUARTILES

von Andrae (1860), Venn (1866), Zachariae (1871), Galton (1881), Edgeworth (1886b, 1887-90), Gore (1889), Edgeworth (1893, 1898), Galton (1899a), Sheppard (1899a), Bowley (1901), Galton (1907b,c), Persons (1909), Yule (1911), Gini (1912), King (1912), Edgeworth (1913), Secrist (1917), Vinci (1918), Niceforo (1919), Dalton (1920), Daniell (1920), Pearson (1920), Czuber (1921), Jones (1921), Julin (1921), Goldhizer (1923), Jackson (1923), Rietz (1924), Lévy (1925), Burgess (1927), Goodrich (1927), Jordan (1927), Gibrat (1930), March (1930), de Montessus de Ballore (1930), d'Addario (1931), von Bortkiewicz (1931), Edgett (1931), Galvani (1931), Hojo (1931), von Mises (1931), Pearson & Pearson (1931), Craig (1932a), Luyten (1932), Pearson & Pearson (1932), Craig (1933), Crowe (1933), Hojo (1933), Rhodes (1933), Yang (1933), Bond (1935), Wold (1935), Crowe (1936), Gumbel (1936a), Mathews (1936), Bresciani-Turroni (1937), Gumbel (1937d, 1937-38), Snedecor (1937), Brenet & Armand (1938), Dodd (1938), Gumbel (1938a,b), Gini (1939), Gumbel (1939a,c,e), Arley & Buch (1940), Pizzetti (1940), Fréchet (1941), Roller (1941), Boldrini (1942), Frechét (1942), Curtiss (1943), Gumbel (1943a,c), Kendall (1943), Mathisen (1943), Bowker (1944), Cramér (1946), Kendall (1946), Bickerstaff (1947), Tanenhaus (1947), Wilks (1948), Benson (1949)

QUASI-MIDRANGES

Faye (1888), Pizzetti (1889, 1892), Láska (1900), Haag (1924), Jordan (1927), Craig (1932b), Gumbel (1944), Cramér (1946), Wilks (1948), Cole (1949)

QUASI-MIDRANGES, GEOMETRIC—SEE GEOMETRIC QUASI-MIDRANGES

QUASI-MIDRANGES, QUADRATIC—SEE QUADRATIC QUASI-MIDRANGES

QUASI-RANGES

Craig (1932b), Calichiopulo (1937), Hertzman (1938), Thompson (1938), Newman (1939), Pearson (1939), Wilks (1941), Kendall (1943), Gumbel (1944), Jones (1946), Mosteller (1946), Bickerstaff (1947), Gumbel (1947), Wilks (1948), Cole (1949), Godwin (1949b)

QUOTIENT, MEAN—SEE MEAN QUOTIENT

RANDOM DIVISION OF AN INTERVAL

Fourier (1826b), Clifford (1866), Whitworth (1867), Halphen (1873), Lemoine (1873), Merriman (1877), Lalanne (1879), van Pesch (1882, 1884), van den Berg (1891), Whitworth (1897), Wölffing (1899), von Bortkiewicz (1915), Mantel (1915), Morant (1921), Bottema (1925), Chadwick (1926), Baticle (1933a,b, 1935), Ruark & Devol (1936), Sukhatme (1936), Lévy (1939), Ruark (1939), Stevens (1939), Feller (1940), Fisher (1940), Garwood (1940), Kendall (1946), Domb (1947), Moran (1947)

RANDOMNESS, TESTS OF

Dodd (1942), Kendall (1946), Wilks (1948)

RANGE

Ptolemaeus (circa 150), Lloyd (1855), Boudin (1865), Merriman (1877), Gore (1889), Yule (1896a), Fechner (1897), Mitscherlich (1903), Charlier (1910), Yule (1911), King (1912), Czuber (1914), Savorgnan (1915), Secrist (1917), Czuber (1918), Niceforo (1919), von Bortkiewicz (1922a,b), von Mises (1923), Neyman (1923), Horton (1924), Tippett (1925), Pearson (1926), Burgess (1927), Student (1927), Bartels (1928), Köppen (1928), Neyman & Pearson (1928), Pearson & Adyanthāya (1928, 1929), Rider (1929), Pearson (1931), Pearson & Pearson (1931), Shewhart (1931), Craig (1932a,b), Dodge (1932), Pearson (1932), Bartlett (1933), Craig (1933), Dodge (1933), McKay & Pearson (1933), Rhodes (1933), Romanovsky (1933), Weinberg (1933), Davies & Pearson (1934), Jeffreys (1934), Birkeland & Frogner (1935), Castellano (1935), Olds (1935), Pearson (1935), Pearson & Haines (1935), Sastry (1935), Nair (1936), Neyman & Pearson (1936), Pearson & Chandra Sekar (1936), Snedecor

(1937), Bose (1938), Bruen (1938), von Eberhard (1938), Wilks (1938), Newman (1939), Pearson (1939), Arley & Buch (1940), Kullback & Frankel (1940), Nair (1940a), Simon (1941), Boldrini (1942), Dodd (1942), Dudding & Jennett (1942), Hartley (1942), Pearson & Hartley (1942), Kendall (1943), Knudsen (1943), Pearson & Hartley (1943), Rajalakshman (1943), Scheffé (1943), Wilks (1943), Barnard (1944), Hartley (1944), Robbins (1944b), Rodgers (1944), Wing, Price & Douglass (1944), Baker (1946), Carlton (1946), Cramér (1946), Daly (1946), Grant (1946), Hoel (1946), Kendall (1946), Mosteller (1946), Winsten (1946), Bickerstaff (1947), Cohan (1947), Duncan (1947), Egudin (1947), Eisenhart, Hastay & Wallis (1947), Elfving (1947), Grubbs & Weaver (1947), Gumbel (1947), Lehmann (1947), Lord (1947), Nair (1947), Plackett (1947), Purcell (1947), Walsh (1947), Cox (1948), Eisenhart & Martin (1948), Gartštein (1948), Nair (1948b), Pillai (1948), Schützenberger (1948), Stevens (1948), Tukey (1948b), Wilks (1948), Cole (1949), Cox (1949), Godwin (1949b), Gumbel (1949), Howell (1949), Nair (1949b), Paulson (1949), Purcell (1949), Scheffé (1949), Shone (1949), Tukey (1949b), Walsh (1949b,c)

RANGE, INTERDECILE—SEE INTERDECILE RANGE

RANGE, INTERQUARTILE—SEE INTERQUARTILE RANGE

RANGE, METHOD USING—SEE METHOD USING THE RANGE

RANGE, MOVING—SEE MOVING RANGE

RANGE, REDUCED—SEE REDUCED RANGE

RANGE, R-PLY—SEE R-PLY RANGE, SEMIRANGE AND/OR MIDRANGE

RANGE, SEMI-INTERQUARTILE—SEE QUARTILE DEVIATION (SEMI-INTERQUARTILE RANGE)

RANGE, STUDENTIZED—SEE STUDENTIZED RANGE

RANKED DIFFERENCE TEST, DUNCAN'S
Duncan (1947), Tukey (1948c, 1949a)

RATIO OF QUARTILES—SEE QUARTILE RATIO

RECTANGULAR DISTRIBUTION, CONTINUOUS—SEE UNIFORM DISTRIBUTION, CONTINUOUS

RECTANGULAR DISTRIBUTION, DISCRETE—SEE UNIFORM DISTRIBUTION, DISCRETE

RECURRENCE INTERVAL
Gumbel (1941a,b), Beard (1942), Gumbel (1942b,c), Kimball (1942), Gumbel (1943a,b,c), Powell (1943), Price (1943), Gumbel (1945a), Thomas (1948), Langbein (1949), Potter (1949a,b), Press (1949)

REDUCED RANGE
Gumbel (1947), Cox (1948), Gartštein (1948), Wilks (1948), Gumbel (1948)

REGIONS, CONFIDENCE—SEE CONFIDENCE REGIONS

REGIONS, TOLERANCE—SEE TOLERANCE REGIONS

REGRESSION, HOUSNER-BRENNAN ESTIMATE OF—SEE HOUSNER-BRENNAN ESTIMATE OF REGRESSION

REGRESSION, LINEAR
Euler (1749), Mayer (1750), Maire & Boscovich (1755), Boscovich (1757, 1760), Lambert (1765b), Lalande (1771), Bernoulli (1785), Laplace (1786, 1793, 1799), Prony (1804), Legendre (1805), Puissant (1805), Svanberg (1805), von Zach (1805), Delambre (1806-10), von Lindenau (1806), Adrain (1808), Gauss (1809), Laplace (1811a,b, 1812), Delambre (1813), Mathieu (1813-14), Legendre (1814), van Beeck Calkoen (1816), Adrain (1818), Fourier (1823), Gauss (1823), Cauchy (1824), Fourier (1824b,c), Ivory (1825), Muncke (1825), Hauber (1830-32), von Riese (1830), Cauchy (1831), Fourier (1831), Encke (1832-34), Poncelet (1835), Dirichlet (1836), Cauchy (1837), Hagen (1837), Bessel & Baeyer (1838), Cournot (1843), Gerling (1843), Ellis (1844), De Morgan (1847), Herschel (1850), Tchebychef (1854), Petzval (1857), von Andrae (1860), Airy (1861), Chauvenet (1863), Boudin (1865), Todhunter (1865), Zachariae (1871), Glaisher (1872), Todhunter (1873), Merriman (1877), Bruns (1882), Schreiber (1882), Edgeworth (1887b,c,f), Turner (1887), Edgeworth (1888), Jordan (1888), Gore (1889), Czuber (1890), Estienne (1890), Runge (1890), Czuber (1891a), Pizzetti (1892), Poincaré (1896), Czuber (1899), Wölffing (1899), Bowley (1901), Czuber (1903), Kirchberger (1903), Saunder (1903), Mansion (1906), Goedseels (1909), de la Vallée Poussin (1909), Charlier (1910), Goedseels (1910), Edgeworth (1911), Goedseels (1911), de la Vallée Poussin (1911), Tits (1912), Goodwin (1913), Mansion (1913), Haar (1918), de la Vallée Poussin (1919), Campbell (1920), Jones (1921), Edgeworth (1923), Jackson (1924), Rietz (1924), Whittaker & Robinson (1924), Goedseels (1925), Lévy (1925), Reilly, Rae & Wheeler (1925), Estienne (1926-27), Friedrich (1927), Bowley (1928), Dufton (1928), Smith (1929), Rhodes (1930), Scarborough (1930), von Mises (1931), Eddington (1933), Smith (1934), Bond (1935), Darmois (1936), Friedrich (1937), Bartlett (1938), Bose (1938), Bruen (1938), Fisher (1938), Eisenhart (1939), Jeffreys (1939), Pitman (1939), Arley & Buch (1940), Lawley (1940), Singleton (1940), Wald (1940), Boldrini (1942), Nair & Shrivastava (1942), Deming (1943), Nair & Banerjee (1943), Wilks (1943), Tintner (1945), Cramér (1946), Grant (1946), Kendall (1946), Tintner (1946), Bickerstaff (1947), Bartlett (1948), Geary (1948), Housner & Brennan (1948), Bartlett (1949), Mather (1949), Potter (1949a,b)

REGRESSION, NONLINEAR
Lambert (1765b), Bernoulli (1785), van Beeck Calkoen (1816), Hauber (1830-32), Cauchy (1831), Tchebychef (1854), Todhunter (1865), Merriman (1877), Gore (1889), Pizzetti (1892), Wölffing (1899), Kirchberger (1903), Goedseels (1909), Edgeworth (1911), de la Vallée Poussin (1919), Jones (1921), Rietz (1924), Scarborough (1930), Smith (1934), Eisenhart (1939), Nair & Shrivastava (1942), Deming (1943), Nair & Banerjee (1943), Cramér (1946), Grant (1946), Kendall (1946), Bickerstaff (1947), Bartlett (1948, 1949), Mather (1949)

REGRESSION, ROLLER'S QUARTILE COEFFICIENT OF—SEE ROLLER'S QUARTILE COEFFICIENT OF REGRESSION

REGRESSION, WEIGHTED—SEE WEIGHTED REGRESSION

RELATIVE VARIABILITY, BOWLEY'S QUARTILE MEASURE OF
Bowley (1901), Lorenz (1905), Persons (1909), Niceforo (1919), Dalton (1920), Julin (1921), von Bortkiewicz (1931), Bresciani-Turroni (1937)

RELATIVE VARIABILITY, MEASURES OF
Bowley (1901), Holmes (1905), Lorenz (1905), Watkins (1905, 1908), Persons (1909), Watkins (1909), Dettori (1912), Czuber (1914), Gini (1914a, 1915a,b), Pietra (1915), Ricci (1916), Gini (1917), Niceforo (1919), Dalton (1920), Fisher (1920), de Pietri-Tonelli (1920), Jones (1921), Julin (1921), Rietz (1924), Winkler (1924), Pietra (1925), Gini (1926), Burgess (1927), Jordan (1927), Tucker (1927), Gumbel

(1928), Gini (1930), von Bortkiewicz (1931), de Finetti (1931a), Galvani (1931, 1932), Gini (1932), Pietra (1932a,b), Yntema (1933), d'Addario (1934b), Mortara (1934b), Castellano (1935), Pietra (1935a,b), Wold (1935), d'Addario (1936), Gini (1936), Mathews (1936), Slade (1936), Bresciani-Turroni (1937, 1939), Gini (1939), de Vergottini (1939), Boldrini (1942), Dudding & Jennett (1942), Gaede (1942), Kendall (1943), Wing, Price & Douglass (1944), Cramér (1946), Kendall (1946), Gurney (1947), Nair (1947), Lane & Lei (1949), Nair (1949a), Potter (1949a)

REPEATING CIRCLES, METHOD OF
Svanberg (1805), Todhunter (1865)

RESTRICTED CLASS OF MULTIVARIATE DISTRIBUTIONS
Mood (1941), Tukey (1947, 1948a), Wilks (1948), Wolfowitz (1949)

RESTRICTED CLASS OF UNIVARIATE DISTRIBUTIONS
Laplace (1781), Gauss (1809), Laplace (1810, 1811a,b, 1812), Cournot (1843), Merriman (1877), Pizzetti (1889, 1892), Estienne (1900a), Whittaker & Robinson (1924), Estienne (1926-27), Fogelson (1930), Kolmogoroff (1931), Craig (1932a), Galvani (1932), Pearson & Pearson (1932), Craig (1933), Eddington (1933), Fisher (1934), Foster (1934), Gumbel (1934a,b,c), Jeffreys (1934), Eyraud (1935), Gumbel (1935-36), Pitman (1936), Gumbel (1937c), Neyman (1937a), Smirnoff (1937b), Waschakidse (1938), Gumbel (1939a,b), Hulme & Symms (1939), Pitman (1939), Fréchet (1940a,b), Kendall (1940), Kolmogoroff (1941), Wald & Wolfowitz (1941), Wald (1942), Bowker (1944), Hoel (1946), Winsten (1946), Elfving (1947), Plackett (1947), Tukey (1947), Cox (1948), Hoeffding (1948), Huzurbazar (1948), Pillai (1948), Wilks (1948), Benson (1949), Cole (1949), Godwin (1949b), Gumbel (1949), Juncosa (1949), Metzler (1949), Shone (1949), Wolfowitz (1949)

RETURN PERIOD
Fuller (1914), Spillman, Tolley & Reed (1916), Tolley (1916), Horton (1924), Foster (1934, 1936), Horton (1936), Saville (1936), Creager (1939), Gumbel (1939b,c), Grant (1940), Gumbel (1940, 1941a,b), Beard (1942), Gumbel (1942b,c), Kimball (1942), Gumbel (1943a,b,c), Powell (1943), Price (1943), Gumbel (1945a,b), Nordquist (1945), Kimball (1946a,b), Gumbel (1948), Thomas (1948), Langbein (1949), Press (1949)

RIGHT TRIANGULAR DISTRIBUTION
Gumbel (1928), Galvani (1931), McKay & Pearson (1933), Pitman (1939), Kimball (1946b), Walsh (1949b)

ROBUST PROCEDURES
Harris & Tukey (1949), Tukey (1949c,d,e,f)

ROHNE'S CRITERION FOR REJECTION OF OUTLIERS—SEE OUTLIERS, ROHNE'S CRITERION FOR REJECTION

ROLLER'S QUARTILE COEFFICIENT OF REGRESSION
Roller (1941)

ROLLER'S QUARTILE COEFFICIENT OF UNIFORMITY
Roller (1941)

R-PLY RANGE, SEMIRANGE AND/OR MIDRANGE
Cole (1949)

SAMPLES, CENSORED—SEE CENSORED SAMPLES

SECH DISTRIBUTION
Eisenhart, Deming & Martin (1948b)

SECH-SQUARE DISTRIBUTION
Eisenhart, Deming & Martin (1948b)

SEISMOLOGY, APPLICATIONS TO
Jeffreys (1932a, 1939), Gumbel (1945a), Nordquist (1945)

SELECTED ORDER STATISTICS—SEE ORDER STATISTICS OF SAMPLE, SELECTED

SELECTED POINTS, METHOD OF—SEE METHOD OF SELECTED POINTS

SEMICIRCULAR DISTRIBUTION
Lambert (1765a), Bernoulli (1778), Euler (1778), Bernoulli (1785), Glaisher (1872), Merriman (1877), Pizzetti (1892)

SEMI-INTERQUARTILE RANGE—SEE QUARTILE DEVIATION (SEMI-INTERQUARTILE RANGE)

SEMI-LOGARITHMIC PAPER
Wilson (1923), Jarvis (1926), Potter (1949b)

SEMIRANGE
Ptolemaeus (circa 150), Gore (1889), Goedseels (1909, 1910, 1911), de la Vallée Poussin (1911), Tits (1912), Mansion (1913), Haar (1918), Goedseels (1925), Barral Souto (1938), Pillai (1948), Tukey (1948b), Cole (1949), van Meer & Plantema (1949)

SEMIRANGE, LOWER
Tukey (1948b)

SEMIRANGE, R-PLY—SEE R-PLY RANGE, SEMIRANGE AND/OR MIDRANGE

SEMIRANGE, UPPER
Tukey (1948b)

SEQUENTIAL TESTS
Eisenhart, Hastay & Wallis (1947), Cox (1949)

SERIES, CYCLICAL—SEE CYCLICAL SERIES

SHEPPARD'S MODIFICATION OF CHAUVENET'S CRITERION—SEE OUTLIERS, SHEPPARD'S MODIFICATION OF CHAUVENET'S CRITERION FOR REJECTION

SIMPLE INDICES OF DISSIMILARITY, CONNECTION AND CONCORDANCE (HOMOPHILY)—SEE INDICES OF DISSIMILARITY, CONNECTION AND CONCORDANCE (HOMOPHILY), SIMPLE

SITUATION, METHOD OF—SEE LEAST ABSOLUTE VALUES, METHOD OF

SIZE EFFECT ON MATERIAL STRENGTH
Chaplin (1880, 1882), Slocum & Hancock (1906), Griffith (1920, 1924), Peirce (1926), Tucker (1927),

Peterson (1930), Logan & Grodsky (1931), Reinkober (1931), Alexandrov & Žurkov (1933), Faulhaber (1933), Faulhaber, Buchholtz & Schulz (1933), Logan (1936), l'Hermite (1939), Weibull (1939a,b), Afanassiev (1940), Gillett (1940), Kontorova (1940), Kontorova & Frenkel (1941), Tucker (1941), Moore & Morkovin (1942-44), Kontorova (1943), Wing, Price & Douglass (1944), Daniels (1945), Fowler (1945), Gurney (1945), Tucker (1945a,b), Freudenthal (1946), Kontorova (1946), Oding (1946), Davidenkov, Shevandin & Wittmann (1947), Fisher & Hollomon (1947), Gensamer, Saibel & Lowrie (1947), Gensamer, Saibel & Ransom (1947), Gurney (1947), Gurney & Pearson (1947), Epstein (1948a,b), Epstein & Brooks (1948), Frankel (1948), Higuchi, Leeper & Davis (1948), Hill & Schmidt (1948), McAdam, Geil, Woodard & Jenkins (1948), Epstein (1949a), Kontorova & Timoshenko (1949), Lévi (1949), van Meer & Plantema (1949), Peterson (1949a,b), Prot (1949a,b,c), Weibull (1949)

SKEW FREQUENCY CURVES, KAPTEYN'S SYSTEM—SEE KAPTEYN'S SYSTEM OF SKEW FREQUENCY CURVES

SKEWNESS, BOWLEY'S QUARTILE MEASURE OF

Bowley (1901), Yule (1911), King (1912), Secrist (1917), Niceforo (1919), Julin (1921), Rietz (1924)

SKEWNESS, MEASURES OF

Pearson (1895), Galton (1896), Yule (1896a), Edgeworth (1898), Láska (1900), Bowley (1901), Charlier (1906), Edgeworth (1911), Yule (1911), King (1912), Fuller (1914), Hazen (1914), Tolley (1916), Becker (1917), Secrist (1917), Niceforo (1919), Daniell (1920), Hall (1921), Jones (1921), Julin (1921), Crum (1923), Foster (1924), Rietz (1924), Amoroso (1925), Pearson (1925), Tippett (1925), Pearson (1926), Peirce (1926), Goodrich (1927), Jordan (1927), Cramér (1928), Pearson & Adyanthāya (1928), Hazen (1930), Lidstone (1930), Edgett (1931), Pearson (1931), Shewhart (1931), d'Addario (1932), Garver (1932), Hotelling & Solomons (1932), Foster (1934), Slade (1934), Sastry (1935), Foster (1936), Saville (1936), Slade (1936), Gumbel (1937d), Snedecor (1937), l'Hermite (1939), Gumbel (1942c), Hartley (1942), Kendall (1943), Gumbel (1944), Reiersöl (1944), Rodgers (1944), Gurney (1945), Baker (1946), Cramér (1946), Greenwood (1946), Kendall (1946), Elfving (1947), Grubbs & Weaver (1947), Gumbel (1947), Gurney (1947), Gurney & Pearson (1947), Kimball (1947), Cox (1948), Tukey (1948b), Benson (1949), Lane & Lei (1949), Shone (1949)

SKEWNESS, PEARSON'S MEASURE OF

Pearson (1895), Yule (1896a, 1911), Secrist (1917), Jones (1921), Rietz (1924), Jordan (1927), Cramér (1946)

SKEW PROBABILITY PAPER—SEE PROBABILITY PAPER, SKEW

SMALLEST SAMPLE VALUE

Svanberg (1805), Mathieu (1813-14), Fourier (1826a,b, 1829), Whitworth (1867), Lexis (1875), Herschel (1878), Chaplin (1880), Bruns (1882), Chaplin (1882), Schreiber (1882), Bertrand (1888c), Jordan (1888), Bertrand (1889), Runge (1890), Czuber (1891a), Vallier (1894), Cranz (1896), Fechner (1897), Whitworth (1897), Czuber (1899), Láska (1900), Slocum & Hancock (1906), Kozák (1907, 1908-10), Dunkel (1909), Goedseels (1909, 1910, 1911), de la Vallée Poussin (1911), Tits (1912), Mansion (1913), Mantel (1915), Frank (1918), Griffith (1920), Hall (1921), von Bortkiewicz (1922a,b), Dodd (1923), von Mises (1923), Neyman (1923), Griffith (1924), Horton (1924), Whittaker & Robinson (1924), Bottema (1925), Goedseels (1925), Irwin (1925b), Pearson (1926), Peirce (1926), Burgess (1927), Friedrich (1927), Galvani (1927), Tucker (1927), Bartels (1928), Birkeland (1928), Fisher & Tippett (1928), Köppen (1928), Neyman & Pearson (1928), Gini & Galvani (1929), Lévy (1929), Pearson & Adyanthāya (1929), Rider (1929), Peterson (1930), Galvani (1931), Pearson (1931), Reinkober (1931), Dodge (1932), de Finetti (1932), Gibrat (1932b), Gumbel (1932), Jeffreys (1932a), Alexandrov & Žurkov (1933), Faulhaber (1933), Faulhaber, Buchholtz & Schulz (1933), Gumbel (1933b,c,d), Rider (1933), Baidaff & Barral Souto (1934), Gumbel (1934a,b,c,e), Smith (1934), Birkeland & Frogner (1935), Fisher (1935),

Gumbel (1935a,b), Olds (1935), Pearson (1935), Smirnoff (1935), Thompson (1935), Foster (1936), Mathews (1936), Nair (1936), Pearson & Chandra Sekar (1936), Pitman (1936), Sukhatme (1936), Friedrich (1937), Pitman (1937), Sukhatme (1937), Barral Souto (1938), Bruen (1938), Cisbani (1938), Dodd (1938), von Eberhard (1938), Wilks (1938), Derevitsky (1939), Gumbel (1939b,f), l'Hermite (1939), Lévy (1939), Pitman (1939), Roy (1939), Stevens (1939), de Vergottini (1939), Wald (1939), Weibull (1939a,b), Afanassiev (1940), Arley (1940), Fisher (1940), Gillett (1940), Gumbel (1940), Kontorova (1940), Zappa (1940), Daniels (1941), Finney (1941), Gumbel (1941b), Hsu (1941b), Kontorova & Frenkel (1941), Paulson (1941), Rajalakshman (1941), Simon (1941), Tucker (1941), Aldanondo (1942), Dodd (1942), Gaede (1942), Gumbel (1942b), Hartley (1942), Moore & Morkovin (1942-44), Nair & Shrivastava (1942), Pearson & Hartley (1942), Roy (1942a,b), Wald (1942), Wilks (1942), Barricelli (1943), Craig (1943), Gnedenko (1943), Gumbel (1943a,b,c), Kendall (1943), Knudsen (1943), Kontorova (1943), Pearson & Hartley (1943), Powell (1943), Price (1943), Rajalakshman (1943), Scheffé (1943), Wilks (1943), Geary (1944), Gumbel (1944), Robbins (1944b), Daniels (1945), Fowler (1945), Gumbel (1945a,b), Gurney (1945), Roy (1945), Tintner (1945), Tucker (1945a,b), Carlton (1946), Cramér (1946), Freudenthal (1946), Kendall (1946), Kimball (1946b), Kontorova (1946), Mosteller (1946), Oding (1946), Roy (1946a,b), Tintner (1946), Bickerstaff (1947), Davidenkov, Shevandin & Wittmann (1947), Egudin (1947), Eisenhart, Hastay & Wallis (1947), Elfving (1947), Fisher & Hollomon (1947), Gensamer, Saibel & Lowrie (1947), Gensamer, Saibel & Ransom (1947), Gurney (1947), Gurney & Pearson (1947), Lehmann (1947), Lord (1947), Purcell (1947), Tanenhaus (1947), Walsh (1947), Anderson (1948), Bartlett (1948), Cox (1948), Eisenhart & Martin (1948), Epstein (1948a,b), Epstein & Brooks (1948), Frankel (1948), Gartštein (1948), Geary (1948), Gumbel (1948), Higuchi, Leeper & Davis (1948), Hill & Schmidt (1948), Huzurbazar (1948), McAdam, Geil, Woodard & Jenkins (1948), Nair (1948a,b), Nanda (1948a,b), Pillai (1948), Schützenberger (1948), Tukey (1948b), Wilks (1948), Birnbaum & Zuckerman (1949), Cole (1949), Epstein (1949a,b), Gumbel (1949), Howell (1949), Juncosa (1949), Kontorova & Timoshenko (1949), Lévi (1949), Mather (1949), McMillan (1949), van Meer & Plantema (1949), Paulson (1949), Peterson (1949a,b), Potter (1949a,b), Prot (1949a,c), Purcell (1949), Smirnov (1949b), Tukey (1949b), Walsh (1949a,b,c), Weibull (1949)

SOCIOLOGY, APPLICATIONS TO—SEE ALSO PAUPERISM, APPLICATIONS TO
Quetelet (1835), Holmes (1892), Galton (1896), Yule (1896a,b), Pareto (1897), Holmes (1905), Lorenz (1905), Watkins (1905, 1908), Gini (1909), Persons (1909), Watkins (1909), Gini (1910), Furlan (1911), Mortara (1911), Gini (1912), King (1912), Porru (1912), Edgeworth (1913), Gini (1914a), Savorgnan (1915), Ricci (1916), Gini (1917, 1918), Vinci (1918), Niceforo (1919), Dalton (1920), Julin (1921), Savorgnan (1921), Winkler (1924), Amoroso (1925), Gini (1926), Saibante (1926), Gumbel (1927, 1928), Lenz & von Verschuer (1928), Gini & Galvani (1929), d'Addario (1930), Gibrat (1930), Gini (1930), de Gleria (1930), d'Addario (1931), von Bortkiewicz (1931), Galvani (1931), Gibrat (1931), d'Addario (1932), Galvani (1932), Gini (1932), Castellano (1933a,b), Rhodes (1933), Yntema (1933), d'Addario (1934a,b), Dominedò (1934), Mortara (1934a,b), Castellano (1935), Pietra (1935a,b), Wold (1935), d'Addario (1936), Gini (1936), Bresciani-Turroni (1937), d'Addario (1939), Bresciani-Turroni (1939), Gini (1939), Davis (1941), Pizzetti (1941b), Boldrini (1942), Kendall (1943), Cramér (1946), Noether (1949)

SPIKE DISTRIBUTION (SINGLE ISOLATED VALUE)
Bruen (1938), Gartštein (1948)

STABLE DISTRIBUTIONS
Brown & Tukey (1946), Juncosa (1949)

STANDARDS OF LENGTH, APPLICATIONS TO
Stampfer (1839), Faye (1888), Mendeleev (1895), Wellisch (1913), Kendall (1946)

STEWART'S METHOD OF TREATING OUTLIERS—SEE OUTLIERS, STEWART'S METHOD OF TREATING

STOCHASTIC PROCESSES, APPLICATIONS TO
Madow (1937), Cisbani (1938), Davis (1941), Cramér (1946), Kendall (1946), Doob (1949), Epstein (1949b), Kac (1949)

STONE'S FIRST (SECOND) CRITERION FOR REJECTION OF OUTLIERS—SEE OUTLIERS, STONE'S FIRST (SECOND) CRITERION FOR REJECTION

STRENGTH-SUMMATION THEORY—SEE STRENGTH THEORY, SUMMATION

STRENGTH THEORY, COMBINATION
Tucker (1945b), van Meer & Plantema (1949)

STRENGTH THEORY, SUMMATION
Tucker (1941, 1945a,b), van Meer & Plantema (1949)

STRENGTH THEORY, WEAKEST-LINK
Chaplin (1880, 1882), Slocum & Hancock (1906), Griffith (1920, 1924), Peirce (1926), Tucker (1927), Peterson (1930), Logan & Grodsky (1931), Reinkober (1931), Alexandrov & Žurkov (1933), Faulhaber (1933), Faulhaber, Buchholtz & Schulz (1933), Logan (1936), l'Hermite (1939), Weibull (1939a,b), Afanssiev (1940), Gillett (1940), Kontorova (1940), Kontorova & Frenkel (1941), Tucker (1941), Moore & Morkovin (1942-44), Kontorova (1943), Daniels (1945), Fowler (1945), Gurney (1945), Tucker (1945a,b), Freudenthal (1946), Kontorova (1946), Oding (1946), Davidenkov, Shevandin & Wittmann (1947), Fisher & Hollomon (1947), Gensamer, Saibel & Lowrie (1947), Gensamer, Saibel & Ransom (1947), Gurney (1947), Gurney & Pearson (1947), Epstein (1948a,b), Epstein & Brooks (1948), Frankel (1948), Higuchi, Leeper & Davis (1948), Hill & Schmidt (1948), McAdam, Geil, Woodard & Jenkins (1948), Epstein (1949a), Kontorova & Timoshenko (1949), Lévi (1949), van Meer & Plantema (1949), Peterson (1949a,b), Prot (1949a,c), Weibull (1949)

STUDENTIZATION
Hartley (1944), Kendall (1946), Tukey (1948c, 1949a)

STUDENTIZED EXTREME DEVIATE
Hartley (1944), Nair (1948a,b), Tukey (1948c, 1949a)

STUDENTIZED LARGEST GAP
Tukey (1948c, 1949a)

STUDENTIZED RANGE
Snedecor (1937) [1956], Newman (1939), Pearson (1939), Hartley (1942), Pearson & Hartley (1943), Hartley (1944), Baker (1946), Kendall (1946), Duncan (1947), Nair (1948a,b), Tukey (1948c), Wilks (1948), Paulson (1949), Tukey (1949a)

STUDENT'S CRITERION FOR REJECTION OF OUTLIERS—SEE OUTLIERS, STUDENT'S CRITERION FOR REJECTION

STUDENT'S T DISTRIBUTION
Student (1931), Thompson (1935), Wilks (1938), Eisenhart (1939), Newman (1939), Pearson (1939), Wald (1940), Kendall (1943), Knudsen (1943), Wilks (1943), Hartley (1944), Cramér (1946), Daly (1946), Kendall (1946), Walsh (1946a), Duncan (1947), Lord (1947), Walsh (1947), Nair (1948a,b), Rao (1948), Tukey (1948b,c), Wilks (1948), Bartlett (1949), Tukey (1949a,b), Walsh (1949a,b)

SUBNORMAL DISTRIBUTIONS
Bohlmann (1909), Daniell (1920)

SUCCESSIVE DIFFERENCES
Bréger (1881), von Bortkiewicz (1931), Bose (1938), Nair & Shrivastava (1942)

SUFFICIENCY (OF STATISTICS)
Fisher (1920, 1922, 1934), Pitman (1936, 1937), Cramér (1946), Kendall (1946), Kimball (1946a), Eisenhart & Solomon (1947), Lehmann (1947), Huzurbazar (1948), Kimball (1949)

SUPERNORMAL DISTRIBUTIONS
Bohlmann (1909), Daniell (1920)

SYMMETRIC DISTRIBUTIONS, CONTINUOUS
Gumbel (1944, 1949), Tukey (1949b), Walsh (1949a,b,c), Wolfowitz (1949)

SYSTEMATIC STATISTICS
Mosteller (1946), Hastings, Mosteller, Tukey & Winsor (1947), Wilks (1948), Nair (1949b), Yamanouchi (1949)

T DISTRIBUTION, STUDENT'S—SEE STUDENT'S T DISTRIBUTION

TAXONOMY, APPLICATIONS TO
Fisher (1938), Rao (1948)

TELEPHONY, APPLICATIONS TO
Kendall (1946)

TERTILES
Burgess (1927), Jeffreys (1932b), Bartlett (1933), Fisher (1934), Jeffreys (1934)

TEST, COCHRAN'S—SEE COCHRAN'S RATIO (TEST STATISTIC)

TEST, KOLMOGOROV-SMIRNOV—SEE KOLMOGOROV-SMIRNOV STATISTIC (OR TEST)

TEST, MULTIPLE T (LSD)—SEE MULTIPLE T (LSD) TEST

TEST, POWER OF—SEE POWER OF A TEST

TESTING HYPOTHESES, BOTH ONE-SIDED AND TWO-SIDED
Neyman & Pearson (1928), Pearson & Adyanthāya (1929), Student (1931), Pearson & Chandra Sekar (1936), Madow (1937), Neyman (1937a), Savur (1937b), Snedecor (1937), Sukhatme (1937), Fisher (1938), Savur (1938), Smirnov (1939a), Wald (1939), Roy (1942a,b), Scheffé (1943), Wilks (1943), Tintner (1945), Cramér (1946), Kendall (1946), Kimball (1947), Feller (1948), Geary (1948), Nair (1948b), Rao (1948), Tukey (1948c), Wilks (1948), Cox (1949), Kac (1949), Maniya (1949), Noether (1949), Smirnov (1949a), Tukey (1949a,b), Walsh (1949a,b,c), Wolfowitz (1949)

TESTING HYPOTHESES, ONE-SIDED—SEE ALSO TESTING HYPOTHESES, BOTH ONE-SIDED AND TWO-SIDED
Fisher (1929), McKay (1935), Sukhatme (1936), Thibaud & Ferber (1937), Bartlett (1938), Fisher & Yates (1938), Wishart (1938), Fisher (1939), Newman (1939), Pearson (1939), Fisher (1940), Chandra Sekar & Francis (1941), Cochran (1941), Davis (1941), Finney (1941), Knudsen (1943), Pearson & Hartley (1943), Daly (1946), Greenwood (1946), Tintner (1946), Walsh (1946a), Bartlett (1947b), Eisenhart, Hastay & Wallis (1947), Eisenhart & Solomon (1947), Jacobson (1947), Lehman (1947), Lord (1947), Walsh (1947), Anderson (1948), Nair (1948a), Hartley (1949), Walsh (1949d)

TESTING HYPOTHESES, TWO-SIDED—SEE ALSO TESTING HYPOTHESES, BOTH ONE-SIDED AND TWO-SIDED
Thompson (1935), Savur (1937a), Smirnoff (1937a), Roy (1939), Smirnov (1939b,c), Paulson (1940), Roy (1940a,b), Kolmogoroff (1941), Paulson (1941), Dodd (1942), Gumbel (1942a), Pearson (1942), Mathisen (1943), Pearson & Hartley (1943), Bowker (1944), Smirnov (1944), Kimball (1946b), von Mises (1947), Bartlett (1948), Smirnov (1948), Chung (1949), Doob (1949)

TESTS, DESTRUCTIVE, APPLICATIONS TO—SEE DESTRUCTIVE TESTS, APPLICATIONS TO

TESTS, EFFICIENCY OF
Tukey (1949b), Walsh (1949a,b,c,d)

TESTS, GOODNESS-OF-FIT—SEE GOODNESS-OF-FIT TESTS

TESTS, K-SAMPLE (K=3 OR MORE)—SEE K-SAMPLE TESTS (K=3 OR MORE)

TESTS, LIKELIHOOD-RATIO—SEE LIKELIHOOD-RATIO TESTS

TESTS OF LINEAR HYPOTHESES—SEE LINEAR HYPOTHESES, TESTS OF

TESTS OF RANDOMNESS—SEE RANDOMNESS, TESTS OF

TESTS, ONE-SAMPLE—SEE ONE-SAMPLE TESTS (OR CONFIDENCE PROCEDURES)

TESTS (OR INTERVALS), BOTH ONE-SIDED AND TWO-SIDED
Pearson & Chandra Sekar (1936), Thompson (1936), Madow (1937), Neyman (1937a,b), Savur (1937b), Snedecor (1937), Sukhatme (1937), Fisher (1938), Savur (1938), Smirnov (1939a), Wald (1939), Kullback & Frankel (1940), Wald & Wolfowitz (1941), Wilks (1941), Wald (1942), Wilks (1942), Pearson & Hartley (1943), Scheffé (1943), Wilks (1943), Scheffé & Tukey (1945), Tintner (1945), Cramér (1946), Daly (1946), Grant (1946), Kendall (1946), Bickerstaff (1947), Tukey (1947), Feller (1948), Geary (1948), Murphy (1948), Nair (1948b), Tukey (1948a,b), Wilks (1948), Cox (1949), Doob (1949), Maniya (1949), Noether (1949), Tukey (1949b), Walsh (1949a,b,c), Wolfowitz (1949)

TESTS (OR INTERVALS), ONE-SIDED—SEE ALSO TESTS (OR INTERVALS), BOTH ONE-SIDED AND TWO-SIDED
Fisher (1929), Pearson & Adyanthāya (1929), McKay (1935), Thompson (1935), Sukhatme (1936), Smirnoff (1937a), Bartlett (1938), Fisher & Yates (1938), Wishart (1938), Fisher (1939), Newman (1939), Pearson (1939), Fisher (1940), Roy (1940a,b), Chandra Sekar & Francis (1941), Cochran (1941), Davis (1941), Finney (1941), Dodd (1942), Gumbel (1942a), Roy (1942b), Knudsen (1943), Robbins (1944a), Kimball (1946b), Tintner (1946), Walsh (1946a), Bartlett (1947b), Eisenhart, Hastay & Wallis (1947), Eisenhart & Solomon (1947), Jacobson (1947), Kimball (1947), Lehmann (1947), Lord (1947), von Mises (1947), Walsh (1947), Anderson (1948), Bartlett (1948), Nair (1948a), Rao (1948), Smirnov (1948), Tukey (1948c), Chung (1949), Hartley (1949), Kac (1949), Smirnov (1949a), Tukey (1949a), Walsh (1949d)

TESTS (OR INTERVALS), TWO-SIDED—SEE ALSO TESTS (OR INTERVALS), BOTH ONE-SIDED AND TWO-SIDED
Student (1931), Neyman & Pearson (1936), Savur (1937a), Thibaud & Ferber (1937), Eisenhart (1939), Roy (1939), Smirnov (1939b,c), Nair (1940a,b), Paulson (1940), Wald (1940), Kolmogoroff (1941), Paulson (1941), von Schelling (1941), Mathisen (1943), Bowker (1944), Scheffé & Tukey (1944), Smirnov (1944), Greenwood (1946), Kimball (1946a), Duncan (1947), Eisenhart, Deming & Martin (1948b), Noether (1948), Schützenberger (1948), Birnbaum & Zuckerman (1949), Kimball (1949)

TESTS, SEQUENTIAL—SEE SEQUENTIAL TESTS

TESTS, TWO-SAMPLE—SEE TWO-SAMPLE TESTS (OR CONFIDENCE PROCEDURES)

TESTS, UNBIASED—SEE UNBIASED TESTS

THEORY OF ERRORS—SEE ERRORS, THEORY OF

THOMPSON'S CRITERION FOR REJECTION OF OUTLIERS—SEE OUTLIERS, THOMPSON'S CRITERION FOR REJECTION

THREADS, STRENGTH OF BUNDLES OF—SEE BUNDLES OF FIBERS (OR THREADS), STRENGTH OF

TIPPETT'S CRITERION FOR REJECTION OF OUTLIERS—SEE OUTLIERS, TIPPETT'S CRITERION FOR REJECTION

TOLERANCE LIMITS
Wilks (1941), Wald (1942), Wilks (1942), Scheffé (1943), Wald (1943), Wilks (1943), Robbins (1944a), Scheffé & Tukey (1944, 1945), Grant (1946), Kendall (1946), Winsten (1946), Cohan (1947), Tukey (1947), Murphy (1948), Schützenberger (1948), Tukey (1948a), Wilks (1948), Birnbaum & Zuckerman (1949), Wolfowitz (1949)

TOLERANCE REGIONS
Wald (1943), Tukey (1947), Murphy (1948), Tukey (1948a), Wilks (1948)

TOPSØE-JENSEN CRITERION FOR REJECTION OF OUTLIERS—SEE OUTLIERS, TOPSØE-JENSEN CRITERION FOR REJECTION

TOXICOLOGY, APPLICATIONS TO
Fisher & Yates (1938), Eisenhart (1939), Kendall (1946)

TRAFFIC CONTROL, APPLICATIONS TO
Greenwood (1946), Kendall (1946)

TRANSFORMATION, PROBABILITY-INTEGRAL
Mosteller (1946), Bickerstaff (1947), Kimball (1947), Gumbel (1949)

TRAPEZOIDAL DISTRIBUTION
Gumbel (1928), von Bortkiewicz (1931), Galvani (1931), Castellano (1933a), Pitman (1936, 1939)

TREATMENT OF OUTLIERS—SEE OUTLIERS, TREATMENT OF

TRIMMED MEANS—SEE ALSO DISCARD AVERAGES
Short (1763), Newcomb (1912), d'Addario (1939), de Vergottini (1939), Nair & Banerjee (1943), Jacobson (1947), Harris & Tukey (1949), Tukey (1949d)

T-SQUARE DISTRIBUTION, HOTELLING'S—SEE HOTELLING'S T-SQUARE DISTRIBUTION

TUKEY'S GAP-STRAGGLER PROCEDURE—SEE GAP-STRAGGLER PROCEDURE, TUKEY'S

TWO-SAMPLE TESTS (OR CONFIDENCE PROCEDURES)
Savur (1937a,b), Snedecor (1937), Sukhatme (1937), Savur (1938), Roy (1939), Smirnov (1939b), Nair (1940a), Roy (1940a,b), Kolmogoroff (1941), Roy (1942a), Mathisen (1943), Pearson & Hartley (1943), Scheffé (1943), Bowker (1944), Smirnov (1944), Roy (1945), Cramér (1946), Kendall (1946), Tintner (1946), Lord (1947), Walsh (1947), Bartlett (1948), Feller (1948), Rao (1948), Wilks (1948), Cox (1949), Doob (1949), Kac (1949), Tukey (1949b), Walsh (1949c), Wolfowitz (1949)

TYPE A DISTRIBUTION—SEE CHARLIER DISTRIBUTION, TYPE A

TYPE B DISTRIBUTION—SEE CHARLIER DISTRIBUTION, TYPE B

U DISTRIBUTION, LORD'S—SEE LORD'S U DISTRIBUTION

UNBIASED ESTIMATORS
Gumbel (1937-38), Sukhatme (1937), Wold (1937), Fisher & Yates (1938), Wilks (1943), Cramér (1946), Kendall (1946), Mosteller (1946), Egudin (1947), Eisenhart & Martin (1948), Hoeffding (1948), Pillai (1948), Yamanouchi (1949)

UNBIASED TESTS
Neyman & Pearson (1936), Wilks (1943), Cramér (1946), Daly (1946), Eisenhart & Solomon (1947), Wilks (1948)

UNIFORM DISTRIBUTION, CONTINUOUS
Bernoulli (1709), Lagrange (1774), Bernoulli (1785), Clifford (1866), Whitworth (1867), Halphen (1873), Lemoine (1873), Todhunter (1873), Helmert (1876b), Merriman (1877), Lalanne (1879), van Pesch (1882, 1884), van den Berg (1891), Pizzetti (1892), Whitworth (1897), Blümcke (1898), Wölffing (1899), Contarino (1914), Mantel (1915), Haar (1918), Dodd (1922), Fisher (1922), Bottema (1925), Chadwick (1926), Gumbel (1928), Neyman & Pearson (1928), Pearson & Adyanthaya (1928, 1929), Rider (1929), Galvani (1931), Pearson & Pearson (1931), Gini (1932), Baticle (1933a,b), Castellano (1933a,b), Craig (1933), McKay & Pearson (1933), Pollard (1934), Baticle (1935), Castellano (1935), Olds (1935), Sastry (1935), Smirnoff (1935), von Mises (1936), Nair (1936), Pitman (1936), Bowley (1937), Gumbel (1937c), Neyman (1937b), Pitman (1937), Smirnoff (1937b), Wold (1937), Bruen (1938), Jeffreys (1938), Wilks (1938), Gumbel (1939d), Lévy (1939), Pitman (1939), Wald (1939), Kullback & Frankel (1940), Nair (1940b), Paulson (1940), Brookner (1941), Rajalakshman (1941), Thompson (1941), Gumbel (1942a), Pearson (1942), Craig (1943), Gumbel (1943a,b,c), Rajalakshman (1943), Geary (1944), Carlton (1946), Cramér (1946), Greenwood (1946), Kendall (1946), Mosteller (1946), Domb (1947), Gumbel (1947), Hastings, Mosteller, Tukey & Winsor (1947), Kimball (1947), Lehmann (1947), Moran (1947), Plackett (1947), Tukey (1947), Eisenhart, Deming & Martin (1948b), Epstein (1948a), Feller (1948), Tukey (1948a), Kac (1949), Smirnov (1949b), Walsh (1949a,b), Weibull (1949)

UNIFORM DISTRIBUTION, DISCRETE
Simpson (1756, 1757), Lagrange (1774), Bernoulli (1785), Trembley (1804), Hauber (1830-32), De Morgan (1864), Merriman (1877), Pizzetti (1892), Lorenz (1905), Gini (1912), Rider (1929), Smith (1929), Baticle (1933a, 1935), Olds (1935), Pitman (1936), Camp (1937), Gini (1939), Ruggles & Brodie (1947)

UNIFORM DISTRIBUTION ON A CIRCLE
Stevens (1939), Greenwood (1946)

UNIFORMITY, ROLLER'S QUARTILE COEFFICIENT OF—SEE ROLLER'S QUARTILE COEFFICIENT OF UNIFORMITY

UNSPECIFIED DISTRIBUTION

Ptolemaeus (circa 150), Galilei (1632), Cotes (1722), Euler (1749), Mayer (1750), Maire & Boscovich (1755), Boscovich (1757, 1760), Lambert (1760), Short (1763), Lambert (1765a,b), Lalande (1771), Bernoulli (1785), Laplace (1786, 1793, 1799), Prony (1804), Legendre (1805), Puissant (1805), Svanberg (1805), von Zach (1805), Delambre (1806-10), von Lindenau (1806), Laplace (1812), Mathieu (1813-14), Legendre (1814), van Beeck Calkoen (1816), Adrain (1818), Anonymous (1821), Fourier (1823), Gauss (1823), Cauchy (1824), Fourier (1824b,c), Muncke (1825), Fourier (1826a,b, 1829), Hauber (1830-32), von Riese (1830), Cauchy (1831), Fourier (1831), Encke (1832-34), Poncelet (1835), Quetelet (1835), Cauchy (1837), Hagen (1837), Bessel & Baeyer (1838), Dove (1838), Stampfer (1839), Cournot (1843), Gerling (1843), Ellis (1844), Herschel (1850), Tchebychef (1854), Lloyd (1855), Airy (1856), Winlock (1856), Petzval (1857), von Andrae (1860), Todhunter (1865), Venn (1866), Todhunter (1873), Helmert (1875a), Merriman (1877), Wilson (1877), Herschel (1878), Bréger (1881), Bruns (1882), Schreiber (1882), Edgeworth (1885), Bertrand (1887a,b), Edgeworth (1887b, 1887-90), Turner (1887), Bertrand (1888a), Galton (1888), Jordan (1888), Galton (1889), Runge (1890), Holmes (1892), Pizzetti (1892), Goedseels & Mansion (1893), Vallier (1894), Mendeleev (1895), Wölffing (1899), Estienne (1900b), Láska (1900), Bowley (1901), Hayford (1902), Czuber (1903), Heydenreich (1903), Kirchberger (1903), Mitscherlich (1903), Saunder (1903), Holmes (1905), Watkins (1905), Mansion (1906), Slocum & Hancock (1906), Galton (1907a,b,c), Hooker (1907), Mazzuoli (1908), Watkins (1908), Dunkel (1909), Goedseels (1909), Persons (1909), de la Vallée Poussin (1909), Watkins (1909), Gini (1910), Goedseels (1910, 1911), Mortara (1911), de la Vallée Poussin (1911), Yule (1911), Dettori (1912), Gini (1912), King (1912), Newcomb (1912), Porru (1912), Tits (1912), Mansion (1913), Wellisch (1913), Czuber (1914), Fuller (1914), Gini (1914a,b), Hazen (1914), Gini (1915a,b), Pietra (1915), Savorgnan (1915), Trachtenberg (1915), Gini (1916a,b), Ricci (1916), Weinberg (1916), Becker (1917), Gini (1917), Secrist (1917), Gini (1918), Haar (1918), Vinci (1918), Niceforo (1919), Persons (1919), de la Vallée Poussin (1919), Campbell (1920), Dalton (1920), Griffith (1920), de Pietri-Tonelli (1920), Jones (1921), Julin (1921), Savorgnan (1921), Edgeworth (1923), Sloane (1923), Griffith (1924), Horton (1924), Jackson (1924), Rietz (1924), Goedseels (1925), Pearson (1925), Pietra (1925), Gini (1926), Jarvis (1926), Peirce (1926), Saibante (1926), Alliaume (1927a,b), Estienne (1927), Friedrich (1927), Galvani (1927), Gumbel (1927), Jordan (1927), Tucker (1927), von Verschuer (1927), Birkeland (1928), Dufton (1928), Köppen (1928), Saibante (1928), Gini & Galvani (1929), de Gleria (1929), Smith (1929), d'Addario (1930), Eells (1930), de Finetti & Paciello (1930), Gini (1930), de Gleria (1930), March (1930), de Montessus de Ballore (1930), Peterson (1930), Rhodes (1930), Ross (1930), Scarborough (1930), Steffensen (1930), von Bortkiewicz (1931), de Finetti (1931a,b), Horst (1931), Insolera (1931), Logan & Grodsky (1931), Reinkober (1931), Steffensen (1931), Student (1931), d'Addario (1932), Craig (1932b), Galvani (1932), Pietra (1932a,b), Alexandrov & Žurkov (1933), Castellano (1933a,b), Crowe (1933), Faulhaber (1933), Faulhaber, Buchholtz & Schulz (1933), Galvani (1933), Gini, Boldrini, Galvani & Venere (1933), Griffin (1933), Gumbel (1933a,b,f,g), Insolera (1933), Julin (1933), Linders (1933), Rhodes (1933), Scates (1933), Watkins (1933), Yntema (1933), Baidaff & Barral Souto (1934), Dominedò (1934), Eyraud (1934b), de Finetti (1934), Foster (1934), de Franchis (1934), Insolera (1934), Mortara (1934a,b), Smith (1934), Anderson (1935), Baidaff & Barral Souto (1935), Barral Souto (1935), Castellano (1935), Insolera (1935), Pietra (1935a,b), Wold (1935), Crowe (1936), Gini (1936), Logan (1936), Mathews (1936), Aitken (1937), Bowley (1937), Bresciani-Turroni (1937), Friedrich (1937), Insolera (1937-38), Smirnoff (1937b), Barral Souto (1938), Bose (1938), Cisbani (1938), Dodd (1938), von Eberhard (1938), Fisher (1938), Gini (1938), Hertzman (1938), Bresciani-Turroni (1939), Creager (1939), Derevitsky (1939), Gini (1939), de Vergottini (1939), Zappa (1939), Dodd (1940), Fréchet (1940b), Gillett (1940), Grant (1940), Pizzetti (1940), Singleton (1940), de Vergottini (1940), Wald (1940), Zappa (1940), Pizzetti (1941a,b,c), von Schelling (1941), Simon (1941), Dodd (1942), Dudding & Jennett (1942), Fréchet (1942), Henry (1942), Livada (1942), Moore & Morkovin (1942-44), Nair & Shrivastava (1942), Deming (1943), Price (1943), Reiersöl (1944), Wing, Price & Douglass (1944), Daniels (1945), Fowler (1945), Tucker (1945a,b), Kontorova (1946), Oding (1946), Gensamer, Saibel & Lowrie (1947), Gensamer, Saibel & Ransom (1947), Kincaid (1947), Bartlett (1948), Frankel (1948), Geary (1948), Gumbel (1948), Hill & Schmidt (1948), Housner

& Brennan (1948), McAdam, Geil, Woodard & Jenkins (1948), Nanda (1948b), Thomas (1948), Bartlett (1949), Langbein (1949), van Meer & Plantema (1949), Peterson (1949a), Potter (1949b)

UPPER CONFIDENCE BOUND—SEE CONFIDENCE BOUNDS

UPPER SEMIRANGE—SEE SEMIRANGE, UPPER

UPPER TOLERANCE LIMIT—SEE TOLERANCE LIMITS

U-STATISTICS
Hoeffding (1948)

VALLIER'S CRITERION FOR REJECTION OF OUTLIERS—SEE OUTLIERS, VALLIER'S CRITERION FOR REJECTION

VARIABILITY, MEASURES OF—SEE DISPERSION, MEASURES OF

VARIABILITY, RELATIVE, MEASURES OF—SEE RELATIVE VARIABILITY, MEASURES OF

VARIABLES INSPECTION
Simon (1941), Grant (1946), Purcell (1947)

VARIANCE, ANALYSIS OF—SEE ANALYSIS OF VARIANCE (ANOVA OR MANOVA)

VARIOUS MULTIVARIATE DISTRIBUTIONS
Rao (1948), Tukey (1948a)

VARIOUS UNIVARIATE DISTRIBUTIONS
Dirichlet (1836), Quetelet (1846), Boudin (1865), Todhunter (1865), Zachariae (1871), Fechner (1874), Helmert (1877b), Jordan (1877a), Merriman (1877), Edgeworth (1883a,b, 1886a, 1887d), Gore (1889), Czuber (1890), Venn (1891), Pizzetti (1892), Fechner (1897), Blümcke (1901), Edgeworth (1905), Yule (1907), Keynes (1911), Yule (1911), Keynes (1921), Dodd (1922, 1923), Winkler (1924), Fréçhet (1927), Bowley (1928), Fisher & Tippett (1928), Craig (1932a), Gini (1932), Castellano (1933a), d'Addario (1934b), Eyraud (1934a), Rosin & Rammler (1934), Anderson (1935), Bond (1935), Fréchet (1935), Gumbel (1935b,c,d), Wold (1935), Horton (1936), von Mises (1936), Ruark & Devol (1936), Saville (1936), Slade (1936), Gumbel (1937a), Neyman (1937b), Fisher (1938), d'Addario (1939), Ruark (1939), Nair (1940a), Wilson (1940), Aldanondo (1942), Boldrini (1942), Price (1943), Robbins (1944b), Bickerstaff (1947), Gurney (1947), Gurney & Pearson (1947), Huzurbazar (1948), Tukey (1948b), Benson (1949), Prot (1949c), Shone (1949), Tukey (1949b), Walsh (1949b,c)

WALSH AVERAGES
Tukey (1949b), Walsh (1949a,c)

WATKINS' LOGARITHMIC CONCENTRATION CURVE—SEE CONCENTRATION CURVE, WATKINS' LOGARITHMIC

WEAKEST-LINK THEORY OF STRENGTH—SEE STRENGTH THEORY, WEAKEST-LINK

WEIBULL DISTRIBUTION—SEE ALSO EXTREME-VALUE DISTRIBUTION, TYPE III SMALLEST
Weibull (1939a,b), Afanassiev (1940), Roller (1941), Tucker (1941), Davidenkov, Shevandin & Wittmann (1947), Epstein (1948a,b), Frankel (1948), Weibull (1949)

WEIGHTED AVERAGES
Cotes (1722), Lalande (1771), Cournot (1843), Merriman (1877), Wright (1884), Edgeworth (1887-90), Pizzetti (1892), Rietz (1924), Burgess (1927)

WEIGHTED REGRESSION
Tintner (1946)

WELLISCH'S CRITERION FOR REJECTION OF OUTLIERS—SEE OUTLIERS, WELLISCH'S CRITERION FOR REJECTION

WINSORIZED MEANS
Newcomb (1912), Harris & Tukey (1949), Tukey (1949d)

WISHART DISTRIBUTION
Fisher (1939), Lawley (1940), Wilks (1943), Anderson (1945), Tintner (1945), Anderson (1946), Kendall (1946), Anderson (1948), Rao (1948)

WRIGHT-HAYFORD CRITERION FOR REJECTION OF OUTLIERS—SEE OUTLIERS, WRIGHT-HAYFORD CRITERION FOR REJECTION

WRIGHT'S CRITERION FOR REJECTION OF OUTLIERS—SEE OUTLIERS, WRIGHT'S CRITERION FOR REJECTION

Z DISTRIBUTION—SEE FISHER'S Z DISTRIBUTION



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| Summaries | Populations | Skewness | Lease absolute values |
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| Citations | Extreme values | Correlation | Maximum likelihood |
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20. ABSTRACT *(Continue on reverse side if necessary and identify by block number)*

Let a sample of size n from a specified or unspecified population be arranged in nondecreasing order. Then the ordered sample values are collectively called the order statistics (plural) of the sample. In another sense of the expression, order statistics (singular) is that branch of the subject of statistics which deals with the mathematical properties of order statistics (plural) and with statistical methods based upon them. Order statistics has often been considered (incorrectly) to be a branch of nonparametric (over)

→ next page

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20. (cont'd)

statistics (or of distribution-free statistics, which differs from nonparametric statistics in ways too subtle to discuss here). In fact, the majority of publications on order statistics deal with the order statistics of specific populations, and many of them involve the use of the sample order statistics to estimate one or more population parameters. The compiler of this bibliography makes a sharp distinction between order statistics and rank statistics, which is a branch of nonparametric statistics. Publications dealing with the ranks of the observations in the sample but not with the values (or expected values) of any of them will not in general be included in this bibliography; however, a few that are needed for comparison of methods based on ranks with alternatives based on order statistics will be included. Similarly, although the method of least squares is not based on order statistics, a substantial number of publications on least squares will be included because most alternative methods are based on order statistics and comparisons with the classical method of least squares are inevitable. Likewise, since the "weakest-link" theory (based on the theory of extreme values, a branch of order statistics) plays an important role in the size effect on material strength, a few publications on the size effect which do not specifically mention the "weakest-link" theory will be included for comparative purposes.

This volume is divided into three sections. The first and largest of these is a chronological bibliography (alphabetical within years) of 937 pre-1950 publications, with summaries, references and citations. At the end of each bibliographic listing, the source(s) of any known abstract(s) and/or review(s) is (are) given. A summary of each publication (or the portion of it relevant to our subject) is given, sometimes including extensive quotations of the author's own words (or translations of them, if the publication is in a language other than English). Comments and/or notes by the compiler are included when appropriate, separate from and following the summary. The 937 publications summarized in this volume form a closed set with respect to the references and citations; references to and citations by other publications in the set are listed chronologically (alphabetically within years) for each of the 937 publications in the set. A list of references to post-1949 literature (to be summarized in later volumes) and a list of abbreviations used for the names of abstracting and reviewing journals are given at the end of Section I. Section II is an alphabetical list of authors, with year(s) of publication for each author in parentheses following his name. In the case of joint authorship, the complete listing is alphabetized according to the name of the first author, and cross references are alphabetized according to the name(s) of the other author(s). Cross references are also given to different forms or spellings of authors' names. Section III is a subject index in which the subjects are listed alphabetically, with cross references when appropriate, and the publications pertinent to each subject are listed chronologically (alphabetically within years).